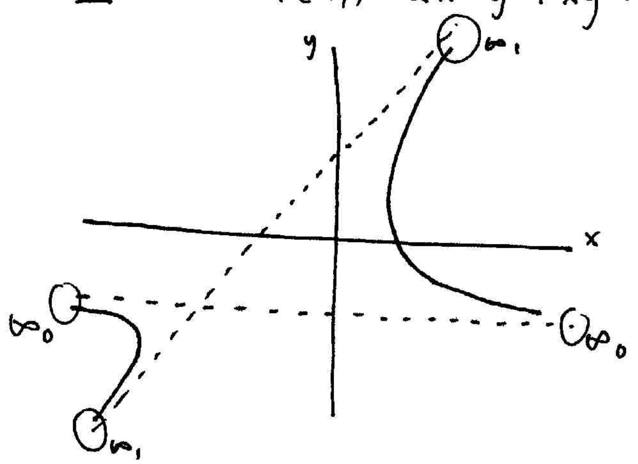
Curve-Sketching in the Projective Plane

Section 1: Definitions

$\mathbb{A}^2(\mathbb{R}) = \{(x,y) \mid x,y \in \mathbb{R}\}$

A <u>curve</u> in $\mathbb{A}^2(\mathbb{R})$ is the solution set for $f(x,y)=0$, f a polynomial.

<u>Ex.</u> $C:\ f(x,y) = 2x - y^2 + xy - 3 = 0$ $\qquad x = \dfrac{y^2+3}{y+2} = y - 2 + \dfrac{7}{y+2}$



Intuitively, we want $\mathbb{P}^2$ to include $\mathbb{A}^2$ <u>and</u> one point at infinity for every slope with which a curve could leave the plane. <u>This</u> curve should go through $m=0$ and $m=1$ infinite points.

$\mathbb{P}^2(\mathbb{R}) = \{(x,y,z) \mid x,y,z \in \mathbb{R}, \text{not all } 0\} \big/ (x,y,z) \sim (kx,ky,kz)$

A <u>curve</u> in $\mathbb{P}^2(\mathbb{R})$ is solution set for $\tilde{f}(x,y,z) = 0$, $\tilde{f}$ a <u>homogeneous</u> polynomial.

<u>Note</u>: "Homogeneous" is necessary for $\tilde{f}(x,y,z)=0$ to be well-defined on $\mathbb{P}^2$.

Ex. $(1,0,-2) = (-1,0,2) = (-3,0,6)$

Let $\tilde{f}(x,y,z) = xz + 2x^2 + 2xy$

$\tilde{f}(1,0,-2) = -2 + 2 + 0 = 0$

$\tilde{f}(-1,0,2) = -2 + 2 + 0 = 0$

$\tilde{f}(-3,0,6) = -18 + 18 + 0 = 0$

$\tilde{f}(kx,ky,kz) = k^2 \tilde{f}(x,y,z)$

Let $\tilde{f}(x,y,z) = 2x^2 + z$

$\tilde{f}(1,0,-2) = 2 - 2 = 0$

$\tilde{f}(-1,0,2) = 2 + 2 = 4$

$\tilde{f}(-3,0,6) = 18 + 6 = 24$

Whether $\tilde{f}$ vanishes depends on representation!

Section 2 : $\mathbb{P}^2$ extends $\mathbb{A}^2$ in the way that we would like.

Theorem: There is a bijection ϕ between $\mathbb{A}^2$ and $\mathbb{P}^2 - \{(a,b,0)\}$ and an association $f \to \tilde{f}$ such that $f(x,y) = 0 \longleftrightarrow \tilde{f}(\phi(x,y)) = 0$.

(Note: Throwing out $(0,1,0)$ and $(1,m,0)$ for each m - the ∞'s !)

Pf/ $\phi(x,y) = (x,y,1)$

Given a polynomial $f(x,y)$ of degree d, take $\tilde{f}(x,y,z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right)$. (Called homogenizing)

Check: ✓

To go back, $(a,b,c) \longrightarrow \left(\frac{a}{c}, \frac{b}{c}\right)$

$\tilde{g} \longrightarrow g(x,y) = \tilde{g}(x,y,1)$ (called dehomogenizing)

Check: ✓

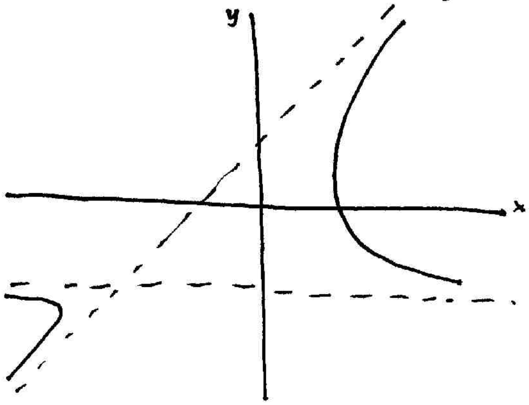
Subtle Point: Can actually de-homogenize in 3 ways!

$$\begin{tikzcd}
 & \mathbb{P}^2_{x,y,z} \arrow[dl, shift left, "{\substack{(a,b,c)\to\left(\frac{a}{c},\frac{b}{c}\right) \\ g(x,y)=\tilde{g}(x,y,1)}}"] \arrow[d, shift left] \arrow[dr, shift left, "{\substack{(a,b,c)\to\left(\frac{b}{a},\frac{c}{a}\right) \\ g(y,z)=g(1,y,z)}}"] & \\
\mathbb{A}^2_{x,y} \arrow[ur, shift left, "{\substack{(x,y)\to(x,y,1) \\ \tilde{f}(x,y,z)=z^d f\left(\frac{x}{z},\frac{y}{z}\right)}}"] & \mathbb{A}^2_{x,z} \arrow[u, shift left] & \mathbb{A}^2_{y,z} \arrow[ul, shift left, "{\substack{(y,z)\to(1,y,z) \\ f(y,z)\to x^d f\left(\frac{y}{x},\frac{z}{x}\right)}}"]
\end{tikzcd}$$

Since x,y,z not all 0, every point of $\mathbb{P}^2$ is finite in one of the $\mathbb{A}^2$'s !! Curve-sketching in $\mathbb{P}^2$ is curve-sketching in the $\mathbb{A}^2$'s & seeing how they glue together.

Section 3: Examples

#1) $f(x,y) = 2x - y^2 + xy - 3 = 0$



Homogenize: $z^2\left(2\frac{x}{z} - \frac{y^2}{z^2} + \frac{x}{z}\frac{y}{z} - 3\right) = 0$

$$2xz - y^2 + xy - 3z^2 = 0$$

All finite points are seen in $\mathbb{A}^2_{xy}$.
So let $z = 0$ to find "infinite" points.

$z=0$: $-y^2 + xy = 0$

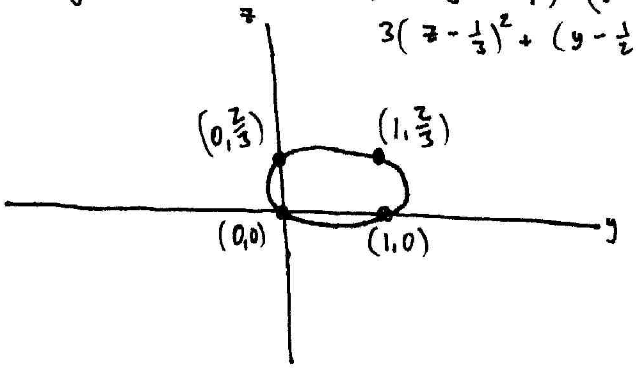
$-y(x-y) = 0$

Two Infinite Points: $x = y$ $(1,1,0)$
$y = 0$ $(1,0,0)$

To see both infinite points, we could look in yz plane.

$2z - y^2 + y - 3z^2 = 0$ $\quad 3\left(z^2 - \frac{2}{3}z + \frac{1}{9}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{7}{12}$

$3\left(z - \frac{1}{3}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{7}{12}$



Note: Don't see any infinite pts here b/c

$x = 0 \to -y^2 - 3z^2 = 0$

Actually are 2 inf. pts. but they are $\mathbb{C}$-valued.

$(0, \sqrt{3}, i)$

$(0, \sqrt{3}, -i)$

Note: Be careful going between $\mathbb{A}^2$'s.

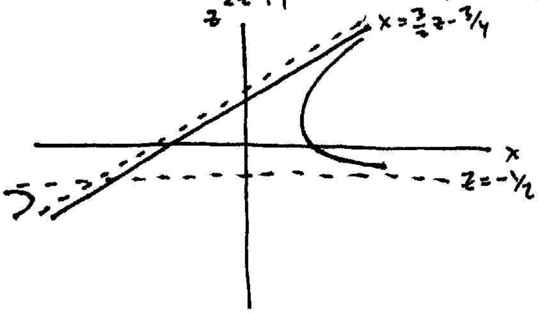
$$\begin{array}{c} y\ z \\ (1, \tfrac{2}{3}) \end{array} \to \begin{array}{c} x\ y\ z \\ (1, 1, \tfrac{2}{3}) \end{array} \xrightarrow{=} \begin{array}{c} x\ y\ z \\ (\tfrac{3}{2}, \tfrac{3}{2}, 1) \end{array} \to \begin{array}{c} x\ y \\ (\tfrac{3}{2}, \tfrac{3}{2}) \end{array}$$

$$\begin{array}{c} x\ y \\ (4, -1) \end{array} \to \begin{array}{c} x\ y\ z \\ (4, -1, 1) \end{array} \xrightarrow{=} \begin{array}{c} x\ y\ z \\ (1, -\tfrac{1}{4}, \tfrac{1}{4}) \end{array} \to \begin{array}{c} y\ z \\ (-\tfrac{1}{4}, \tfrac{1}{4}) \end{array}$$

Because of $(1,0,0)$ will have at least 1 infinite pt. in xz plane.

$y = 1$: $2xz - 1 + x - 3z^2 = 0$

$$x = \frac{3z^2 + 1}{2z + 1} = \frac{3}{2}z - \frac{3}{4} + \frac{7/4}{2z+1}$$



$y = 0 \to 2xz - 3z^2 = 0$

$z(2x - 3z) = 0$

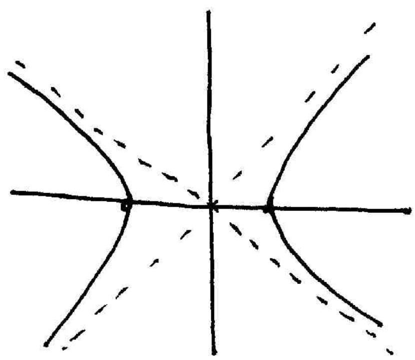
Two Infinite Points: $(1, 0, 0)$ "$m = 0$"
$(3, 0, 2)$ "$m = 2/3$"

Section 4 Bezout's Thm

Theorem: Two projective curves, C_1 and C_2, of degrees m & n, always have mn intersections.

Caveats: (1) Must be projective.
(2) Must consider complex valued pts.
(3) Must count multiplicity. (Up to how many derivatives do curves agree?)

Ex. Intersect $C: x^2-y^2=1$ with various lines. (∞'s are $(1,1,0)$ and $(1,-1,0)$)



L_1: $y=-\frac{x}{2}+1$ (Two finite real)

$x^2-(1-\frac{x}{2})^2=1$

$x^2-1+x-\frac{x^2}{4}=1$

$\frac{3}{4}x^2+x-2=0$

$3x^2+4x-8=0$

$x=\frac{-4\pm\sqrt{16+96}}{6}=\frac{-4\pm4\sqrt{7}}{6}=-\frac{2}{3}\pm\frac{2}{3}\sqrt{7}$

$x+2y=2z$

No infinite int.

b/c $(2,\cancel{1,0})$

L_2: $y=-2x$ (None?)

$x^2-4x^2=1$

$-3x^2=1$

$x=\pm i/\sqrt{3}$ Two complex. No infinite b/c $(1,\cancel{-2,0})$

L_3: $y=x+3$

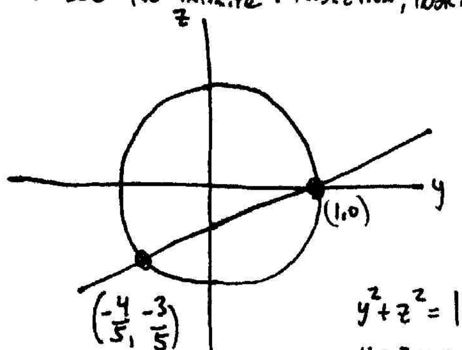
$x^2-(x+3)^2=1$

$-6x-9=1$

$x=-\frac{5}{3}$ $y=\frac{4}{3}$

$z=0\to(1,1,0)$ ✓

To see the infinite intersection, look in yz.

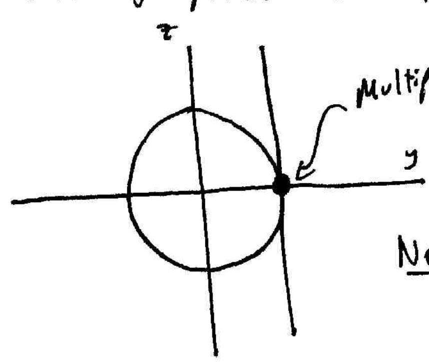


L_4: $y=x$

No finite intersections (Not even complex)

$(1,1,0)$ is only infinite pt. Multiplicity?

Look in yz-plane. $y=1$ & $y^2+z^2=1$



Note: Degree 2 + Degree 1 $\to$ 2 intersections!

Ex. Compare / Intersect

$C_1: \; x^2+y^2-2xy-x-y=0$
$C_2: \; x^2-y^2=1$
$C_3: \; y=x$

Conclusion: Affine curves aren't compact. So bad stuff can hide in the places where the curve leaves the plane. Projective curves are compact. Hence there is nowhere for the bad stuff to hide. Many accountability theorems.

- Bezout's Thm
- Nonconstant map of projective curves is surjective.
- Optimization Results over $\mathbb{R}$ (Ex. Max/Min values of $xy - \frac{1}{y-5}$ over $x^2+y^2=9$)
- Grandaddy of Them ALL: GAGA

~~Every analytic (meromorphic) function on a projective~~ alg

Every analytic projective curve is analytically isomorphic to an algebraic one.

(Things like $y=\sin x$ and $y=e^x$ don't "projectify")