CALCULUS

SINGLE VARIABLE

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PREFACE

Acknowledgements typically appear at the end of a preface. In this case, however, my indebtedness to Professor Marion Berger for her invaluable input throughout the development of this text is such that I am compelled to express my gratitude for her contributions at the beginning: Thank you, dear colleague and friend.

That said:

Our text consists of two volumes. Volume I addresses those topics typically covered in standard Calculus I and Calculus II courses; which is to say, the Single Variable Calculus. Multivariable Calculus is covered in Volume II.

Our primary goal all along has been to write a readable text, without compromising mathematical integrity. Along the way you will encounter numerous Check Your Understanding boxes designed to challenge your understanding of each newly-introduced concept. Complete solutions to the problems in those boxes appear in Appendix A, but please don't be in too much of a hurry to look at those solutions. You should **TRY** to solve the problems on your own, for it is only through **ATTEMPTING** to solve a problem that one grows mathematically. In the words of Descartes:

> ***WE NEVER UNDERSTAND A THING SO WELL, AND MAKE IT OUR OWN, WHEN WE LEARN IT FROM ANOTHER, AS WHEN WE HAVE DISCOVERED IT FOR OURSELVES.***

You will encounter a few graphing calculator glimpses in the text. In the final analysis, however, one can not escape the fact that:

MATHEMATICS DOES NOT RUN ON BATTERIES

CHAPTER 1
PRELIMINARIES

§1. SETS AND FUNCTIONS

A set may be defined by listing its elements inside braces, as in:

$$\{3, \sqrt{7}, -14\}$$

A set may also be specified by means of some property or condition, as with:

$$\{x|1 < x < 5\}$$

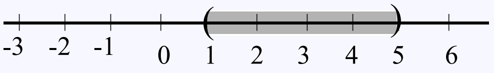
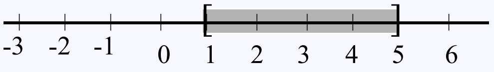
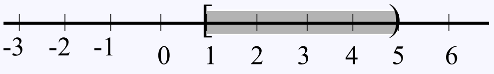
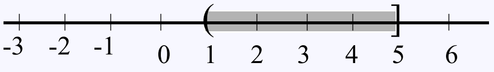
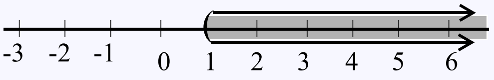
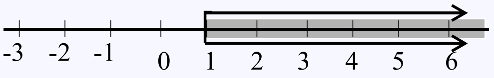
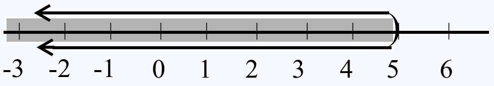
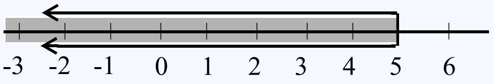
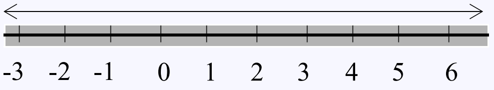
read: *such that*

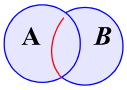
which represents the set of real numbers, *x*, that are greater than 1 and less than 5.

For a given set *X*, the expression $x \in X$ is used to indicate that x is an element of, or is contained, in *X*. For example:

$$3 \in \{3, \sqrt{7}, -14\} \quad \text{while} \quad 9 \notin \{x|1 < x < 5\}$$

The following table illustrates some notation that can be used to denote the interval subsets of the number line:

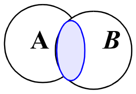
	Interval Notation	**Geometrical Representation**
All real numbers strictly between 1 and 5 (not including 1 or 5).	$(1, 5) = \{x\|1 < x < 5\}$ excluding 1 and 5	
All real numbers between 1 and 5, including both 1 and 5.	$[1, 5] = \{x\|1 \le x \le 5\}$ including 1 and 5	
All real numbers between 1 and 5, including 1 but not 5.	$[1, 5) = \{x\|1 \le x < 5\}$ including 1 and excluding 5	
All real numbers between 1 and 5, including 5 but not 1.	$(1, 5] = \{x\|1 < x \le 5\}$ excluding 1 and including 5	
All real numbers greater than 1.	$(1, \infty) = \{x\|x > 1\}$ the infinity symbol	
All real numbers greater than or equal to 1.	$[1, \infty) = \{x\|x \ge 1\}$	
All real numbers less than 5.	$(-\infty, 5) = \{x\|x < 5\}$	
All real numbers less than or equal to 5.	$(-\infty, 5] = \{x\|x \le 5\}$	
The set of all real numbers.	$(-\infty, \infty) = \{x\|-\infty < x < \infty\}$	



$A \cup B$

For any two sets A and B, the **union** of A and B is defined to be that set, denoted by $A \cup B$, which consists of all elements that are either in A or in B, including the elements in both A and B. That is:

$$A \cup B = \{x | x \in A \textbf{ OR } x \in B\}$$

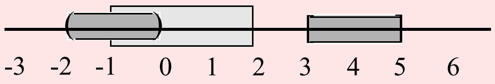
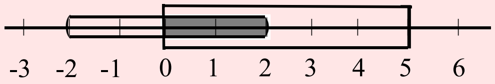


$A \cap B$

The **intersection** of A and B, written $A \cap B$, is the set consisting of the elements common to both A and B. That is:

$$A \cap B = \{x | (x | x \in A \textbf{ AND } x \in B)\}$$

For example:

$(-2, 0) \cup [-1, 2] \cup [3, 5]$ $= (-2, 2] \cup [3, 5]$	
$(-2, 2) \cap [0, 5] = [0, 2)$	

FUNCTIONS

We will be concerned with functions, f, which assign a real number $f(x)$ to a given real number x. Such functions can often be described by mathematical expressions; as with:

$$f(x) = 2x + 5$$

Note that the variable x is a placeholder; a "box" that can hold any meaningful expression. For example:

$$f(\mathbf{x}) = 2\mathbf{x} + 5$$

$$f\boxed{} = 2\boxed{} + 5$$

$$f(\mathbf{3}) = 2 \cdot \mathbf{3} + 5 = 11$$

$$f(\boldsymbol{c}) = 2 \cdot \boldsymbol{c} + 5 = 2c + 5$$

$$f(\mathbf{3}\boldsymbol{t}) = 2 \cdot \mathbf{3}\boldsymbol{t} + 5 = 6t + 5$$

$$f(\boldsymbol{x}^{\mathbf{2}} + \mathbf{3}) = 2(\boldsymbol{x}^{\mathbf{2}} + \mathbf{3}) + 5 = 2x^2 + 11$$

CHECK YOUR UNDERSTANDING 1.1

For $f(x) = 3x - 5$, determine:

(a) $f(-2)$ (b) $f(t+1)$ (c) $f(-2x+1)$ (d) $f\left(\frac{-2}{x}\right)$

Answers:

(a) -11 (b) $3t - 2$

(c) $-6x - 2$ (d) $\frac{-6-5x}{x}$

EXAMPLE 1.1 For $f(x) = -3x^2 + 6x - 1$ and $h \neq 0$, determine $\frac{f(x+h) - f(x)}{h}$.

SOLUTION:

$$\frac{\overbrace{f(x+h)}-f(x)}{h} = \frac{\overbrace{[-3(x+h)^2+6(x+h)-1]}-(-3x^2+6x-1)}{h}$$

$$= \frac{[-3(x^2+2xh+h^2)+6x+6h-1]+3x^2-6x+1}{h}$$

$$= \frac{-3x^2-6xh-3h^2+6x+6h-1+3x^2-6x+1}{h}$$

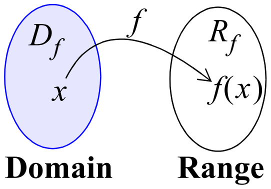
$$= \frac{-6xh-3h^2+6h}{h} = \frac{\cancel{h}(-6x-3h+6)}{\cancel{h}} = -6x-3h+6$$

CHECK YOUR UNDERSTANDING 1.2

For $f(x) = \frac{x}{x+1}$ and $h \neq 0$, simplify $\frac{f(x+h)-f(x)}{h}$.

Answer:
$$\frac{1}{(x+h+1)(x+1)}$$

THE DOMAIN AND RANGE OF A FUNCTION



The **domain** of a function f is the set, D_f, on which f "acts," and its **range** is the set R_f of the function values (see margin).

When not specified, the domain of a function defined by an expression is **understood** to be the set of all numbers for which the given expression is defined. For example:

Since x^2 is defined for all numbers, the domain of $f(x) = x^2$ is the set of all numbers: $D_f = (-\infty, \infty)$. Since the function can not assume negative values, and assumes every nonnegative value: $R_f = [0, \infty)$.

Since one can not take the square root of a negative number (in the real number system), the domain of the function $g(x) = \sqrt{x}$ is the set of all numbers greater than or equal to 0: $D_g = [0, \infty)$, with $R_f = [0, \infty)$ (the function can only assume nonnegative values).

Since division by zero is not defined, the domain of the function $h(x) = \frac{1}{x-1}$ is the set $D_h = (-\infty, 1) \cup (1, \infty)$.

The range of h is not so easy to determine at this point. You will be able to do so once you know how to graph such a function.

CHECK YOUR UNDERSTANDING 1.3

(a) Determine the domain and range of $f(x) = \sqrt{x+3}$.

(b) Determine the domain of $g(x) = \frac{1}{(x+1)(x-2)}$.

Answers:

(a) $D_f = [-3, \infty)$; $R_f = [0, \infty)$

(b) $D_g = (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

PIECEWISE-DEFINED FUNCTIONS

Functions such as $f(x) = x^2$ and $g(x) = x + 1$ are described by a single algebraic expression. For whatever reasons, one may wish to consider a function which acts like $f(x) = x^2$ for $-2 \le x < 0$ and like $g(x) = x + 1$ for $x \ge 0$. Such a function is said to be a **piecewise-defined** function and is represented in the following manner:

$$h(x) = \begin{cases} x^2 & \text{if } -2 \le x < 0 \\ x + 1 & \text{if } x \ge 0 \end{cases}$$

Combining the "if" parts of the definition of *h*, we see that the domain is $D_h = [-2, \infty)$. To evaluate *h* at a particular *x* you must first determine which of the two rules applies. For example:

$$h(-\mathbf{1}) \underset{\uparrow}{=} (-\mathbf{1})^2 = 1, \quad \text{and} \quad h(\mathbf{9}) \underset{\uparrow}{=} \mathbf{9} + 1 = 10$$

top rule since $-2 \le -\mathbf{1} < 0$

bottom rule since $\mathbf{9} \ge 0$

EXAMPLE 1.2 Evaluate the function:

$$f(x) = \begin{cases} 3x - 5 & \text{if } x < 0 \\ \dfrac{1}{x+1} & \text{if } x \ge 0 \end{cases}$$

at $x = -1$, at $x = 0$, and at $x = 3$.

SOLUTION: Since -1 is less than 0, it falls under the jurisdiction of the formula on the top line, and we have:

$$f(-1) = 3(-1) - 5 = -3 - 5 = -8$$

Both 0 and 3 are greater than or equal to 0, thus:

$$f(0) = \frac{1}{0+1} = \frac{1}{1} = 1 \quad \text{and} \quad f(3) = \frac{1}{3+1} = \frac{1}{4}$$

CHECK YOUR UNDERSTANDING 1.4

Evaluate the function:

$$f(x) = \begin{cases} -4x + 1 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 5 \\ -2x & \text{if } 5 < x < 10 \end{cases}$$

at: $x = -1$, $x = 1$, $x = 5$, $x = 7$. Is $f(10)$ defined? If not, why not?

Answer:
$f(-1) = 5$, $f(1) = 1$
$f(5) = 25$, $f(7) = -14$
No, 10 is not in the domain of the function.

A particularly important piecewise-defined function is the absolute value function, $f(x) = |x|$, where:

DEFINITION 1.1
ABSOLUTE VALUE

The **absolute value** of a number a is that number $|a|$ given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

You are invited to establish the following results in the exercises:

THEOREM 1.1

For any numbers a and b:

(a) $|ab| = |a||b|$ and, if $b \neq 0$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(b) **(Triangle Inequality)** $|a + b| \leq |a| + |b|$

You can, and should, interpret $|a|$ as representing the distance (number of units) between the numbers a and 0 on the number line. For example, both 5 and -5 are 5 units from the origin, and we have:

$$|5| = 5 \text{ and } |-5| = 5$$

When you subtract one number from another the result is either **plus or minus** the distance (number of units) between those numbers on the number line. For example, $9 - 1 = 8$ while $1 - 9 = -8$. In either case, the absolute value of the difference is 8, the distance between the two numbers:

$$|9 - 1| = |8| = 8 \text{ and } |1 - 9| = |-8| = 8$$

In general:

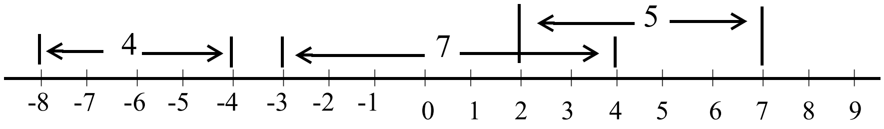
DEFINITION 1.2
DISTANCE

The **distance** between a and b on the number line is given by

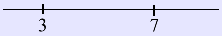
$$|a - b|$$

note the negative sign

In particular, $|(-8) - (-4)| = |-8 + 4| = 4$ is the distance between -8 and -4, $|(-3) - (4)| = |-3 - 4| = 7$ is the distance between -3 and 4, and $|2 - 7| = 5$ is the distance between 2 and 7:



Answers:

(a)


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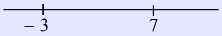
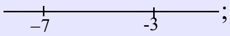
(b)


; 10

(c)


; 10

(d)


; 4

CHECK YOUR UNDERSTANDING 1.5

Plot the two numbers on the number line and determine the distance between them; both visually and by using Definition 1.2.

(a) 3 and 7 (b) -3 and 7 (c) 3 and -7 (d) -3 and -7

THE ARITHMETIC OF FUNCTIONS

The following definition is the natural extension of addition, subtraction, multiplication, and division of numbers to functions:

DEFINITION 1.3
COMBINING FUNCTIONS

The sum, difference, product, and quotient of two functions f and g are defined as follows:

$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ [providing } g(x)\neq 0]$$

For any constant c: $(cf)(x) = cf(x)$

Noting that the functions $f+g$, $f-g$, and fg can be evaluated at x if and only if **both** f and g can be evaluated at x, we see that the domains of those three functions coincide with the intersection of the domain of f with that of g:

$$D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g$$

In determining the domain of $\frac{f}{g}$ one must also exclude those x's where $g(x) = 0$:

$$D_{\frac{f}{g}} = \{x|x \text{ is in } (D_f \cap D_g) \text{ and } g(x)\neq 0\}$$

Finally, the domain of the function cf is the same as that of f:

$$D_{cf} = D_f$$

EXAMPLE1.3

For $f(x) = \sqrt{x}$ and $g(x) = x-1$, determine the functions $f+g$, $f-g$, fg, $\frac{f}{g}$, and $5g$, along with their domains.

SOLUTION: Appealing to Definition 1.3, we have:

$$(f+g)(x) = f(x)+g(x) = \sqrt{x}+x-1$$
$$(f-g)(x) = f(x)-g(x) = \sqrt{x}-(x-1) = \sqrt{x}-x+1$$
$$(fg)(x) = f(x)g(x) = \sqrt{x}(x-1) = x^{\frac{3}{2}}-\sqrt{x}$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x-1}$$
$$5g(x) = 5(x-1) = 5x-5$$

Noting that $D_f = [0, \infty)$ and $D_g = (-\infty, \infty)$ we conclude that:

$$D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g = [0, \infty) \cap (-\infty, \infty) = [0, \infty)$$

$$D_{\frac{f}{g}} = \{x | x \text{ is in } [0, \infty) \text{ and } x - 1 \neq 0\} = [0, 1) \cup (1, \infty)$$

$$D_{5g} = D_g = (-\infty, \infty)$$

CHECK YOUR UNDERSTANDING 1.6

For $f(x) = x - 3$ and $g(x) = \frac{1}{x-3}$, determine the functions $f + g$, $f - g$, fg, $\frac{f}{g}$, and $5g$, along with their domains.

Answer:

$$(f+g)(x) = \frac{x^2 - 6x + 10}{x - 3}$$

$$(f-g)(x) = \frac{x^2 - 6x + 8}{x - 3}$$

$$(fg)(x) = \frac{x-3}{x-3} = 1 \text{ if } (x \neq 3)$$

$$\left(\frac{f}{g}\right)(x) = x^2 - 6x + 9;$$

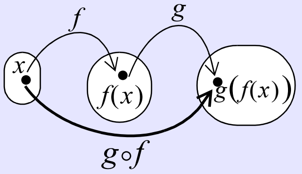
$$(5g)(x) = \frac{5}{x-3}$$

all domains are $(-\infty, 3) \cup (3, \infty)$

COMPOSITION OF FUNCTIONS

If $h(x) = (x-3)^2$ then $h(8) = 25$. You get that answer by first subtracting 3 from 8: $(x - 3) = (8 - 3) = \mathbf{5}$, and then squaring the result: $(\mathbf{5})^2 = 25$. In other words, you **first** apply the function $f(x) = x - 3$, **and then** apply the function $g(x) = x^2$ to that result. This operation of first performing one function, and then another on that result, is called composition, and is denoted by $(g \circ f)(x)$:

COMPOSITION



DEFINITION 1.4

The **composition** $(g \circ f)(x)$ is given by:

$$(g \circ f)(x) = g(f(x))$$

first apply f

and **then** apply g

[Assuming $f(x)$ is in the domain of g]

EXAMPLE 1.4

Determine $(g \circ f)(2)$ and $(f \circ g)(2)$ for:

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = 2x - 3$$

SOLUTION:

$$(g \circ f)(2) = g(\mathbf{f(2)}) = g(\mathbf{2^2 + 1}) = g(\mathbf{5}) = 2 \cdot \mathbf{5} - 3 = 7$$

$$(f \circ g)(2) = f(\mathbf{g(2)}) = f(\mathbf{2 \cdot 2 - 3}) = f(\mathbf{1}) = \mathbf{1}^2 + 1 = 2$$

Note that composition is **not** a commutative operation:

$$(g \circ f)(2) \neq (f \circ g)(2)$$

EXAMPLE 1.5 Determine $(f\circ g)(x)$ and $(g\circ f)(x)$ for:

$$f(x) = 3x-1 \text{ and } g(x) = -2x^2-3x+1$$

SOLUTION:

$$\begin{aligned}(f\circ g)(x) \overset{\text{Definition 1.4}}{=} f(\mathbf{g(x)}) = f(-\mathbf{2x^2}-\mathbf{3x}+\mathbf{1}) &\overset{f(\mathbf{x}) = 3\mathbf{x}-1}{=} 3(-\mathbf{2x^2}-\mathbf{3x}+\mathbf{1})-1\\ &= -6x^2-9x+3-1\\ &= -6x^2-9x+2\end{aligned}$$

$$\begin{aligned}(g\circ f)(x) = g(f(x)) = g(3x-1) &= -2(3x-1)^2-3(3x-1)+1\\ &= -2(9x^2-6x+1)-9x+3+1\\ &= -18x^2+3x+2\end{aligned}$$

EXAMPLE 1.6 Express the function $h(x) = \sqrt{x^2-1}+5$ as a composition $h = g\circ f$.

SOLUTION: There are many choices for f and g. Perhaps the most natural one is to first do: $f(x) = x^2-1$, then take the square root of that result and add 5: $g(x) = \sqrt{x}+5$. In other words:

$$h(x) = (g\circ f)(x) \text{ where } f(x) = x^2-1 \text{ and } g(x) = \sqrt{x}+5$$

An alternate solution:
$h(x) = (g\circ f)(x)$ where:
$f(x) = \sqrt{x^2-1}$
and $g(x) = x+5$

CHECK YOUR UNDERSTANDING 1.7

(a) For $f(x) = x^2+2x-2$ and $g(x) = 4x+3$:

(i) Evaluate $(f\circ g)(-2)$ (ii) Determine $(f\circ g)(x)$

(b) Express $h(x) = \dfrac{x^2}{x^2+3}$ as a composition $h = g\circ f$.

Answers:
(a-i) 13 (a-ii) $16x^2+32x+13$
(b) One possible answer:
$f(x) = x^2$ and $g(x) = \dfrac{x}{x+3}$

EXERCISES

Exercises 1-6. Determine the domain of the given function.

1. $f(x) = 3x^2 + x$
2. $g(x) = \frac{x+3}{x-3}$
3. $k(x) = \frac{x}{(x-1)(x+100)}$
4. $k(x) = x + \sqrt{x-2}$
5. $h(x) = \sqrt{x+7}$
6. $f(x) = \frac{\sqrt{x+3}}{x(x-1)}$

Exercises 7-12. Determine $(f+g)(2)$, $(f-g)(2)$, $(fg)(2)$, $\left(\frac{f}{g}\right)(2)$, $(2f)(2)$, $(f \circ g)(2)$, and $(g \circ f)(2)$ for the given functions f and g.

7. $f(x) = 2x + 3,\ g(x) = -x + 1$
8. $f(x) = -x + 5,\ g(x) = x - 3$
9. $f(x) = x^2 + x,\ g(x) = -x + 2$
10. $f(x) = -x^2 + 2,\ g(x) = x + 1$
11. $f(x) = \frac{1}{x+5},\ g(x) = x + 3$
12. $f(x) = (x-1)^2,\ g(x) = \frac{x}{x-3}$

Exercises 13-18. Determine $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$, $(2f)(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$ for the given functions f and g.

13. $f(x) = x + 3,\ g(x) = -2x + 1$
14. $f(x) = -x + 2,\ g(x) = 2x + 10$
15. $f(x) = x^2 + x - 1,\ g(x) = x + 3$
16. $f(x) = 2x^2 + 1,\ g(x) = -2x + 3$
17. $f(x) = \frac{1}{-x+2},\ g(x) = x^2 + 3$
18. $f(x) = -x + 3,\ g(x) = \frac{1}{x}$

Exercises 19-22. (a) Determine $(f \circ g)(x)$ for the given functions f and g.
(b) Evaluate $(f \circ g)(2)$ both with and without using the result of (a).

19. $f(x) = 2x + 5,\ g(x) = -3x - 1$
20. $f(x) = 5x - 7,\ g(x) = 7x + 5$
21. $f(x) = x^2 + x,\ g(x) = x + 1$
22. $f(x) = -2x + 3,\ g(x) = 2x^2 + 2x + 1$

Exercises 23-28. Evaluate the given function at 2, $2 + h$, and at $x + h$.

23. $f(x) = 3x + 9$
24. $f(x) = -5x + 2$
25. $f(x) = -x^2 + x + 1$
26. $f(x) = 2x^2 + x + 2$
27. $f(x) = \frac{x^2}{2x+3}$
28. $f(x) = \frac{-x}{x^2+2}$

Exercises 29-34. Simplify the algebraic expression $\frac{f(x+h)-f(x)}{h}$ for the given function f.

29. $f(x) = 3x+9$
30. $f(x) = -5x+2$
31. $f(x) = -x^2+x+1$
32. $f(x) = 2x^2+x+2$
33. $f(x) = \frac{x^2}{2x+3}$
34. $f(x) = \frac{-x}{x^2+2}$

35. Evaluate the function $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x+1 & \text{if } x \ge 1 \end{cases}$ at $x = 0$ and at $x = 1$.

36. Evaluate the function $f(x) = \begin{cases} 3x-5 & \text{if } x < 0 \\ \frac{1}{x+1} & \text{if } x \ge 0 \end{cases}$ at $x = -1$ and at $x = 1$.

37. Evaluate the function $f(x) = \begin{cases} 3x-5 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 5 \\ -2x & \text{if } 5 \le x < 10 \end{cases}$ at: $x = -1$, $x = 1$, $x = 7$, and at $x = 10$ (Are you sure? What is the domain of f?).

38. For $f(x) = \begin{cases} x+2 & \text{if } x < 1 \\ x+1 & \text{if } x \ge 1 \end{cases}$ and $g(x) = \begin{cases} 2x-1 & \text{if } x < 0 \\ x-5 & \text{if } x \ge 0 \end{cases}$ determine:

(a) $(f+g)(0)$ (b) $(g \circ f)(1)$ (c) $(f \circ g)(2)$

Exercises 39-41. (Theory) Prove:

39. Theorem 1.1(a)
40. Theorem 1.1(b)
41. For any numbers a and b: $|a-b| \ge |a| - |b|$

Exercises 42-47. (Even and Odd Functions)

f is an **even function** if $f(-x) = f(x)$ for every $x \in D_f$.

f is an **odd function** if $f(-x) = -f(x)$ for every $x \in D_f$.

Determine if the given function is even, odd, or neither even nor odd.

42. $f(x) = 3x^2$
43. $f(x) = 3x^3$
44. $f(x) = 3x^3+x$
45. $f(x) = 3x^2+1$
46. $f(x) = 3x^4+x^2+5$
47. $f(x) = x^3+x+1$

§2. ONE-TO-ONE FUNCTIONS AND THEIR INVERSES

Though a function $y = f(x)$ can not assign more than one value of y to each x in its domain, it can assign the same y-value to different x's. The function $f(x) = x^2$, for example, assigns the number 4 to both 2 and -2, and we say that f **maps** 2 and -2 onto 4.

Of particular interest are those functions that map different values of x onto different values of y:

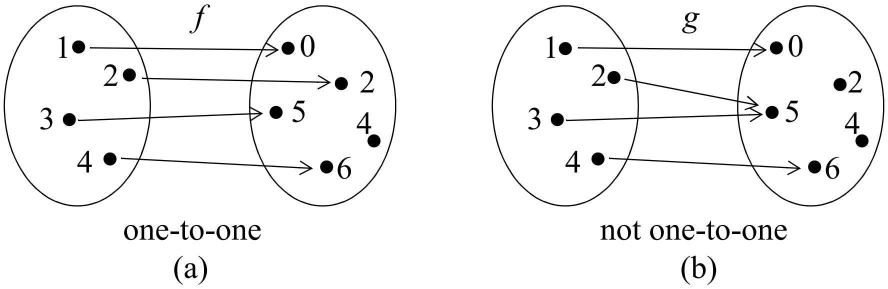
DEFINITION 1.5
ONE-TO-ONE

A function f is **one-to-one** if for all a and b in D_f:

$$\text{If } f(a) = f(b) \text{ then } a = b.$$

Equivalently: If $a \neq b$, then $f(a) \neq f(b)$.

The function f, represented in Figure 1.1(a), **is** one-to-one since no two elements in its domain, $\{1, 2, 3, 4\}$, are mapped to the same element in its range $\{0, 2, 5, 6\}$. The function g, of Figure 1.1(b), is **not** one-to-one since 2 and 3 are both mapped to 5.



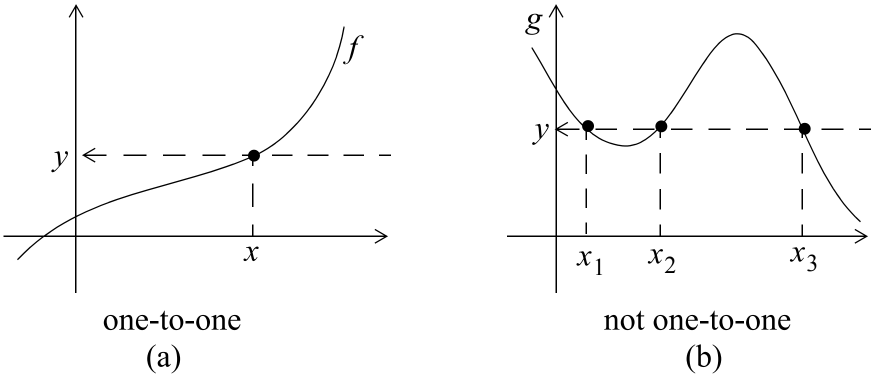
one-to-one
(a)

not one-to-one
(b)

Figure 1.1

Looking at more traditional graphs of functions, we see that the function f of Figure 1.2(a) is one-to-one (no two x's map onto the same y), while the function g of Figure 1.2(b) is not (different x's map onto the same y).

Horizontal Line Test:
No horizontal line can intersect the graph of a one-to-one function at more than one point.



one-to-one
(a)

not one-to-one
(b)

Figure 1.2

EXAMPLE 1.7

Show that the function $f(x) = \frac{x}{5x+2}$ is one-to-one.

Solution: Appealing to Definition 1.5, we begin with $f(a) = f(b)$, and show that this can only hold if $a = b$:

$$\begin{tikzcd} f(a) = f(b) \arrow[d, squiggly] \\ a = b \end{tikzcd}$$

$$\begin{aligned} f(a) &= f(b) \\ \frac{a}{5a+2} &= \frac{b}{5b+2} \\ a(5b+2) &= b(5a+2) \\ 5ab+2a &= 5ab+2b \\ 2a &= 2b \\ a &= b \end{aligned}$$

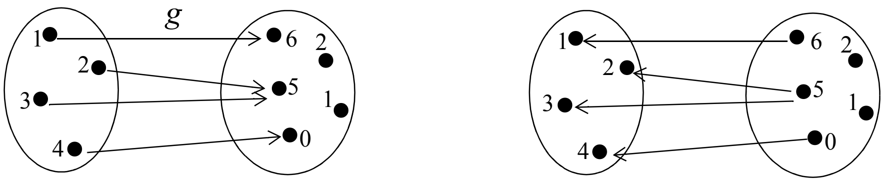
CHECK YOUR UNDERSTANDING 1.8

Show that the function $f(x) = \frac{x}{x+1}$ is one-to-one.

Answer: See page A-2.

INVERSE FUNCTIONS

An attempt to reverse the direction of the arrows in Figure 1.3(a), representing the action of the non-one-to-one function g, would not yield a function. As is shown in Figure 1.3(b), the number 5 would be mapped onto two numbers, 2 and 3, and a function can not assign more than one value to each number in its domain.



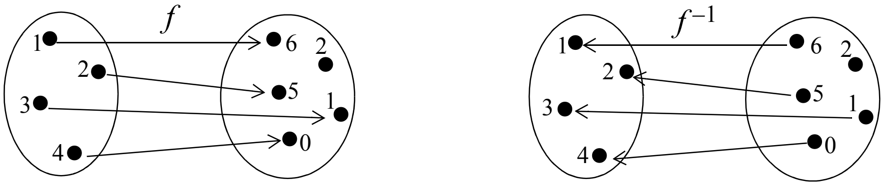
not one-to-one
(a)

not a function
(b)

Figure 1.3

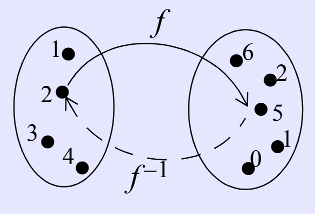
Reversing the arrows of the one-to-one function f of Figure 1.4(a) **does** lead to a function [see Figure 1.4(b)]. That function is called the **inverse** of f and is denoted by the symbol f^{-1}. As you can see, the domain of f^{-1} is the range of f: $\{0, 1, 5, 6\}$, and the range of f^{-1} is the domain of f: $\{1, 2, 3, 4\}$.

Do not confuse f^{-1} with $[f(x)]^{-1} = \frac{1}{f(x)}$.



(a) (b)

Figure 1.4



The relationship between the functions f and f^{-1} depicted in the margin reveals the fact that each function “undoes” the work of the other. For example:

$$(f^{-1}\circ f)(\mathbf{2}) = f^{-1}(f(2)) = f^{-1}(5) = \mathbf{2}$$

and

$$(f\circ f^{-1})(\mathbf{5}) = f(f^{-1}(5)) = f(2) = \mathbf{5}$$

In general:

Only one-to-one functions have inverses (see Figure 1.3).

DEFINITION 1.6

INVERSE FUNCTIONS

The **inverse** of a one-to-one function f with domain D_f and range R_f is that function f^{-1} with domain R_f and range D_f such that:

$$(f^{-1}\circ f)(x) = x \text{ for every } x \text{ in } D_f$$

and

$$(f\circ f^{-1})(x) = x \text{ for every } x \text{ in } R_f$$

EXAMPLE 1.8

Find the inverse of the one-to-one function:

$$f(x) = \frac{x}{5x+2}$$

SOLUTION: We offer two methods for your consideration.

Start with: $f(f^{-1}(x)) = x$

For notational convenience, substitute t for $f^{-1}(x)$: $f(t) = x$

Since $f(x) = \frac{x}{5x+2}$: $\frac{t}{5t+2} = x$

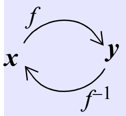
Solve for t:

$$t = (5t+2)x$$
$$t = 5tx + 2x$$
$$t - 5tx = 2x$$
$$t(1-5x) = 2x$$
$$t = \frac{2x}{1-5x}$$

Substituting $f^{-1}(x)$ back for t: $f^{-1}(x) = \frac{2x}{1-5x}$

To say that $y = f(x)$ is to say that $f^{-1}(y) = x$.

So, start with: $y = \frac{x}{5x+2}$



And solve for x in terms of y:

$$(5x+2)y = x$$
$$5xy + 2y = x$$
$$5xy - x = -2y$$
$$x(5y-1) = -2y$$
$$x = \frac{-2y}{5y-1} = \frac{2y}{1-5y} = f^{-1}(y)$$

To obtain the inverse function expressed in terms of the variable x (instead of y), interchange x and y: $y = \frac{2x}{1-5x} = f^{-1}(x)$

As a check, we verify directly that $(f^{-1} \circ f)(x) = x$:

Since $f(x) = \dfrac{x}{5x+2}$

Since $f^{-1}(x) = \dfrac{2x}{1-5x}$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{x}{5x+2}\right) = \frac{2\left(\frac{x}{5x+2}\right)}{1-5\left(\frac{x}{5x+2}\right)}$$

Multipy numerator and denominator by $5x+2$:

$$= \frac{2x}{5x+2-5x} = \frac{2x}{2} = x$$

CHECK YOUR UNDERSTANDING 1.9

Determine the inverse of the one-to-one function:

$$f(x) = \frac{x}{x+1}$$

Verify, directly, that $(f \circ f^{-1})(x) = x$ and that $(f^{-1} \circ f)(x) = x$.

Answer: $f^{-1}(x) = \dfrac{x}{1-x}$ and see page A-3,

GRAPH OF AN INVERSE FUNCTION

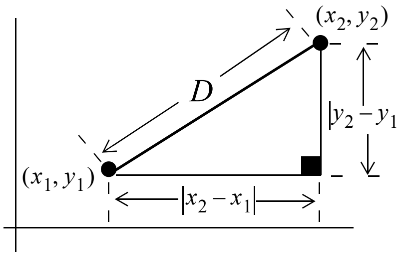
We end this section with a result which relates the graph of a one-to-one function with that of its inverse. In that endeavor, we will use the following distance formula:

THEOREM 1.2
DISTANCE BETWEEN TWO POINTS

The distance D between the points (x_1, y_1) and (x_2, y_2) in the plane is given by:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

PROOF:

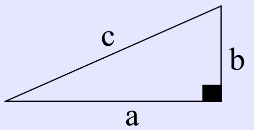


Pythagorean Theorem:

$$D^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or: $D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

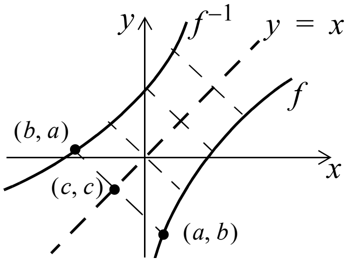
Pythagorean Theorem



$$c^2 = a^2 + b^2$$

THEOREM 1.3

The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.



Proof: Since $f(a) = b$ if and only if $f^{-1}(b) = a$, to say that (a, b) is on the graph of f, is to say that (b, a) is on the graph of f^{-1}. Since the slope of the line joining (b, a) and (a, b) is $\frac{a-b}{b-a} = -1$, that line segment is perpendicular to the line $y = x$ (which has slope 1). Moreover, using Theorem 1.2, we see that the point (c, c) on that line segment is equidistant from (b, a) and (a, b):

$$\overset{\text{distance between } (b, a) \text{ and } (c, c)}{\downarrow}\qquad\qquad \overset{\text{distance between } (a, b) \text{ and } (c, c)}{\downarrow}$$

$$\sqrt{(b-c)^2 + (a-c)^2} = \sqrt{(a-c)^2 + (b-c)^2}$$

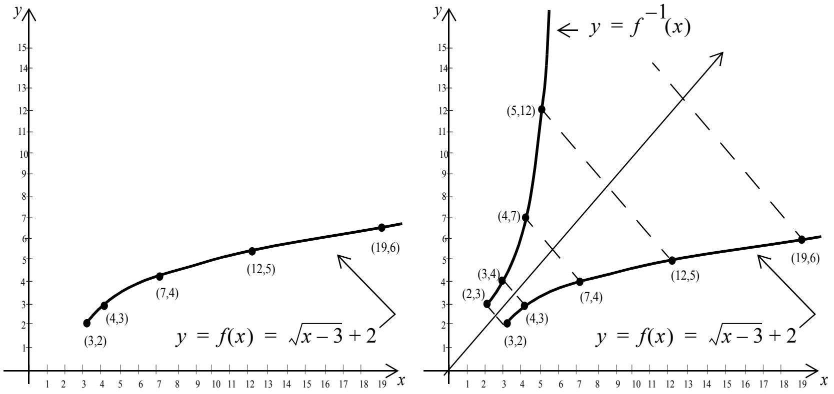
EXAMPLE 1.9

(a) Sketch the graph of the function $f(x) = \sqrt{x-3} + 3$. Specify its domain and range.

(b) Use Theorem 1.3 to obtain the graph of its inverse $f^{-1}(x)$. Specify its domain and range.

(c) Find $f^{-1}(x)$.

Solution: (a) The graph of the function $f(x) = \sqrt{x-3} + 2$ appears in Figure 1.5(a). From that graph, we see that the domain of f is $[3, \infty)$, and that its range is $[2, \infty)$.



(a) (b)

Figure 1.5

(b) Reflecting the graph of f about the line $y = x$ we arrive at the graph of f^{-1} [Figure 1.5(b)]. Note that the domain of f^{-1} is the range of f, namely: $[2, \infty)$; and that the range of f^{-1} is the domain of f, namely: $[3, \infty)$.

(c) Proceeding as in Example 1.8 (right-hand side), we find $f^{-1}(x)$:

Start with: $y = \sqrt{x-3}+2$

Solve for x in terms of y:

$$\begin{aligned}\sqrt{x-3} &= y-2\\ x-3 &= (y-2)^2\\ x-3 &= y^2-4y+4\\ x &= y^2-4y+7\end{aligned}$$

To obtain the inverse function expressed in terms of the variable x (instead of y), interchange x and y: $y = x^2-4x+7 = f^{-1}(x)$

with domain: $[\mathbf{2}, \infty)$

CHECK YOUR UNDERSTANDING 1.10

Show that the function $f(x) = \sqrt{x}-2$ is one-to-one. Indicate its domain and range. Find its inverse and indicate its domain and range. Sketch the graph of both functions on the same set of axes.

Answer: See page A-3.

EXERCISES

Exercises 1-9. Prove that the given function is one-to-one.

1. $f(x) = -5x - 1$
2. $f(x) = 6x + 5$
3. $f(x) = x^3 + 1$
4. $f(x) = \frac{1}{x}$
5. $f(x) = \frac{4}{2x - 3}$
6. $f(x) = \frac{3x}{2x + 1}$
7. $f(x) = \sqrt{x + 1} + 2$
8. $f(x) = \frac{4}{\sqrt{x + 1}}$
9. $f(x) = \sqrt{\frac{3x}{2x + 1}}$

Exercises 10-13. Show that the given function is not one-to-one by finding two different values of x: a and b, such that $f(a) = f(b)$.

10. $f(x) = x^2 + 1$
11. $f(x) = x^2 - x - 6$
12. $f(x) = \frac{x^2 + 3}{x}$
13. $f(x) = \frac{x}{x^2 + 1}$

Exercises 14-22. Find the inverse of the given one-to-one function, and verify directly that $(f \circ f^{-1})x = x$ and $(f^{-1} \circ f)x = x$.

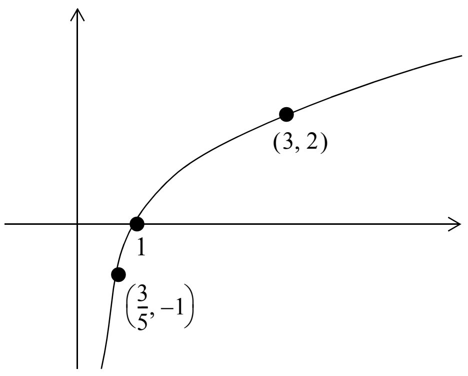
14. $f(x) = 2x - 3$
15. $f(x) = -x + 1$
16. $f(x) = \frac{1}{x}$
17. $f(x) = \frac{3}{2x - 5}$
18. $f(x) = \frac{1}{x - 1}$
19. $f(x) = \frac{5x + 2}{x - 3}$
20. $f(x) = x^3 - 1$
21. $f(x) = 2\sqrt{x + 3}$
22. $f(x) = \sqrt{2x + 3}$

Exercises 23-25. Sketch the graph of the given one-to-one function along with the inverse of that function on the same set of axes.

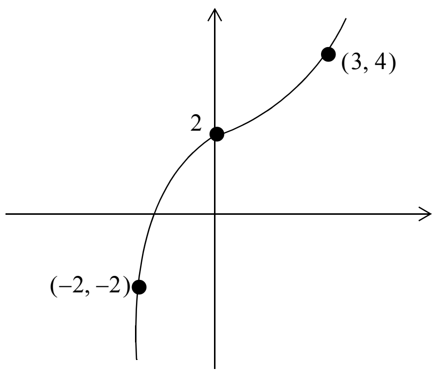
23. $f(x) = 2x - 3$
24. $f(x) = -x + 1$
25. $f(x) = \sqrt{x - 1} + 2$

Exercises 26-27. Sketch the graph of the inverse function $f^{-1}(x)$ from the given function of f.

26.



27.



§3. EQUATIONS AND INEQUALITIES

We begin by noting that one solves linear inequalities in exactly the same fashion as linear equations, with one notable exception:

> WHEN MULTIPLYING OR DIVIDING BOTH SIDES OF AN INEQUALITY BY A **NEGATIVE** QUANTITY, **REVERSE** THE DIRECTION OF THE INEQUALITY SIGN.

To illustrate:

Equation	**Inequality**
$3x-5 = 5x-7$	$3x-5<5x-7$
$3x-5x = -7+5$	$3x-5x<-7+5$
$-2x = -2$	$-2x<-2$ ← *reverse* dividing by a negative number
$x = 1$	$x>1$

CHECK YOUR UNDERSTANDING 1.11

Solve:

$$\frac{3x}{5}-\frac{2+5x}{3}-1 = \frac{-x-1}{15} \quad \text{and} \quad \frac{3x}{5}-\frac{2+5x}{3}-1<\frac{-x-1}{15}$$

Suggestion: Begin by multiplying both sides of the inequality by 15 so as to eliminate all denominators.

Answer:
$x = -\frac{8}{5}$ and $x>-\frac{8}{5}$

POLYNOMIAL EQUATIONS

Solving a polynomial equation often hinges on the important fact that:

> A product is zero **if**, and **only if**, one of its factors is zero.

EXAMPLE 1.10 Solve:

(a) $x^3+x^2-6x = 0$

(b) $x^3+x^2-5x = 0$

(c) $3x^3+5x^2-6x-10 = 0$

SOLUTION:

(a) $$x^3+x^2-6x = 0$$

Pull out the common factor, x: $x(x^2+x-6) = 0$

factor further: $x(x+3)(x-2) = 0$

A product is zero if, and **only if**, one of its factors is zero: $x = 0$ or $x = -3$ or $x = 2$

Quadratic Formula:
If: $ax^2+bx+c = 0$
Then: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

(b)

$$\begin{aligned} x^3+x^2-5x &= 0 \\ x(x^2+x-5) &= 0 \\ x = 0 \text{ or } x &= \frac{-1 \pm \sqrt{21}}{2} \quad \text{(see margin)} \end{aligned}$$

(c) If you look closely at the polynomial $3x^3+5x^2-6x-10$ you can see that by grouping the first two terms together: $3x^3+5x^2 = x^2(\mathbf{3x+5})$, and the last two terms together: $-6x-10 = -2(\mathbf{3x+5})$, the common factor $(\mathbf{3x+5})$ emerges:

$$\begin{aligned} 3x^3+5x^2-6x-10 &= 0 \\ (3x^3+5x^2)+(-6x-10) &= 0 \\ x^2(\mathbf{3x+5})-2(\mathbf{3x+5}) &= 0 \\ (\mathbf{3x+5})(x^2-2) &= 0 \\ (3x+5)(x+\sqrt{2})(x-\sqrt{2}) &= 0 \\ x = -\frac{5}{3} \text{ or } x &= \pm\sqrt{2} \end{aligned}$$

Answers: (a) $x = \frac{1}{2}, -4$

(b) $x = \frac{2 \pm \sqrt{13}}{3}$

(c) $x = -1, \pm 4$

CHECK YOUR UNDERSTANDING 1.12

Solve:

(a) $2x^2+7x-4 = 0$

(b) $3x^2-4x-3 = 0$

(c) $x^3+x^2-16x = 16$

ZEROS AND FACTORS OF A POLYNOMIAL

A **zero of a polynomial** $p(x)$ is a number which when substituted for the variable x yields zero. For example, $\mathbf{-1}$ is a zero of the polynomial $p(x) = x^3-3x-2$, since:

$$p(\mathbf{-1}) = (-1)^3-3(-1)-2 = -1+3-2 = \mathbf{0}$$

It is easy to see that if $(x-c)$ is a factor of the polynomial $p(x)$, then c is a zero of that polynomial (margin). The converse is also true:

If
$p(x) = (x-c)q(x)$
then:
$p(\mathbf{c}) = (c-c)q(x) = \mathbf{0}$

THEOREM 1.4
ZEROS AND FACTORS

If c is a zero of a polynomial then $(x-c)$ is a factor of the polynomial.

The following example illustrates how the above theorem can be used to solve certain polynomial equations.

EXAMPLE 1.11 Solve:

$$2x^3-3x^2-8x-3 = 0$$

SOLUTION: We simply observe (see margin) that -1 is a zero of the polynomial $p(x) = 2x^3 - 3x^2 - 8x - 3$:

$$p(-1) = 2(-1)^3 - 3(-1)^2 - 8(-1) - 3 = 0$$

Theorem 1.4 assures us that $x - (-1) = x + 1$ is a factor of $2x^3 - 3x^2 - 8x - 3$, and as you can easily check:

$$\begin{array}{r|l} & 2x^2 - 5x - 3 \\ \hline x + 1 & 2x^3 - 3x^2 - 8x - 3 \end{array}$$

Leading us to: $2x^3 - 3x^2 - 8x - 3 = (x + 1)(2x^2 - 5x - 3)$

Returning to our equation we have:

$$2x^3 - 3x^2 - 8x - 3 = 0$$
$$(x + 1)(2x^2 - 5x - 3) = 0$$
$$(x + 1)(2x + 1)(x - 3) = 0$$

Solution: $x = -1$, $x = -\frac{1}{2}$, $x = 3$.

The following result provides a method for determining the rational zeros of a given polynomial: Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial of degree n with integer coefficients. Each rational zero of $p(x)$ (reduced to lowest terms) is of the form $\frac{b}{c}$, where b is a factor of the constant coefficient a_0, and c is a factor of the leading coefficient a_n.

CHECK YOUR UNDERSTANDING 1.13

Solve by finding a zero:

$$x^3 + 2x^2 - 7x + 4 = 0$$

Answer: $x = 1, x = -4$

POLYNOMIAL INEQUALITIES

Each of the following expressions is zero at $x = 5$:

$$(x - 5) \qquad (x - 5)^7 \qquad (x - 5)^2 \qquad (x - 5)^8$$

In those cases where $(x - 5)$ is raised to an odd power we will say that 5 is an **odd-zero**, and that it is an **even-zero** when $(x - 5)$ is raised to an even power. For example:

5 is an odd-zero of $(x - 5)$, $(x - 5)^3$, $(x - 5)^5$, $(x - 5)^7$, ...

5 is an even-zero of $(x - 5)^2$, $(x - 5)^4$, $(x - 5)^6$, $(x - 5)^8$, ...

Note that the sign of $x - 5$ will change as one moves from one side of 5 to the other on the number line (it is positive to the right of 5 and negative to the left of 5). It follows that if you raise $(x - 5)$ to any odd power, say $(x - 5)^7$, then the change of sign “will survive.” On the other hand, if $(x - 5)$ is raised to an even power, say $(x - 5)^8$, then the change of sign will “not survive.” In general:

Traversing the zero c of $(x - c)^n$:
If c is an **odd-zero** (i.e: n is odd): **Sign Changes**.
If c is an **even-zero** (i.e: n is even): **Sign does NOT Change.**

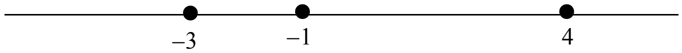
We now show how the above information can be used to solve a polynomial inequality when expressed in factored form.

EXAMPLE 1.12 Solve:

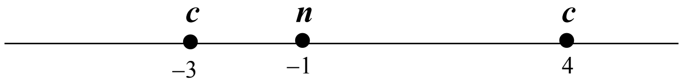
$$(x+3)^3(x+1)^2(4-x)<0$$

SOLUTION:

Step 1. Locate the zeros of $(x+3)^3(x+1)^2(4-x)$ on the number line:



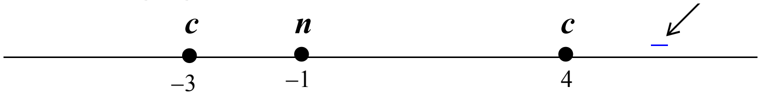
Place the letter ***c*** above the odd-zeros -3 and 4 to remind you that the SIGN of the product $(x+3)^3(x+1)^2(4-x)$ will **c**hange as one traverses those zeros, and place the letter ***n*** above the even-zero -1 to remind you that the sign of the product will **n**ot change about that even-zero:



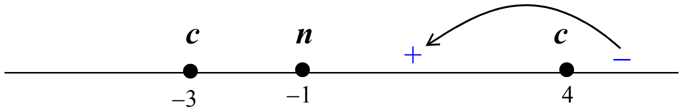
Step 2. You can get the "SIGN-ball rolling" by determining the SIGN of $(x+3)^3(x+1)^2(4-x)$ at any number other than $-3, -1$, or 4. For our part, we simply ask ourselves:

What is the SIGN to the right of the last zero, say at a million?

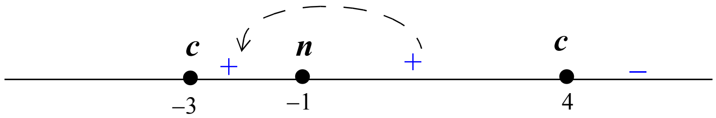
At a million, SIGN is **negative** [for only the factor $(4-x)$ is negative at that point]. Bringing us to:



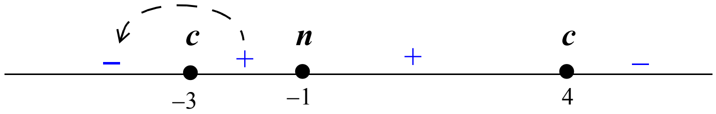
Step 3. (Walk the sign to the left) The ***c*** above 4 indicates that the sign will change about 4 (from negative to positive):



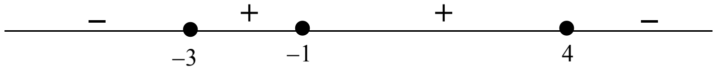
The ***n*** above -1 indicates that the sign will not change as you traverse -1 (will remain positive):



Finally, the ***c*** above -3 indicates a sign changes:



Here is the end result:



SIGN $(x+3)^3(x+1)^2(4-x)$

Figure 1.6

Step 4. Since we are solving $(x+3)^3(x+1)^2(4-x)<0$, we read off the intervals where the polynomial is negative (the "**–**" intervals): $(-\infty,-3)\cup(4,\infty)$.

NOTE: The information in Figure 1.6 also enables us to solve the inequalities:

$(x+3)^3(x+1)^2(4-x)>0$: $(-3,-1)\cup(-1,4)$

$(x+3)^3(x+1)^2(4-x)\geq 0$: $[-3,4]$

$(x+3)^3(x+1)^2(4-x)\leq 0$: $(-\infty,-3]\cup[4,\infty)\cup\{-1\}$

CHECK YOUR UNDERSTANDING 1.14

Solve:

(a) $(x-3)(x+2)(-x+5)<0$ (b) $(x+1)^2(x+2)^3(x-4)^2\geq 0$

Answers:
(a) $(-2,3)\cup(5,\infty)$
(b) $[-2,\infty)$

EXAMPLE 1.13

Solve:

$$(x+3)^2(x^2-8x+15)(3x^2-4x-3)>0$$

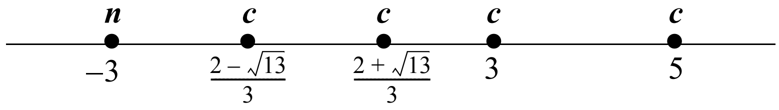
SOLUTION: We begin by factoring:

$$(x+3)^2(x^2-8x+15)(3x^2-4x-3)>0$$
$$(x+3)^2(x-3)(x-5)(3x^2-4x-3)>0$$

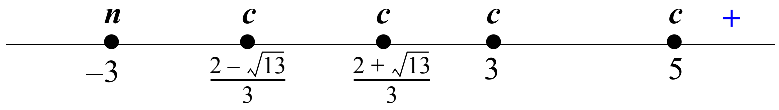
The zeros are $x=-3, x=3, x=5$, and $x=\frac{2\pm\sqrt{13}}{3}$ [CYU 1.12(b)].

Of the five zeros, only -3 is an even-zero (margin). So, the sign of the polynomial will **n**ot change about -3, but will **change** about the rest:

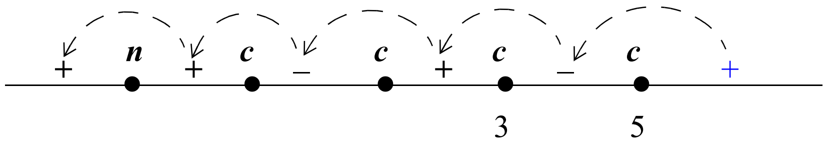
Note: If a quadratic polynomial has a positive discriminant then it has two distinct zeros, and they are both odd-zeros.



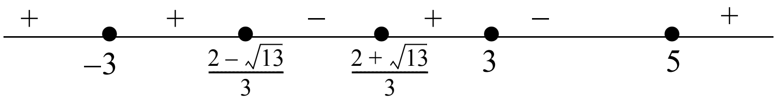
At a million, the polynomial is easily seen to be **positive**:



Proceed to the left, changing the sign each time you traverse a ***c***, and not changing the sign when traversing an ***n***:



Giving us:



SIGN $(x+3)^2(x^2-8x+15)(3x^2-4x-3)$

Since we are solving

$$(x+3)^2(x^2-8x+15)(3x^2-4x-3) > 0$$

we read off the intervals where the polynomial is positive (the "+ intervals"):

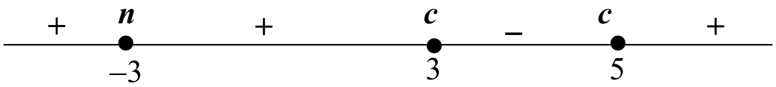
$$(-\infty, -3) \cup \left(-3, \frac{2-\sqrt{13}}{3}\right) \cup \left(\frac{2+\sqrt{13}}{3}, 3\right) \cup (5, \infty)$$

EXAMPLE 1.14 Solve:

$$(x+3)^2(x^2-8x+15)(3x^2-4x+3) > 0$$

SOLUTION: We replaced the factor $(3x^2-4x-3)$ of the previous example which has two zeros with the factor $(3x^2-4x+3)$ which has **no zeros**: negative discriminant: $b^2-4ac = 16-36 = -20$. All else remains as in the previous example, leading us to:

Only the zeros of the polynomial are represented on the number line. [$(3x^2-4x+3)$ has no zeros.]



SIGN $(x+3)^2(x^2-8x+15)(3x^2-4x+3)$

Solution of $(x+3)^2(x^2-8x+15)(3x^2-4x+3) > 0$:
$(-\infty, -3) \cup (-3, 3) \cup (5, \infty)$

Answer:
$(-\infty, -2] \cup \left[\frac{1-\sqrt{21}}{2}, 1\right] \cup \left[\frac{1+\sqrt{21}}{2}, \infty\right)$

CHECK YOUR UNDERSTANDING 1.15

Solve:

$$(x^2+x-2)(x^2-x+5)(-x^2+x+5) \le 0$$

RATIONAL EQUATIONS

Just as a **rational number** is an expression of the form $\frac{m}{n}$, where m and n are integers with $n \neq 0$, so then a **rational expression** (in the variable x) is an algebraic expression of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$.

A rational equation of the form $\frac{p_1(x)}{q_1(x)} = \frac{p_2(x)}{q_2(x)}$ can be solved by multiplying both sides of the equation by the least common denominator

(LCD) of the rational expressions in that equation, and then solving the resulting polynomial equation. It is important to remember, however, that while you can't "lose" a root of an equation by multiplying both sides by any quantity:

> MULTIPLYING BOTH SIDES OF AN EQUATION BY A QUANTITY WHICH CAN BE ZERO MAY INTRODUCE EXTRANEOUS SOLUTIONS. **CHECK YOUR ANSWERS.**

Consider the following example:

EXAMPLE 1.15 Solve:

$$\frac{2x^2}{x^2-x-6} = \frac{2}{x^2+2x} - \frac{1}{x}$$

SOLUTION: Factor all expressions:

$$\frac{2x^2}{(x+2)(x-3)} = \frac{2}{x(x+2)} - \frac{1}{x}$$

Clear denominators by multiplying both sides of the equation by $\mathbf{x(x+2)(x-3)}$, the LCD of the three rational expressions:

$$\frac{2x^2}{(x+2)(x-3)} \cdot \mathbf{x(x+2)(x-3)} = \frac{2}{x(x+2)} \cdot x\mathbf{(x+2)(x-3)} - \frac{1}{x} \cdot \mathbf{x(x+2)(x-3)}$$

$$\begin{aligned} 2x^2(\mathbf{x}) &= 2(\mathbf{x-3}) - 1(\mathbf{x+2})(\mathbf{x-3}) \\ 2x^3 &= 2x-6-(x^2-x-6) \\ 2x^3+x^2-3x &= 0 \\ x(2x+3)(x-1) &= 0 \\ x = 0,\ x = -\frac{3}{2},\ x &= 1 \end{aligned}$$

At this point, we see that the only **possible** solutions are 0, $-\frac{3}{2}$, and 1. Any candidate which causes a denominator in the original equation to be zero must be discarded. Discarding 0 (as it renders the denominator of $\frac{1}{x}$ to be zero), we conclude that $-\frac{3}{2}$ and 1 are the only solutions of the given equation.

CHECK YOUR UNDERSTANDING 1.16

Solve:

(a) $\frac{x-2}{x^2-4} - \frac{5}{4} = \frac{1}{x-3}$ (b) $3\left(\frac{x}{x^2+1}\right) + \left(\frac{x^2+1}{x}\right) = 4$

Suggestion: Make the substitution: $y = \frac{x}{x^2+1}$

Answers: (a) -1

(b) $\frac{3 \pm \sqrt{5}}{2}$

RATIONAL INEQUALITIES

One can use the SIGN method previously introduced to solve rational inequalities. Consider the following examples.

EXAMPLE 1.16 Solve:

$$\frac{2}{x-2}+\frac{x}{x+1}+1\geq 0$$

SOLUTION: Combine terms, and factor:

$$\frac{2(x+1)+x(x-2)+(x-2)(x+1)}{(x-2)(x+1)}\geq 0$$

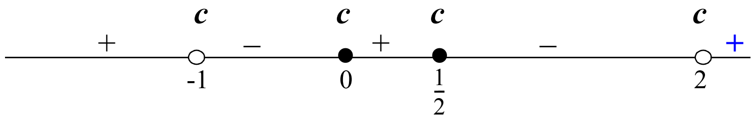
$$\frac{2x+2+x^2-2x+x^2-x-2}{(x-2)(x+1)}\geq 0$$

$$\frac{2x^2-x}{(x-2)(x+1)}\geq 0$$

$$\frac{x(2x-1)}{(x-2)(x+1)}\geq 0$$

We are adopting the convention of placing a "hole" where the denominator is zero (function not defined).

Locate the zeros of either the numerator or denominator on the number line, positioning a ***c*** above each, as all are odd-zeros. Noting that the rational expression is positive to the right of 2, we placed a "+" over the right-most interval, and then moved to the left, changing the sign each time we crossed over an odd-zero:



SIGN $\frac{x(2x-1)}{(x-2)(x+1)}$

Reading off the intervals with "+" signs, and adding the numbers where the numerator is zero (the black dots), we see that:

$$\frac{2}{x-2}+\frac{x}{x+1}+1\geq 0:\quad(-\infty,-1)\cup[0,\tfrac{1}{2}]\cup(2,\infty)$$

EXAMPLE 1.17 Solve:

$$\frac{3}{x}-2\leq 5x$$

SOLUTION: (**WARNING**) All too often, when confronted with such an inequality, one is tempted to begin by multiplying both sides by x, as one typically does with rational equations:

$$\frac{3}{x}-2\leq 5x$$

WRONG: when $x<0$ $\quad 3-2x\leq 5x^2$

DON'T DO IT! The resulting inequality will **not** be equivalent to the original one when x is a **negative** number (as you know, if you multiply both sides of an inequality by a negative number, you must reverse the inequality sign). Therefore, if you're set on clearing the denominator, then you will have to consider two cases: (1) if $x > 0$, and (2) if $x < 0$. A simpler approach is to bring all terms to the left, and proceed as in the previous example

$$\frac{3}{x} - 2 \le 5x$$

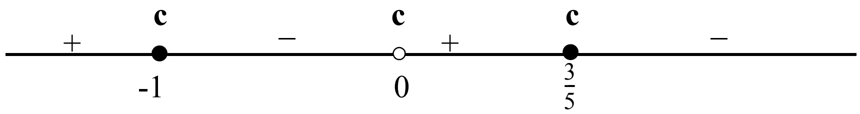
$$\frac{3}{x} - 2 - 5x \le 0$$

$$\frac{3 - 2x - 5x^2}{x} \le 0$$

$$\frac{-5x^2 - 2x + 3}{x} \le 0$$

$$\frac{(-5x + 3)(x + 1)}{x} \le 0$$

Noting that each zero is odd, and that the rational expression is negative to the right of the last zero, we have:



SIGN $\frac{(-5x + 3)(x + 1)}{x}$

Reading off the "–" intervals, and where the numerator is zero (the black dots in the SIGN chart), we arrive at the solution set of the given inequality:

$$[-1, 0) \cup [\tfrac{3}{5}, \infty)$$

CHECK YOUR UNDERSTANDING 1.17

Determine the solution set of the given inequality.

(a) $\frac{x + 2}{x^2 - 2x - 3} \ge 0$ (b) $x < \frac{1}{3x + 2}$

Answers:
(a) $[-2, -1) \cup (3, \infty)$
(b) $(-\infty, -1) \cup \left(-\frac{2}{3}, \frac{1}{3}\right)$

EXERCISES

Exercises 1-20. Solve.

1. (a) $3x-5 = 2x+7$ (b) $3x-5<2x+7$ (c) $3x-5\geq 2x+7$

2. (a) $\frac{3x-5}{2} = \frac{2x+7}{-6}$ (b) $\frac{3x-5}{2}<\frac{2x+7}{-6}$ (c) $\frac{3x-5}{2}\geq\frac{2x+7}{-6}$

3. (a) $\frac{1}{3}-\frac{2x-1}{6} = \frac{1}{2}-\frac{2(3x+2)}{3}$ (b) $\frac{1}{3}-\frac{2x-1}{6}\leq\frac{1}{2}-\frac{2(3x+2)}{3}$

4. (a) $x^3 = 4x$ (b) $x^3<4x$ (c) $-x^3<4x$ (d) $-x^3\geq 4x$

5. (a) $x^4-x^3-6x^2 = 0$ (b) $x^4-x^3-6x^2\leq 0$ (c) $x^4-x^3-6x^2>0$

6. (a) $x^4-x^3-5x^2 = 0$ (b) $x^4-x^3-5x^2\leq 0$ (c) $x^4-x^3-5x^2>0$

7. (a) $x^3-2x-1 = 0$ (b) $x^3-2x-1\leq 0$ (c) $x^3-2x-1>0$

8. (a) $9x^3-9x^2+x-1 = 0$ (b) $9x^3-9x^2+x-1\leq 0$ (c) $9x^3-9x^2+x-1>0$

9. (a) $(1-x)(2x+3)(x+2) = 0$ (b) $(1-x)(2x+3)(x+2)<0$

10. (a) $(1-x)^2(2x+3)^3(x+2) = 0$ (b) $(1-x)^2(2x+3)^3(x+2)>0$

11. (a) $(1-x)^{21}(2x+3)^{30}(x+2)^2 = 0$ (b) $(1-x)^{21}(2x+3)^{30}(x+2)^2\leq 0$

12. (a) $(1-x)^2(x+5)^3(5-x)^3 = 0$ (b) $(1-x)^2(x+5)^3(5-x)^3>0$

13. (a) $-x^4(2x+3)^3(-x+1)^5 = 0$ (b) $-x^4(2x+3)^3(-x+1)^5\leq 0$

14. (a) $(x^2-4)^3-64 = 0$ (b) $(x^2-4)^3-64<0$ (c) $(x^2-4)^3-64\geq 0$

15. (a) $3(x^2-1)^2 = 10(x^2-1)+8$ (b) $3(x^2-1)^2\leq 10(x^2-1)+8$

16. (a) $(3x+2)^4-2(3x+2)^2 = 3$ (b) $(3x+2)^4-2(3x+2)^2>3$

17. (a) $\dfrac{1}{x+1} = \dfrac{1}{2x}$ (b) $\dfrac{1}{x+1} > \dfrac{1}{2x}$ (c) $\dfrac{1}{x+1} \le \dfrac{1}{2x}$

18. (a) $\dfrac{1}{x+1} = \dfrac{x-1}{2x}$ (b) $\dfrac{1}{x+1} < \dfrac{x-1}{2x}$ (c) $\dfrac{1}{x+1} > \dfrac{x-1}{2x}$

19. (a) $x = \dfrac{2}{x} + 1$ (b) $x \ge \dfrac{2}{x} + 1$ (c) $x \le \dfrac{2}{x} + 1$

20. (a) $\dfrac{x+3}{2x+4} = \dfrac{4x}{x^2-x-6}$ (b) $\dfrac{x+3}{2x+4} < \dfrac{4x}{x^2-x-6}$ (c) $\dfrac{x+3}{2x+4} \ge \dfrac{4x}{x^2-x-6}$

Exercises 21-22. Exhibit a polynomial equation with the given solution set.

21. $\{1, 2, 3\}$ 22. $\{-1, 0, 2, 4\}$

Exercises 23-25. Exhibit a polynomial inequality with the given solution set.

23. $(-\infty, 1) \cup (2, 5)$ 24. $(-\infty, 1] \cup [2, 5]$ 25. $(-\infty, 1) \cup (2, 5) \cup (9, \infty)$

Exercises 26-28. Exhibit a rational inequality with the given solution set.

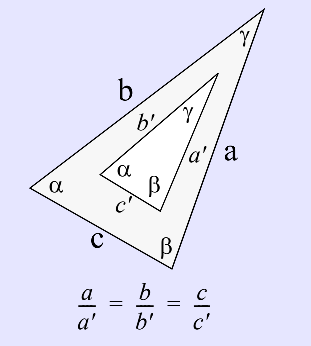
26. $(-\infty, 1) \cup [2, 5]$ 27. $(-\infty, 1] \cup (2, 5]$ 28. $(-\infty, 1) \cup [2, 5] \cup (9, \infty)$

§4. TRIGONOMETRY

An **angle** is formed by two line segments having a common endpoint. The line segments are the **sides** of the angle and the common endpoint is the **vertex**. Lower case Greek letters will be used to denote angles; particularly the letters α (alpha), β (beta), γ (gamma), and θ (theta).

The most commonly used unit of angle measurement is the **degree** (denoted “ ° ”). A $90°$ angle is said to be a **right angle**, and one strictly between $0°$ and $90°$ is called an **acute angle**.

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES



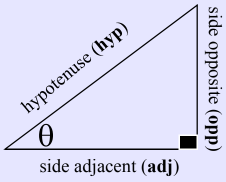
Two triangles are said to be **similar** if the angles of one of them are the same as those of the other (margin). While similar triangles have the same shape, they need not be of the same size; however:

The ratio of corresponding sides of similar triangles are equal (margin).

Since the sum of the angles in any triangle equals $180°$, if two angles in one triangle equal two angles in another, then their third angles are also equal, and the triangles are similar. In particular, any right triangle with acute angle θ has to be similar to any other right triangle with the same acute angle θ. This enables us to define the trigonometric functions of an acute angle θ in terms of ratios of lengths of sides of **any** right triangle containing θ:

DEFINITION 1.7

TRIGONOMETRIC FUNCTIONS



Let θ be an acute angle. The functions sine, cosine, tangent, cosecant, secant, and cotangent of θ (abbreviated sin, cos, tan, csc, sec, and cot, respectively), are defined as follows:

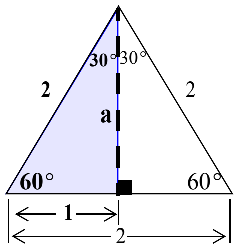
$$\sin\theta = \frac{\text{opp}}{\text{hyp}},\ \cos\theta = \frac{\text{adj}}{\text{hyp}},\ \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}},\ \sec\theta = \frac{\text{hyp}}{\text{adj}},\ \cot\theta = \frac{\text{adj}}{\text{opp}}$$

where opp, adj, and hyp are the lengths of the opposite side, adjacent side and hypotenuse, respectively.

TWO IMPORTANT RIGHT TRIANGLES

If one acute angle of a right triangle measures $45°$, then so does the other: $90° - 45° = 45°$. This means that the legs of the triangle are equal in length. Such a triangle is said to be **isosceles.** Since all isosceles right triangles are similar, any one of them can be used to compute the values of the trigonometric functions of a $45°$ angle. The one in Figure 1.7(a), with legs of length 1 unit and, consequently, with hypotenuse of length $\sqrt{1^2 + 1^2} = \sqrt{2}$ will be called the $45°$ reference triangle. The $30°/60°$ reference triangle is depicted in Figure 1.7(b).



The 30°/60° reference triangle was obtained by folding the above **equilateral triangle** (all sides of equal length) in half along the dashed line. Using the Pythagorean Theorem, we then found the length of the leg opposite the 60° angle:

$$2^2 = 1 + a^2, \text{ or } a = \sqrt{3}$$

45° reference triangle
(a)

30°/60° reference triangle
(b)

Figure 1.7

CHECK YOUR UNDERSTANDING 1.18

Complete the table of values:

θ	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
30°				2		
45°		$\frac{1}{\sqrt{2}}$				
60°			$\sqrt{3}$			

Answer: See page A-6.

RADIAN MEASURE

The **radian measure** of an angle with vertex at the center of a circle is the ratio of the length of the arc subtending that angle to the radius of the circle. Since that ratio is independent of the radius of the circle, it can be used as a measure of the angle.

Radian measure, being the ratio of two lengths, is a real number and is **not** associated with a unit (like degrees). Nonetheless, when referring to radian measure the word "radian" is often used to refer to that measure. To convert from degrees to radians, or the other way around, use the "bridge:"

$$90° = \frac{\pi}{2} \text{ radians or: } \quad 180° = \pi \text{ radians} \quad (*)$$

EXAMPLE 1.18 (a) Convert $\theta = 45°$ to radian measure.

(b) Convert $\theta = \frac{3\pi}{2}$ to degree measure.

SOLUTION: Using (*):

$$\text{(a)}\quad \theta = 45° = 45\cancel{°}\cdot\frac{\pi\text{ radians}}{180\cancel{°}} = \frac{\pi}{4}\text{ radians}$$

$$\text{(b)}\quad \theta = \frac{3\pi}{2} = \left(\frac{3\cancel{\pi}}{2}\ \cancel{\text{radians}}\right)\left(\frac{180°}{\cancel{\pi}\ \cancel{\text{radians}}}\right) = 270°$$

CHECK YOUR UNDERSTANDING 1.19

(a) Convert $\theta = 120°$ to radians.

(b) Convert $\theta = \frac{\pi}{6}$ radians to degrees.

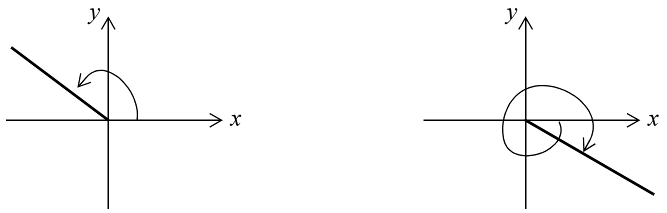
Answers: (a) $\frac{2\pi}{3}$ (b) $30°$

ORIENTED ANGLES

It is often useful to think of an angle θ as evolving from the following dynamic process:

> A fixed ray or half-line, called the **initial side** of the angle, is rotated about an endpoint *O*, called the **vertex** of the angle, to a final destination, called the **terminal side** of the angle. If the rotation is counterclockwise, then the angle is said to be **positive**, and if clockwise then it is **negative**. Because of the sign associated with it, such angles are said to be **oriented angles**.

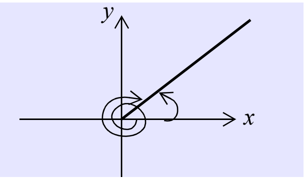
Typically, one positions an angle in the Cartesian plane, with its vertex at the origin and its initial side along the positive x-axis. In such a setting, the angle is said to be in **standard position.** Two angles in standard position are depicted in Figure 1.8. The one in Figure 1.8(a) is positive (counterclockwise rotation) and that in (b) is negative (clockwise rotation).



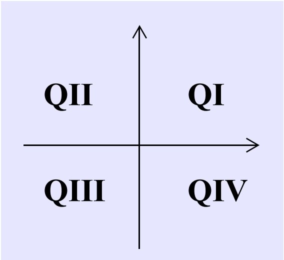
positive angle
(a)

negative angle
(b)

Figure 1.8



When the terminal sides of two angles in standard position coincide, then the angles are said to be **coterminal**. The two angles depicted in the adjacent figure are coterminal (one positive and the other negative).



The Cartesian plane is divided into four quadrants QI through QIV (see margin). When the terminal side of an angle in standard position lies within one of the four quadrants, the angle is said to **lie in that quadrant**. In particular, the angle of Figure 1.8(a) lies in the second quadrant (QII), the angle in Figure 1.8(b) lies in the fourth quadrant (QIV). No quadrant is associated with an angle whose terminal side lies on a coordinate axis, such as $90°$ or π radians. Such angles are called **quadrantal** angles.

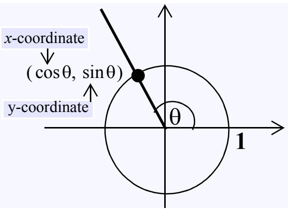
TRIGONOMETRIC FUNCTIONS OF ORIENTED ANGLES

We begin by defining the sine and cosine of any oriented angle:

DEFINITION 1.8

SINE AND COSINE FUNCTIONS

For any angle θ, $(\cos\theta, \sin\theta)$ is the point of intersection of the terminal side of θ with the **unit** circle.



Note that this definition coincides with the previous one when θ is an acute angle:

From right-triangle definition (see margin):

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1}$$

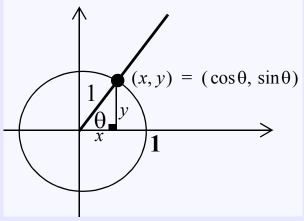
same $\longleftrightarrow$

same $\longleftrightarrow$

From Definition 1.8:

$$y = \sin\theta$$

$$x = \cos\theta$$



The hypotenuse has length 1 as it coincides with the radius of the circle.

The remaining four trigonometric functions of oriented angles are defined in terms of the sine and cosine functions:

DEFINITION 1.9

TANGENT, COSECANT, SECANT, COTANGENT

For any θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

When neither $\sin\theta$ nor $\cos\theta$ is zero:

$$\tan\theta = \frac{1}{\cot\theta},\ \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta},\ \cos\theta = \frac{1}{\sec\theta}$$

TRIGONOMETRIC VALUES OF QUADRANTAL ANGLES

Figure 1.9 shows the four points of intersection of the unit circle and the x- and y-axes: $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$. These points lie on the terminal side of quadrantal angles (angles whose terminal side lies on an axis). The sine and cosine of such angles are easily determined, as is illustrated in the following examples.

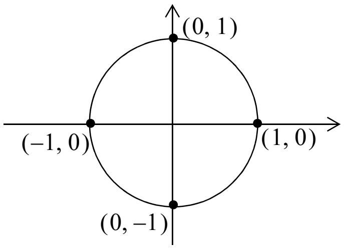
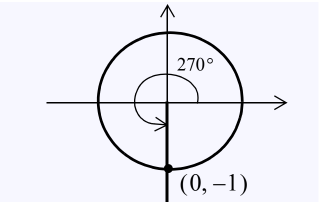


Figure 1.9

EXAMPLE 1.19 Determine the value of $\sin 270°$.

SOLUTION: Placing the angle in standard position and reading off the ***y*-coordinate** of the intersection of its terminal side with the unit circle, we find that:

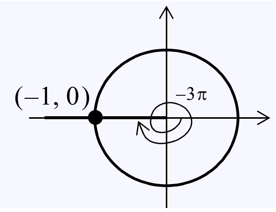
$$\sin 270° = -1$$



EXAMPLE 1.20 Evaluate $\tan(-3\pi)$.

SOLUTION: Placing the angle in standard position we see that $\sin(-3\pi) = 0$ (*y*-coordinate), and that $\cos(-3\pi) = -1$ (*x*-coordinate). Consequently:

$$\tan(-3\pi) = \frac{\sin(-3\pi)}{\cos(-3\pi)} = \frac{0}{-1} = 0$$



CHECK YOUR UNDERSTANDING 1.20

Complete the following table:

θ		sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
degrees	radians						
0°	0	0	1	0			undef
90°							
	π						
270°							

Answer: See page A-6.

TRIGONOMETRIC VALUES OF NON-QUADRANTAL ANGLES

We now turn our attention to that of determining trig θ for certain non-quadrantal angles θ, where "trig" represents any one of the six trigonometric functions. This will involve two steps: finding (1) the sign of trig θ and (2) the magnitude of trig θ.

To determine the signs of trig θ you need only remember that $(\cos\theta, \sin\theta)$ is the point of intersection of the terminal side of θ with the unit circle:

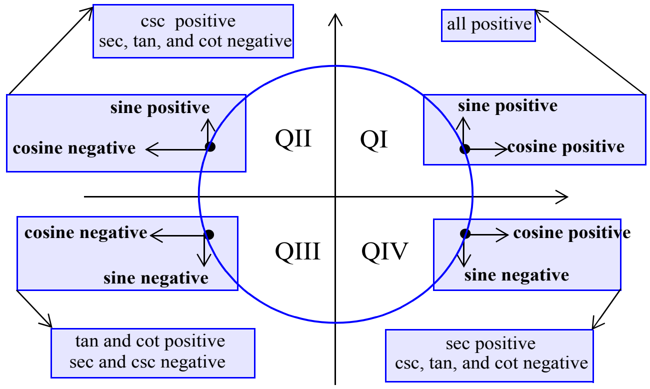
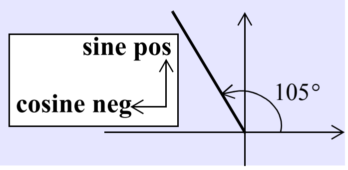


Figure 1.10

EXAMPLE 1.21 Determine the sign of the six trigonometric functions of $\theta = 825°$

SOLUTION: Since $825 = 2(360) + 105$, the angle $825°$ is coterminal with $105°$, and therefore lies in the second quadrant, where the sine is positive and the cosine is negative. Consequently:



$\boxed{\sin 825° > 0}$ $\boxed{\cos 825° < 0}$

$\csc 825° > 0$

$\tan 825° < 0$

$\downarrow$

$\cot 825° < 0$

$\sec 825° < 0$

Answer: cosine and secant are positive, the others are negative.

CHECK YOUR UNDERSTANDING 1.21

Determine the sign of the six trigonometric functions of $\theta = -\frac{29\pi}{7}$.

As you have seen, determining the sign of $\text{trig}\,\theta$ is an easy matter, once the quadrant of θ has been determined. Our next concern is with the magnitude of $\text{trig}\,\theta$, and we begin by defining the **reference angle**, θ_r, of any non-quadrantal angle θ, to be that **acute angle** formed by the terminal side of θ and the x-axis:

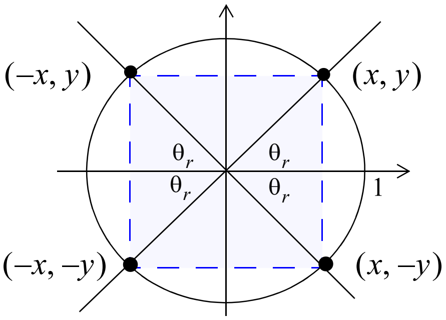


Figure 1.11

Consider, now, any angle θ with reference angle θ_r. The terminal side of θ is one of the four depicted in Figure 1.11. By symmetry, the coordinates of the points of intersection of the terminal sides with the unit circle only differ in sign:

$$(x, y), (-x, y), (-x, -y), (x, -y)$$

Thus $\cos\theta = x$ or $-x$, and $\sin\theta = y$ or $-y$, depending on the quadrant in which θ lies. Noting that $(x, y) = (\cos\theta_r, \sin\theta_r)$, we have:

$$\left.\begin{aligned} |\cos\theta| &= \cos\theta_r \\ |\sin\theta| &= \sin\theta_r \end{aligned}\right\} \text{ consequently: } \boxed{|\textbf{trig}\,\theta| = \textbf{trig}\,\theta_r}$$

We can now find the exact value of $\operatorname{trig}\theta$ for any θ whose reference angle is $30°$, $45°$, or $60°$. It is a two-step process:

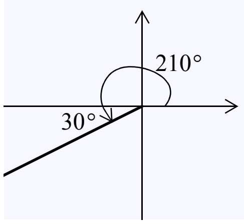
Step 1. **Determine the sign of** $\operatorname{trig}\theta$. Locate the quadrant in which θ lies, and then refer to Figure 1.10.

Step 2. **Determine the magnitude of** $\operatorname{trig}\theta$. Find the reference angle θ_r, and then use the fact that $|\operatorname{trig}\theta| = \operatorname{trig}\theta_r$

EXAMPLE 1.22 Evaluate $\cos 570°$.

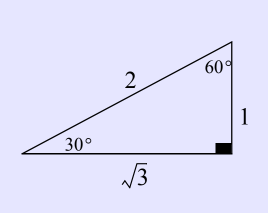
SOLUTION:

STEP 1: Noting that $570° - 360° = 210°$, we conclude that $\theta = 570°$ lies in QIII, where the cosine is **negative**.



Step 2: Noting that the reference angle of θ is $\theta_r = 210° - 180° = 30°$, we conclude that $|\cos 570°| = \cos 30°$. Thus:

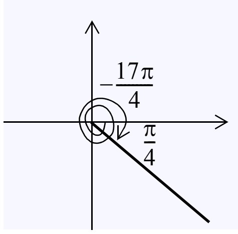
$$\cos 570° = \overset{\boxed{\text{Step 1}}}{-}\overset{\boxed{\text{Step 2}}}{\cos 30°} = -\underset{\text{see margin}}{\frac{\text{adj}}{\text{hyp}}} = -\frac{\sqrt{3}}{2}$$



EXAMPLE 1.23 Evaluate $\cot\left(-\frac{17\pi}{4}\right)$.

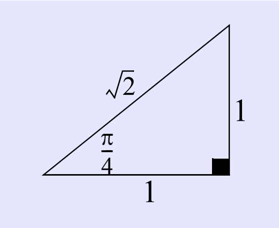
SOLUTION:

STEP 1: Noting that $-\frac{17\pi}{4} = -4\pi - \frac{\pi}{4}$, we conclude that $\theta = -\frac{17\pi}{4}$ lies in QIV, where the cotangent is **negative** ($\cot\theta = \frac{\cos\theta}{\sin\theta}$).



Step 2: Noting that the reference angle of θ is $\theta_r = \frac{\pi}{4}$, we conclude that $\left|\cot\left(-\frac{17\pi}{4}\right)\right| = \cot\frac{\pi}{4}$. Thus:

$$\cot\left(-\frac{17\pi}{4}\right) = \overset{\boxed{\text{Step 1}}}{-}\overset{\boxed{\text{Step 2}}}{\cot\frac{\pi}{4}} = -\underset{\text{see margin}}{\frac{\text{adj}}{\text{hyp}}} = -1$$



CHECK YOUR UNDERSTANDING 1.22

Determine the exact value of:

(a) $\sin(-840°)$ (b) $\cot\frac{11\pi}{4}$ (c) $\sec\left(-\frac{25\pi}{6}\right)$

Answers: (a) $-\frac{\sqrt{3}}{2}$

(b) -1 (c) $\frac{2}{\sqrt{3}}$

TRIGONOMETRIC FUNCTIONS OF A REAL VARIABLE

Though it may be comfortable to think of the trigonometric functions as acting on angles, a different interpretation is called for in the calculus where one is concerned with functions defined on real numbers. As you can see from the following definition, the transition from trigonometric functions of angles to trigonometric functions of numbers hinges on the fact that the radian measure of an angle, being a ratio of two lengths, is actually a real number.

DEFINITION 1.10
TRIGONOMETRIC FUNCTIONS OF A REAL VARIABLE

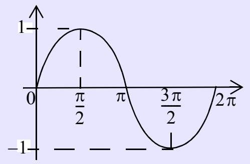
For any real number x:

$$\underset{\substack{\uparrow\\ \text{a number}}}{\operatorname{trig} x} = \underset{\substack{\uparrow\\ \text{an angle}}}{\operatorname{trig}\theta}$$

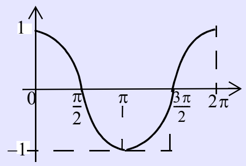
where θ is the angle with **radian measure** x.

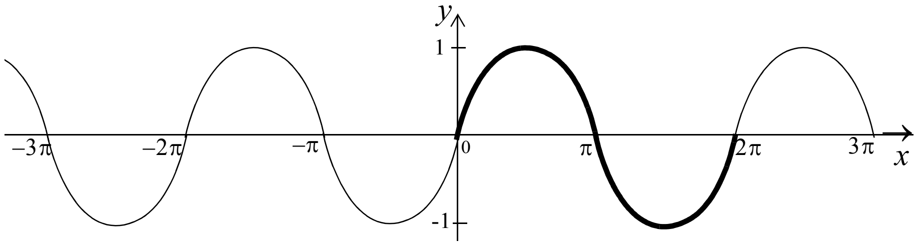
Figure 1.12 displays the graphs of the sine and cosine functions. As you can see, both are periodic with period 2π: the bold-faced portion of each graph just keeps repeating itself. In other words: $\sin(x+2k\pi) = \sin x$ and $\cos(x+2k\pi) = \cos x$ for every integer k.

Principal period of the sine function:

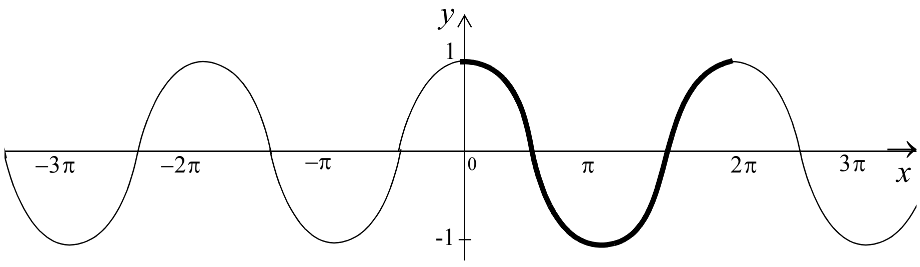


Principal period of the cosine function:





$f(x) = \sin x$



$f(x) = \cos x$

(b)

Figure 1.12

The graph of the tangent function appears in Figure 1.13. Note that the tangent function has period π: with principal period the bold-faced portion of the graph over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

A **vertical asymptote** for the graph of a function is represented by a dashed vertical line about which the graph tends to either plus or minus infinity. In particular the graph of the tangent function has vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

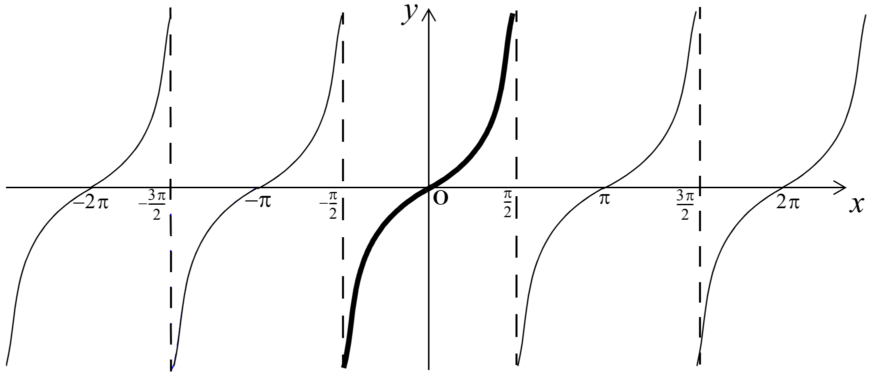


Figure 1.13

TRIGONOMETRIC IDENTITIES

An **identity** is an equation that holds for every value of the variable(s) for which both sides of the equation are defined. At this point, we content ourselves with acknowledging several important trigonometric identities.

THEOREM 1.5 For all numbers x and y:

PYTHAGOREAN IDENTITY

(i) $\sin^2 x + \cos^2 x = 1$

ADDITION IDENTITIES

(ii)
$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \end{aligned}$$

(iii)
$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \end{aligned}$$

DOUBLE-ANGLE IDENTITIES

(iv) $\sin 2x = 2\sin x \cos x$

(v) $\cos 2x = \cos^2 x - \sin^2 x$

HALF-ANGLE IDENTITIES

(vi) $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

(vii) $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

POWER-REDUCTION IDENTITIES

(viii) $\sin^2 x = \frac{1-\cos 2x}{2}$

(ix) $\cos^2 x = \frac{1+\cos 2x}{2}$

EXERCISES

Exercises 1-8. Convert to radian measure.

1. $\theta = 30°$
2. $\theta = -45°$
3. $\theta = 60°$
4. $\theta = 90°$
5. $\theta = 120°$
6. $\theta = 135°$
7. $\theta = -150°$
8. $\theta = 360°$

Exercises 9-16. Convert to degree measure.

9. $\theta = \frac{\pi}{4}$
10. $\theta = \frac{\pi}{3}$
11. $\theta = -\frac{\pi}{6}$
12. $\theta = \frac{3\pi}{4}$
13. $\theta = \frac{\pi}{2}$
14. $\theta = -\frac{5\pi}{4}$
15. $\theta = \frac{7\pi}{6}$
16. $\theta = \frac{11\pi}{3}$

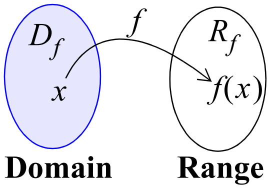
Exercises 17-32. Evaluate.

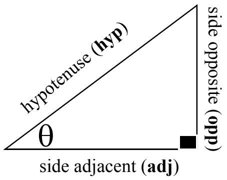
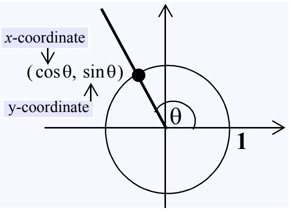
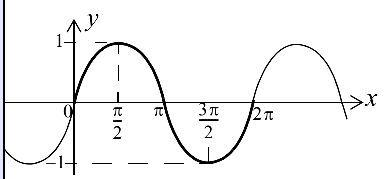
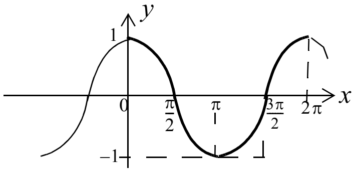
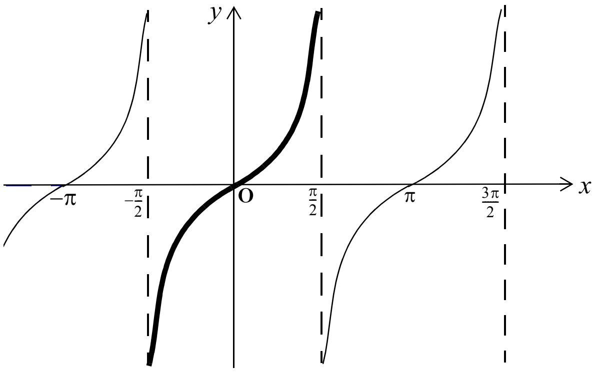
17. $\sin 810°$
18. $\cos 5\pi$
19. $\cot\frac{17\pi}{2}$
20. $\csc(-180°)$
21. $\sec 11\pi$
22. $\cot 540°$
23. $\tan(-360°)$
24. $\sin\left(-\frac{15\pi}{2}\right)$
25. $\sin 135°$
26. $\cos\frac{5\pi}{4}$
27. $\tan\frac{17\pi}{3}$
28. $\csc(-45°)$
29. $\sec\frac{11\pi}{6}$
30. $\cot 240°$
31. $\tan(-510°)$
32. $\sin\left(-\frac{15\pi}{6}\right)$

Exercises 33-41. Use the trigonometric identities of Theorem 1.5 to simply the expression.

33. $\frac{2\sin^2 x}{\sin 2x}$
34. $\frac{2\cos^2 x}{\sin 2x}$
35. $\frac{(\sin x + \cos x)^2 - 1}{\sin 2x}$
36. $\frac{\sin^2 x}{\cos x} - \sec x$
37. $\frac{\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{1 + \cos 2x}$
38. $\frac{1 + \cos 2x}{\sin 2x}$
39. $\frac{\tan x + \cot x}{\csc 2x}$
40. $\frac{(\sin x - \cos x)^2 - 1}{\sin 2x}$
41. $(\sec x + \tan x)^2(\sec x - \tan x)$

CHAPTER SUMMARY

ABSOLUTE VALUE AND DISTANCE	The **absolute value** of a, denoted by $\lvert a\rvert$, is given by: $\lvert a\rvert = \begin{cases} a & \text{if} \quad a \ge 0 \\ -a & \text{if} \quad a < 0 \end{cases}$ $\lvert a\rvert$ represents the distance (number of units) between the numbers a and 0 on the number line. The **distance** between a and b is given by $\lvert a - b\rvert$.
DOMAIN AND RANGE OF A FUNCTION	Roughly speaking, the **domain** of a function f is the set, D_f, on which f "acts," and its **range** is the set R_f of the function values. 
THE ARITHMETIC OF FUNCTIONS	The sum, difference, product, and quotient of two functions f and g are defined as follows: $(f+g)(x) = f(x) + g(x)$ $(f-g)(x) = f(x) - g(x)$ $(fg)(x) = f(x)g(x)$ $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ [providing $g(x) \neq 0$] For any constant c: $(cf)(x) = cf(x)$
COMPOSITION	The **composition** $(g\circ f)(x)$ is given by: $(g\circ f)(x) = g(f(x))$ **first** apply f and **then** apply g
ONE-TO-ONE FUNCTION AND ITS INVERSE	A function f is **one-to-one** if for all a and b in D_f: $f(a) = f(b) \Rightarrow a = b$ The **inverse** of a one-to-one function f with domain D_f and range R_f is that function f^{-1} with domain R_f and range D_f such that: $(f^{-1}\circ f)(x) = x$ for every x in D_f and $(f\circ f^{-1})(x) = x$ for every x in R_f

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES	 $\sin\theta = \frac{\text{opp}}{\text{hyp}}, \cos\theta = \frac{\text{adj}}{\text{hyp}}, \tan\theta = \frac{\text{opp}}{\text{adj}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}, \sec\theta = \frac{\text{hyp}}{\text{adj}}, \cot\theta = \frac{\text{adj}}{\text{opp}}$
TRIGONOMETRIC FUNCTIONS OF ARBITRARY ANGLES	For any angle θ, $(\cos\theta, \sin\theta)$ is the point of intersection of the terminal side of θ with the **unit** circle.  Then: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$
TRIGONOMETRIC FUNCTIONS OF A REAL VARIABLE	For any real number x: $\underset{\substack{\uparrow \\ \text{a number}}}{\text{trig}\,x} = \underset{\substack{\uparrow \\ \text{an angle}}}{\text{trig}\,\theta}$ where θ is the angle with **radian measure** x.
GRAPHS OF THE SINE, COSINE, AND TANGENT FUNCTIONS	 $y = \sin x$  $y = \cos x$  $y = \tan x$

TRIGONOMETRIC IDENTITIES	For all numbers x and y: (i) $\sin^2 x + \cos^2 x = 1$ (ii) $\sin(x+y) = \sin x \cos y + \cos x \sin y$ $\sin(x-y) = \sin x \cos y - \cos x \sin y$ (iii) $\cos(x+y) = \cos x \cos y - \sin x \sin y$ $\cos(x-y) = \cos x \cos y + \sin x \sin y$ (iv) $\sin 2x = 2 \sin x \cos x$ (v) $\cos 2x = \cos^2 x - \sin^2 x$ (vi) $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$ (vii) $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$ (viii) $\sin^2 x = \frac{1-\cos 2x}{2}$ (ix) $\cos^2 x = \frac{1+\cos 2x}{2}$

CHAPTER 2

LIMITS AND CONTINUITY

§1. THE LIMIT: AN INTUITIVE INTRODUCTION

At the very heart of the calculus is the concept of a limit, and here is an example:

$$\lim_{x \to 2}(3x+5)$$

It is read: The limit as x approaches 2 of the function $3x+5$.

It represents: That number which $3x+5$ approaches as the value of x approaches 2.

Clearly, as x gets closer and closer to 2, $3x$ will get closer and closer to 6, and $3x+5$ will consequently approach 11. We therefore write:

$$\lim_{x \to 2}(3x+5) = 11$$

By the same token,

$$\lim_{x \to 3}\frac{x}{x^2+5} = \frac{3}{14}$$

(as x approaches 3, the numerator approaches 3, and the denominator approaches $3^2+5 = 14$.)

CHECK YOUR UNDERSTANDING 2.1

Determine the given limit.

(a) $\lim_{x \to -1}(4x^2+x)$ (b) $\lim_{x \to 2}\frac{x+3}{x+2}$ (c) $\lim_{x \to 3}[x(3x^2+1)]$

Answer: (a) 3 (b) $\frac{5}{4}$ (c) 84

At this point, you might be wondering what all of the fuss is about. Up to now, it was totally natural to simply plug the given number into the function to arrive at the limit, right? Yes, but consider:

$$\lim_{x \to 2}\frac{x^2+x-6}{x^2-4}$$

Attempting to substitute 2 for x in the numerator and denominator brings us to the **meaningless** expression "$\frac{0}{0}$". However, if you let the value of x get closer and closer to 2; say $x = 1.99$, $x = 2.001$, $x = 1.9999$, $x = 2.00001$, and so on, you will find that $\frac{x^2+x-6}{x^2-4}$ will indeed approach a particular number. To find that number, we turn to a related algebra problem:

You can use your calculator to see what happens, but at some point, say for $x = 1.99999999$, you may receive an error message, since most calculators think that $1.99999999 = 2$. Poor things.

Simplify: $\dfrac{x^2+x-6}{x^2-4}$.

Solution: $\dfrac{x^2+x-6}{x^2-4} = \dfrac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \dfrac{x+3}{x+2}$

But the above is not totally correct, for one should really write:

We remind you that one cannot “cancel a 0.”

$$\frac{x^2+x-6}{x^2-4} = \frac{(x+3)(x-2)}{(x+2)(x-2)} \overset{\uparrow}{=} \frac{x+3}{x+2} \text{ if } x \neq 2 \qquad (*)$$

conditional equality

In the limit process however, the variable x **approaches 2** — it can get as close to 2 as you wish but it is **<u>never</u> equal to 2**; and we do indeed have:

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4} = \lim_{x\to 2}\frac{(x+3)(x-2)}{(x+2)(x-2)} \underset{\uparrow}{=} \lim_{x\to 2}\frac{x+3}{x+2} = \frac{5}{4}$$

not conditional

We’ve encountered two types of limits:

Those like $\lim\limits_{x\to 3}\dfrac{x}{x^2+5}$ and $\lim\limits_{x\to 2}\dfrac{x+3}{x+2}$, which can be determined by simply plugging in the indicated x-value.

And the more interesting type, like $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x^2-4}$, which cannot be evaluated at $x = 2$.

EXAMPLE 2.1 Evaluate:

$$\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^2+2x-15}$$

SOLUTION: Since both the numerator and denominator are zero at $x = 3$, $(x-3)$ must be a factor of both polynomials:

$$\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^2+2x-15} = \lim_{x\to 3}\frac{x(x^2-2x-3)}{(\mathbf{x-3})(x+5)} = \lim_{x\to 3}\frac{x\cancel{(\mathbf{x-3})}(x+1)}{\cancel{(\mathbf{x-3})}(x+5)}$$

$$= \lim_{x\to 3}\frac{x(x+1)}{(x+5)} = \frac{3(3+1)}{3+5} = \frac{12}{8} = \frac{3}{2}$$

CHECK YOUR UNDERSTANDING 2.2

Determine the given limit.

(a) $\lim\limits_{x\to 1}\dfrac{x^2+3x-4}{x^2-1}$ (b) $\lim\limits_{x\to 2}\dfrac{x^3-2x^2+2x-4}{x^2-x-2}$

Answer: (a) $\frac{5}{2}$ (b) 2

EXAMPLE 2.2

Evaluate: $$\lim_{x\to -2}\frac{\frac{1}{2}+\frac{1}{x}}{x+2}$$

Note: We carry the limit symbol until the limit is performed. An analogous situation:

$$3\cdot 2+4 = \underset{\uparrow}{6}+4 = 10$$

you write the "+" until the sum is performed

SOLUTION: The problem is with the zero in the denominator (when $x = -2$). Our goal is to alleviate that problem:

$$\lim_{x\to -2}\frac{\frac{1}{2}+\frac{1}{x}}{x+2} = \lim_{x\to -2}\frac{\frac{x+2}{2x}}{x+2} = \lim_{x\to -2}\left(\frac{x+2}{2x}\cdot\frac{1}{x+2}\right)$$

$$= \lim_{x\to -2}\frac{1}{2x} \underset{\uparrow}{=} \frac{1}{2(-2)} = -\frac{1}{4}$$

see margin

EXAMPLE 2.3

Evaluate:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

SOLUTION: We have to do something to get rid of that bothersome 0 in the denominator (when $x = 0$). Out of desperation, we rationalize the numerator:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$(a-b)(a+b) = a^2-b^2$: $$= \lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0}\frac{x}{\mathbf{x}(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{2}$$

The numerator got nicer while the denominator got uglier. But ugliness is in the eyes of the beholder. The truly "ugly" thing was that **x** in the denominator which vanishes.

CHECK YOUR UNDERSTANDING 2.3

Determine the given limit.

(a) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$ (b) $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x^2-4}$

Answer: (a) 1 (b) $\frac{1}{16}$

You can determine $\lim_{x\to 1}\frac{x^2+3x-4}{x^2+1}$ by simply substituting 1 for x in the expression $\frac{x^2+3x-4}{x^2+1}$: $\frac{(1)^2+3(1)-4}{(1)^2+1} = \frac{0}{2} = 0$. We now turn that problem upside down, and consider:

$$\lim_{x\to 1}\frac{x^2+1}{x^2+3x-4}$$

Observing that as x gets closer and closer to 1, the denominator of $\frac{x^2+1}{x^2+3x-4}$ gets closer and closer to 0 while the numerator approaches 2, we can conclude that the quotient must get arbitrarily large in magnitude: **LIMIT DOES NOT EXIST.**

Here is another situation where a limit fails to exist:

The function $f(x) = \begin{cases} 3x+1 & \text{if } x<2 \\ x^2 & \text{if } x>2 \end{cases}$ does not have a limit at $x = 2$. Why not? Because:

> As x approaches 2 from the left, the top rule is in effect, and $f(x)$ approaches $3 \cdot 2 + 1 = 7$. On the other hand, as x approaches 2 from the right, the bottom rule is in effect, and $f(x)$ approaches $2^2 = 4$.

ONE-SIDED LIMITS

One says that the **left-hand limit** of the above function equals 7 and that the **right-hand limit** equals 4; written:

$$\lim_{x \to 2^-} f(x) = 7 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 4$$

You can easily convince yourself that the following assertion holds:

THEOREM 2.1 $\lim\limits_{x \to c} f(x)$ exists if and only if $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ both exist and are equal; and, if they are, then:

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

You are invited to establish this result in the next section.

In particular, the function $f(x) = \begin{cases} 3x+1 & \text{if } x<2 \\ x^2 & \text{if } x>2 \end{cases}$, does not have a limit at 2 since, as we have seen, $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$.

How about the function $g(x) = \begin{cases} 3x+1 & \text{if } x<2 \\ x^2+3 & \text{if } x>2 \end{cases}$ — does it have a limit at 2? Yes, and it equals 7:

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} g(x) = 7$$

Finally, does the function $h(x) = \begin{cases} 3x+1 & \text{if } x<2 \\ 100 & \text{if } x=2 \\ x^2+3 & \text{if } x<2 \end{cases}$ have a limit at 2?

Yes, it is again 7; for the limit is "not concerned" with what happens at 2, but only what happens as x **approaches** 2!

CHECK YOUR UNDERSTANDING 2.4

Determine if the given limit exists, and if does, evaluate it.

(a) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ (b) $\lim\limits_{x \to 3} \dfrac{x - 3}{x^2 - 6x + 9}$

(c) $\lim\limits_{x \to 3} f(x)$, for $f(x) = \begin{cases} 2x - 1 & \text{if } x < 3 \\ x + 5 & \text{if } x > 3 \end{cases}$

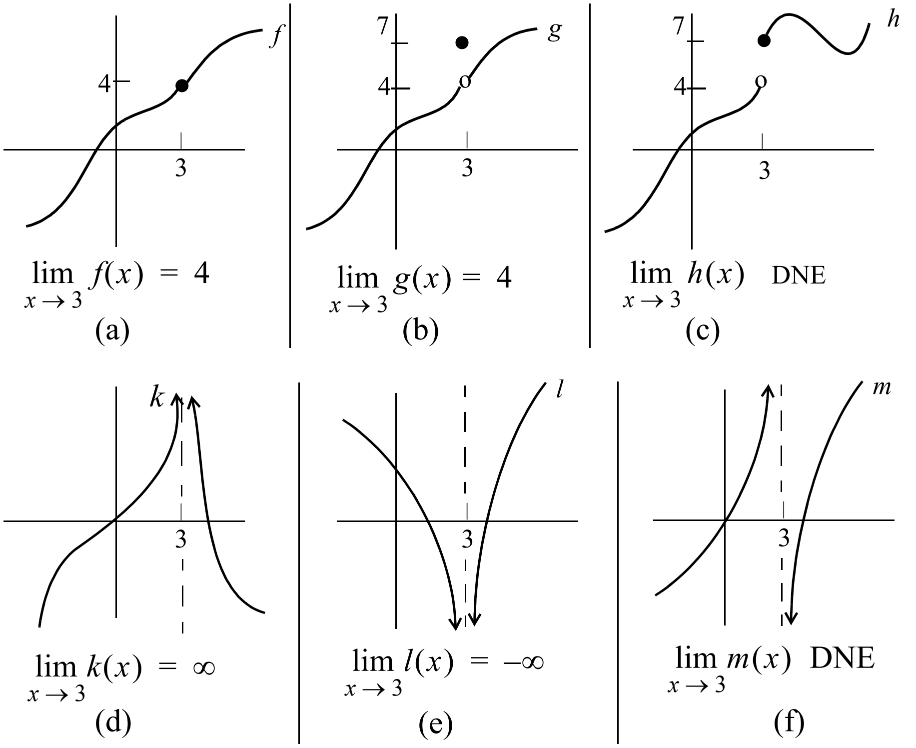
(d) $\lim\limits_{x \to 3} g(x)$, for $g(x) = \begin{cases} x - 1 & \text{if } x \le 3 \\ 2 & \text{if } x > 3 \end{cases}$

Answer: (a) 6 (b) DNE (c) DNE (d) 2

GEOMETRICAL INTERPRETATION OF THE LIMIT CONCEPT

Consider the functions depicted in Figure 2.1.

The solid dot above 3 in (a) depicts the value of the function at 3: $f(3) = 4$. Similarly in (b) $g(3) = 7$, and in (c) $h(3) = 7$.



$\lim\limits_{x \to 3} f(x) = 4$
(a)

$\lim\limits_{x \to 3} g(x) = 4$
(b)

$\lim\limits_{x \to 3} h(x)$ DNE
(c)

$\lim\limits_{x \to 3} k(x) = \infty$
(d)

$\lim\limits_{x \to 3} l(x) = -\infty$
(e)

$\lim\limits_{x \to 3} m(x)$ DNE
(f)

Figure 2.1

Note that "∞" and "$-\infty$" are not numbers. The notation $\lim\limits_{x \to c} k(x) = \infty$ in (d) is used to indicate that the function k takes on arbitrarily large positive values as x approaches 3 from either side.
$\lim\limits_{x \to 3} l(x) = -\infty$ indicates that the function l takes on arbitrarily large negative values as x approaches 3 from either side [see (e)].
The function m in (f) does not have a limit at 3, as the function values approach $+\infty$ as x approaches 3 from the left, and $-\infty$ as it approaches 3 from the right. One can write:

$$\lim_{x \to 3^-} m(x) = \infty$$

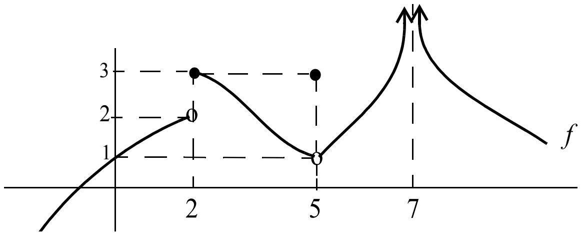
and $\lim\limits_{x \to 3^+} m(x) = -\infty$

(Formal definition appears at the end of the next section.)

Looking at the function in (a), we see that as x approaches 3, from **either the left or the right**, the function values (y-values) approach the number 4. Thus: $\lim\limits_{x \to 3} f(x) = 4$. The function in (b) differs from that of (a) only at $x = 3$, where it has a "hiccup." But that anomaly has absolutely no effect on the limit, since the limit does not care about what happens at 3 — it only cares about what happens as x **approaches** 3. Thus: $\lim\limits_{x \to 3} g(x) = 4$. (The same conclusion could be drawn, even if g were not defined at 3.) The function h in (c) does **not** have a limit at $x = 3$, since $\lim\limits_{x \to 3^-} h(x) = 4$ while $\lim\limits_{x \to 3^+} h(x) = 7$. A discussion of the limit situation depicted in Figure (d), (e), and (f) is offered in the margin.

CHECK YOUR UNDERSTANDING 2.5

Referring to the graph of the function *f* below, determine if the given limit exists, or is infinite. If it exists, indicate its value.

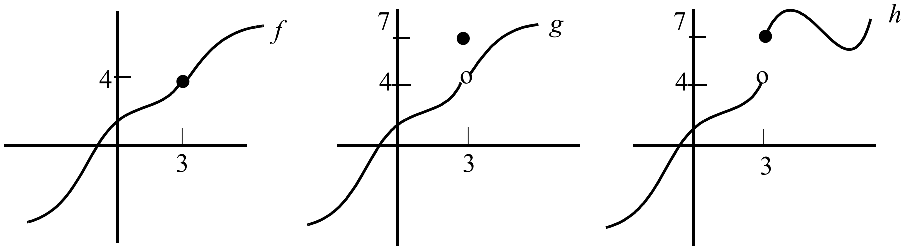


(a) $\lim\limits_{x \to 0} f(x)$ (b) $\lim\limits_{x \to 2} f(x)$ (c) $\lim\limits_{x \to 5} f(x)$ (d) $\lim\limits_{x \to 7} f(x)$

Answers:
(a) 1 (b) DNE (c) 1
(d) DNE: $\lim\limits_{x \to 7} f(x) = \infty$

CONTINUITY

Let's reconsider the functions:



$\lim\limits_{x \to 3} f(x) = 4$ (a) $\lim\limits_{x \to 3} g(x) = 4$ (b) $\lim\limits_{x \to 3} h(x)$ Does Not Exist (c)

Figure 2.2

While the function in (c) does not have a limit as x approaches 3, both the functions in (a) and (b) do. The limit, oblivious of what happens at 3, cannot tell you that the function g in (b) behaves in a somewhat peculiar fashion at $x = 3$. Another concept, one more sensitive than that of the limit, is called for:

DEFINITION 2.1
CONTINUITY

A function *f* is **continuous** at c if:

$$\lim_{x \to c} f(x) = f(c)$$

A function that is not continuous at *c*, is said to be **discontinuous** at that point.

In other words, for the function *f* to be continuous at c, three things must happen:

(1) The function must be defined at *c*.

(2) The limit must exist at *c*.

(3) That limit must equal the function value at *c*.

Returning to Figure 2.2, we see that:

The function f in (a) is continuous at 3
(limit equals 4, and $f(3) = 4$)

The function g in (b) is not continuous at 3
(limit equals 4, but $g(3) = 7$).

The function h in (c) is also not continuous at 3
(limit does not even exist)

REMOVABLE DISCONTINUITY

If a function f has a limit at c but that limit is not equal to $f(c)$ (perhaps because the function is not even defined at c), then f is said to have a **removable discontinuity** at c. [The function in Figure 2.2(b) has a removable discontinuity at 3.

JUMP DISCONTINUITY

If the left- or right-hand limit of f fails to exist at c, then the function is said to have an **essential discontinuity** at that point. It can be shown that the function $f(x) = \sin\frac{1}{x}$ has an essential discontinuity at 0.

If the left- and right-hand limits of a function f exist at c but are not equal to each other, then f is said to have a **Jump discontinuity** at that point [The function in Figure 2.2(c) has a jump discontinuity at 3].

EXAMPLE 2.4 Determine if the function

$$f(x) = \begin{cases} 2x & \text{if } x < 3 \\ 3x + 1 & \text{if } 3 \le x \le 4 \\ x^2 - 3 & \text{if } 4 < x < 5 \\ 4x + 2 & \text{if } 5 < x \end{cases}$$

has a discontinuity at:

(a) $x = 3$ (b) $x = 4$ (c) $x = 5$

If so, is it removable or a jump discontinuity?

SOLUTION:

(a) Since $\lim\limits_{x \to 3^-} f(x) = 2 \cdot 3 = 6$ and $\lim\limits_{x \to 3^+} f(x) = 3 \cdot 3 + 1 = 10$, the limit does not exist at 3, and the function has a jump discontinuity at that point.

(b) Since $\lim\limits_{x \to 4^-} f(x) = 3 \cdot 4 + 1 = 13$, $\lim\limits_{x \to 4^+} f(x) = 4^2 - 3 = 13$,

and since $f(4) = 3 \cdot 4 + 1 = 13$, the function is continuous at 4.

(c) Since the function is not defined at 5, it cannot possibly be continuous at that point. But is the discontinuity removable? Yes:

$$\lim_{x \to 5^-} f(x) = 5^2 - 3 = 22 \text{ and } \lim_{x \to 5^+} f(x) = 4 \cdot 5 + 2 = 22$$

Answers:
(a) Jump discontinuity.
(b) Removable discontinuity.

CHECK YOUR UNDERSTANDING 2.6

Is the given function continuous at $x = 2$? If not, does it have a removable or jump discontinuity at the point?

(a) $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 1.001 & \text{if } x \geq 2 \end{cases}$ (b) $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ 25 & \text{if } x = 2 \\ x^2 - 1 & \text{if } x > 2 \end{cases}$

Finally, we note that a **continuous function** is a function that is continuous at every point in its domain. Roughly speaking, a function is continuous if it can be graphed without lifting the writing utensil. The function $f(x) = x^2$ is continuous everywhere, as is every polynomial function. Rational functions are continuous wherever they are defined.

EXERCISES

Exercises 1-29. Evaluate the given limit, if it exists.

1. $\lim\limits_{x \to 3} \dfrac{x^2-5}{x+3}$

2. $\lim\limits_{x \to 5} \dfrac{x^2-5}{x+3}$

3. $\lim\limits_{x \to 5} \dfrac{x^2-5}{x+5}$

4. $\lim\limits_{x \to 5} \dfrac{x^2-5}{x-5}$

5. $\lim\limits_{x \to 5} \dfrac{x^2-25}{x^2-3x-10}$

6. $\lim\limits_{x \to -5} \dfrac{x^2-25}{x^2+4x+5}$

7. $\lim\limits_{x \to 2} \dfrac{x^2+3x-10}{x^2-4x+4}$

8. $\lim\limits_{x \to -2} \dfrac{x^2+4x+4}{x^2+3x+2}$

9. $\lim\limits_{x \to -1} \dfrac{2x^3+5x^2+3x}{x^2-3x-4}$

10. $\lim\limits_{x \to 1} \dfrac{x^2-1}{x^2-2x+1}$

11. $\lim\limits_{x \to 1} \dfrac{x^2-2x+1}{x^2-1}$

12. $\lim\limits_{x \to 1} \dfrac{x^2-1}{x^3-1}$

13. $\lim\limits_{x \to 1} \dfrac{x^2-1}{x^3-x^2+2x-2}$

14. $\lim\limits_{x \to -2} \dfrac{x^2+x-2}{x^3+x^2-4x-4}$

15. $\lim\limits_{x \to 1} \dfrac{x^2-1}{x^3-2x^2+1}$

16. $\lim\limits_{x \to 4} \dfrac{2-\sqrt{x}}{x^2-16}$

17. $\lim\limits_{x \to 3} \dfrac{\sqrt{1+x}-2}{x-3}$

18. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-1}{x^2}$

19. $\lim\limits_{x \to 0} \dfrac{x^2-1}{\sqrt{1+x}-1}$

20. $\lim\limits_{x \to 2} \dfrac{\dfrac{1}{x+2}-\dfrac{1}{4}}{\dfrac{x^2}{x-1}-4}$

21. $\lim\limits_{x \to 0} \left(\dfrac{1}{x}-\dfrac{1}{x\sqrt{x+1}}\right)$

22. $\lim\limits_{x \to 2} \left(\dfrac{1}{x-2}+\dfrac{1}{\sqrt{x^2-4}}\right)$

23. $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

24. $\lim\limits_{h \to 0} \dfrac{(x+h)^2-x^2}{h}$

25. $\lim\limits_{h \to 0} \dfrac{(x+h)^3-x^3}{h}$

26. $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

27. $\lim\limits_{x \to 0} \dfrac{\sin^2 x+\sin x}{\cot x}$

28. $\lim\limits_{x \to 0} \dfrac{\cos^2 x-1}{\cos^2 x+\cos x-2}$

29. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\sin x}{\cos x}$

Exercises 30-33. Determine if the given limit exists. If it does, indicate its value. Is the function continuous at the given point? If not, is the discontinuity removable or is it a jump discontinuity?

30. $\lim\limits_{x \to 2} f(x)$ where:

$$f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$$

31. $\lim\limits_{x \to 2} f(x)$ where:

$$f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

32. $\lim\limits_{x \to 2} f(x)$ where:

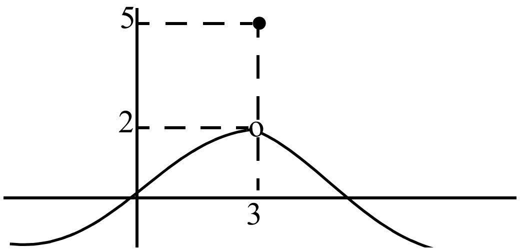
$$f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ 2 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

33. $\lim\limits_{x \to 2} f(x)$ where:

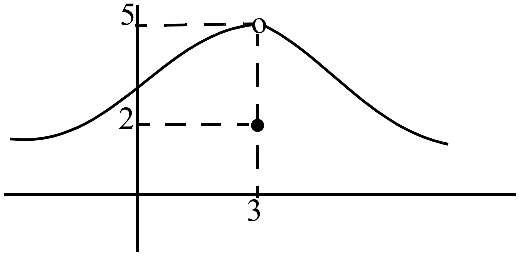
$$f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

Exercises 34-37. (Geometrical Interpretation) Referring to the graph of the function *f*, determine if the given limit exists or is infinite. If it exists, indicate its value. Is the function continuous at the given point? If not, is the discontinuity removable, a jump discontinuity, or an essential discontinuity?

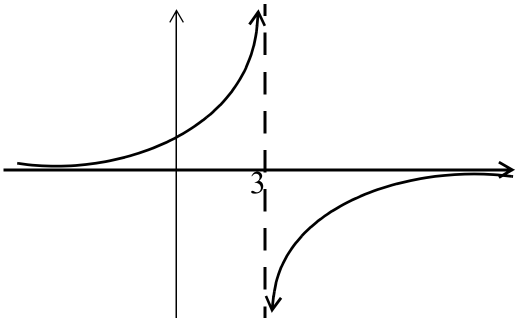
34. $\lim\limits_{x \to 3} f(x)$



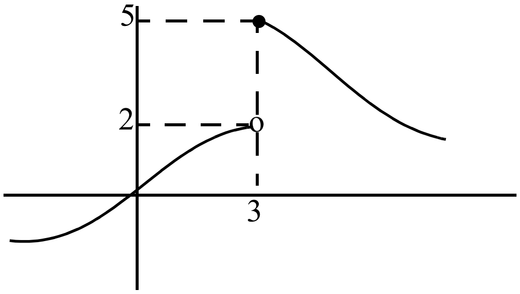
35. $\lim\limits_{x \to 3} f(x)$



36. $\lim\limits_{x \to 3} f(x)$



37. $\lim\limits_{x \to 3} f(x)$



Exercise 38-42. (Theory) Sketch the graph of a function f satisfying the given conditions.

38. $f(1) = 5$ and $\lim\limits_{x \to 1} f(x) = 5$

39. $f(3) = 1$ and $\lim\limits_{x \to 3} f(x) = -1$

40. $f(1) = 5$ and $\lim\limits_{x \to 1} f(x) = 6$.

41. $f(1) = 5$ and $\lim\limits_{x \to 1} f(x) = 5$.

42. *f* is:
   (i) Continuous at 1.
   (ii) Defined at 2 and has a removable discontinuity at 2.
   (iii) Is not defined at 3 and has a removable discontinuity at 3.
   (iv) Is defined at 4 and has a jump discontinuity at 4.

§2. THE DEFINITION OF A LIMIT

The intuitive notion of the limit concept is valuable, but hardly rigorous. It's fine to say that $\lim\limits_{x \to c} f(x) = L$ tells us that the function values $f(x)$ get arbitrarily close to L as long as x is sufficiently close to c, but what exactly does "arbitrarily close," and "sufficiently close" mean? The time has come to place the limit concept on a firm foundation:

The Greek letters ε and δ ("epsilon" and "delta," respectively) are traditionally used in the definition of the limit.

DEFINITION 2.2

$$\lim_{x \to c} f(x) = L \text{ if:}$$

For any given $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\text{if } 0 < |x - c| < \delta \text{ then } |f(x) - L| < \varepsilon$$

Left-Hand Limit:

$\lim\limits_{x \to c^-} f(x) = L$ if:

For any given $\varepsilon > 0$ there exists $\delta > 0$ such that:

if $c - \delta < x < c$ then $|f(x) - L| < \varepsilon$

Right-Hand Limit:

$\lim\limits_{x \to c^+} f(x) = L$ if:

For any given $\varepsilon > 0$ there exists $\delta > 0$ such that:

if $c < x < c + \delta$ then $|f(x) - L| < \varepsilon$

We remind you that, geometrically, the absolute value expression $|a - b|$ denotes the **distance between *a* and *b*** on the number line. Now look at the last line in the above definition. It says exactly what needs to be said:

$$\textbf{if } \underset{\boxed{\text{if } x \neq c \text{ is within } \delta \text{ units of } c}}{\underset{\uparrow}{0 < |x - c| < \delta}} \quad \textbf{then} \quad \underset{\boxed{f(x) \text{ is within } \varepsilon \text{ units of } L}}{\underset{\uparrow}{|f(x) - L| < \varepsilon}}$$

then

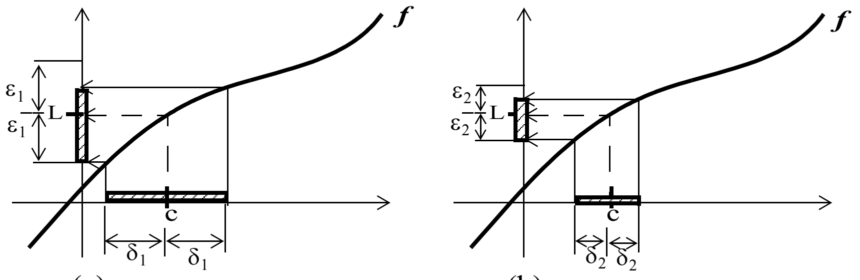
In the exercises you are invited to establish the following result which asserts that limits, if they exist, are unique:

THEOREM 2.2
UNIQUENESS THEOREM

If $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} f(x) = M$, then:

$$L = M.$$

To show that $\lim\limits_{x \to c} f(x) = L$ we need to find, **for any given** $\varepsilon > 0$, a positive number δ such that for every x within δ units of c (excluding c itself) $f(x)$ falls within ε units of L. Generally, the smaller the given ε, the smaller the corresponding δ. Consider, for example, the function f depicted in Figure 2.3. Note that while "everything in the δ_1 neighborhood of c in Figure 2.3(a) maps into the ε_1 neighborhood of L," a smaller δ (labeled δ_2) had to be chosen to accommodate the smaller ε (labeled ε_2) of Figure 2.3(b).



(a) (b)

Figure 2.3

EXAMPLE 2.5

Prove that $\lim\limits_{x \to 3} 2x + 5 = 11$.

SOLUTION: For a given $\varepsilon > 0$ we are to find $\delta > 0$ such that:

$$\left.\begin{aligned}
&0 < |x-3| < \delta \Rightarrow |f(x) - 11| < \varepsilon\\
&0 < |x-3| < \delta \Rightarrow |(2x+5) - 11| < \varepsilon\\
&0 < |x-3| < \delta \Rightarrow |2x-6| < \varepsilon\\
&0 < |x-3| < \delta \Rightarrow 2|x-3| < \varepsilon\\
&0 < |x-3| < \delta \Rightarrow |x-3| < \frac{\varepsilon}{2}
\end{aligned}\right\}\text{same}$$

While a choice of δ for which $0 < |x-3| < \delta \Rightarrow |(2x+5) - 11| < \varepsilon$ in the top line above may not be so apparent, it is trivial to find a δ that works in the rewritten form (bottom line):

$$0 < |x-3| < \delta \Rightarrow |x-3| < \frac{\varepsilon}{2}\text{; namely: } \delta = \frac{\varepsilon}{2}.$$

$$\text{For certainly: } 0 < |x-3| < \frac{\varepsilon}{2} \Rightarrow |x-3| < \frac{\varepsilon}{2}\ !$$

$\delta = \frac{\varepsilon}{2}$ is the largest δ that "works." Any positive number less than $\frac{\varepsilon}{2}$ can also be used.

CHECK YOUR UNDERSTANDING 2.7

Prove that

$$\lim_{x \to 4} 5x + 1 = 21$$

Answer: See page A-9.

EXAMPLE 2.6

Prove that $\lim\limits_{x \to 3} x^2 = 9$.

SOLUTION: For a given $\varepsilon > 0$ we are to find $\delta > 0$ such that:

$$\begin{aligned}
&0 < |x-3| < \delta \Rightarrow |x^2 - 9| < \varepsilon\\
&0 < |x-3| < \delta \Rightarrow |(x+3)(x-3)| < \varepsilon\\
&0 < |x-3| < \delta \Rightarrow |x+3||x-3| < \varepsilon
\end{aligned}$$

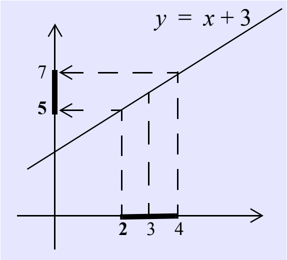
The proof will be complete once we find a $\delta > 0$ for which:

$$0 < |x-3| < \delta \Rightarrow |x+3||x-3| < \varepsilon \quad (*)$$

While it is tempting to choose $\delta = \frac{\varepsilon}{|x+3|}$, that temptation must be suppressed, for δ has to be a positive number and not a function of x. Since we are interested in what happens near $x = 3$, we decide to focus on the interval:

$$(2, 4) = \{x \,|\, |x-3| < 1\}$$

Within that interval $|x+3| < 7$ (see margin).



Consequently, within that interval:

$$|x+3||x-3| < 7|x-3|$$

Taking δ to be the smaller of the two numbers 1 and $\frac{\varepsilon}{7}$ [written: $\delta = \min\left(1, \frac{\varepsilon}{7}\right)$], we are assured that $|x+3| < 7$ and that $\delta \le \frac{\varepsilon}{7}$. Thus:

$$0 < |\mathbf{x-3}| < \delta \Rightarrow |x^2-9| = |x+3||\mathbf{x-3}| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

CHECK YOUR UNDERSTANDING 2.8

Show that:

$$\lim_{x \to 2} (x^2+1) = 5$$

Answer: See page A-9.

PROPERTIES OF LIMITS

The following theorem formalizes results that you have been taking for granted all along.

The theorem also holds for one-sided limits. In particular, if:

$$\lim_{x \to c^+} f(x) = L$$

$$\text{and } \lim_{x \to c^+} g(x) = M$$

then:

$$\lim_{x \to c^+} [f(x) + g(x)] = L + M.$$

THEOREM 2.3

LIMIT THEOREMS

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then:

(a) $\lim_{x \to c} [f(x) + g(x)] = L + M$.

(b) $\lim_{x \to c} [f(x) - g(x)] = L - M$.

(c) $\lim_{x \to c} [f(x) \cdot g(x)] = LM$.

(d) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \ne 0$.

(e) $\lim_{x \to c} [af(x)] = aL$ for any number a.

IN WORDS:	
	(a) The limit of a sum is the sum of the limits.
	(b) The limit of a difference is the difference of the limits.
	(c) The limit of a product is the product of the limits.
	(d) The limit of a quotient is the quotient of the limits (providing the limit of the denominator is not zero).
	(e) The limit of a constant times a function is the constant times the limit of the function.

PROOF: We prove (a) and (c). You are invited to establish (e) in CYU 2.9 below, and (b) and (d) in the exercises.

Throughout this development we are assuming that the variable x will always be contained in the domain of both f and g.

(a) For a given $\varepsilon > 0$ we are to find $\delta > 0$ such that:

$$0 < |x-c| < \delta \Rightarrow |(f+g)(x) - (L+M)| < \varepsilon$$
$$0 < |x-c| < \delta \Rightarrow |f(x) + g(x) - L - M| < \varepsilon$$
$$0 < |x-c| < \delta \Rightarrow |\boldsymbol{[f(x) - L] + [g(x) - M]}| < \varepsilon \ (*)$$

For any two numbers a and b:
$$|a+b| \le |a| + |b|$$
(Triangle Inequality)

By virtue of the triangle inequality (see margin) we have:

$$|[f(x) - L] + [g(x) - M]| \le |f(x) - L| + |g(x) - M|$$

It follows that (*) will hold for any $\delta > 0$ for which $0 < |x-c| < \delta$ implies that **BOTH** $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$. Let's find such a δ:

Since $\lim\limits_{x \to c} f(x) = L$, there is a $\delta_1 > 0$ such that

$$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}.$$

Since $\lim\limits_{x \to c} g(x) = M$ there is a $\delta_2 > 0$ such that

$$0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2} \quad .$$

Taking δ to be the smaller of δ_1 and δ_2 we have:

$0 < |x-c| < \delta$, implies tha **BOTH** $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$.

(c) For a given $\varepsilon > 0$ we are to find $\delta > 0$ such that

$$0 < |x-c| < \delta \Rightarrow |(fg)(x) - LM| < \varepsilon$$
$$0 < |x-c| < \delta \Rightarrow |f(x)g(x) - LM| < \varepsilon$$

see margin: $0 < |x-c| < \delta \Rightarrow |f(x)g(x) \boldsymbol{- f(x)M + f(x)M} - LM| < \varepsilon$

$$0 < |x-c| < \delta \Rightarrow |f(x)[g(x) - M] + M[f(x) - L]| < \varepsilon$$

We want to work $|f(x) - L|$ and $|g(x) - M|$ into the picture, and do so by introducing the **clever zero**:
$$\boldsymbol{-f(x)M + f(x)M}$$
within the expression:
$$|(fg)(x) - LM|$$

The triangle inequality tells us that:

$$|f(x)[g(x) - M] + M[f(x) - L]| \le |f(x)||g(x) - M| + |M||f(x) - L|$$

We now set our sights on finding a $\delta > 0$ for which both:

(A) $0 < |x-c| < \delta \Rightarrow |f(x)||g(x) - M| < \frac{\varepsilon}{2}$

and **(B)** $0 < |x-c| < \delta \Rightarrow |M||f(x) - L| < \frac{\varepsilon}{2}$

For (A): Since $\lim\limits_{x \to c} f(x) = L$, we can choose δ_1 such that:

$$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \underset{\substack{\uparrow \\ \text{could choose any positive number}}}{1}$$

Then: $$1 > |f(x) - L| \underset{\substack{\uparrow \\ \text{Exercise 41, page 10}}}{\geq} |f(x)| - |L| \Rightarrow |f(x)| < |L| + 1 \quad (^{**})$$

So that: $$|f(x)||g(x) - M| < (1 + |L|)|g(x) - M|$$

Since $\lim\limits_{x \to c} g(x) = M$, we can choose $\delta_2 > 0$ such that:

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2(|L| + 1)} \quad (^{***})$$

Letting $\delta_A = \min(\delta_1, \delta_2)$, we find that (A) is satisfied:

$$0 < |x - c| < \delta_A \Rightarrow |f(x)||g(x) - M| \underset{(***)}{\overset{(**)}{<}} (|L| + 1) \cdot \frac{\varepsilon}{2(|L| + 1)} = \frac{\varepsilon}{2}$$

For (B): Since $\lim\limits_{x \to c} f(x) = L$, there exists a $\delta_B > 0$ such that:

$$0 < |x - c| < \delta_B \Rightarrow |f(x) - L| < \frac{\varepsilon}{2(|M| + 1)} \leftarrow \text{(margin)}$$

Then:

$$0 < |x - c| < \delta_B \Rightarrow |M||\boldsymbol{f(x)} - L| < |M| \cdot \frac{\varepsilon}{\mathbf{2}(|M| + \mathbf{1})} < \frac{\varepsilon}{2}$$

We used $|M| + 1$ instead of $|M|$ in the denominator, as $|M|$ might be zero.

End result: For $\delta = \min(\delta_A, \delta_B)$, both **(A)** and **(B)** are satisfied.

CHECK YOUR UNDERSTANDING 2.9

Prove Theorem 2.3(e).

Answer: See page A-9.

CONTINUITY

Since the concept of continuity rests on the limit concept, and since the limit concept has rigorously been defined, a rigorous definition of continuity follows nearly free of charge.

From Definition 2.1: A function f is **continuous** at c if:

$$\lim_{x \to c} f(x) = f(c)$$

Equivalently:For any given $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\text{if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \varepsilon$$

(Why $|x - c| < \delta$ rather than $0 < |x - c| < \delta$?)

A function that is not continuous at c, is said to be **discontinuous** at that point.

A function that is continuous at every point in an interval is said to be **continuous in that interval**.

A function that is continuous throughout its domain is said to be a **continuous function**.

(Continuous from the left)
Left-Hand Continuity at c:
$\lim\limits_{x \to c^-} f(x) = f(c)$
(Continuous from the right)
Right-hand Continuity at c:
$\lim\limits_{x \to c^+} f(x) = f(c)$

If a function is defined only on one side of an endpoint of an interval, such as is the case with the function $f(x) = \sqrt{x}$ which is only defined on the interval $[0, \infty)$, we then understand continuity at that endpoint to mean continuity from the right (or continuous from the left; whichever is appropriate).

Theorem 2.3 readily extends to accommodate continuity:

THEOREM 2.4 If f and g are continuous at c then so are the functions:

(a) $f+g$ (b) $f-g$ (c) fg

(d) $\frac{f}{g}$ [providing $g(c) \neq 0$] (e) af

PROOF: Each of the above results follows directly from its corresponding limit theorem. Consider the following proof of (c):

$$\lim_{x \to c} (fg)(x) = \lim_{x \to c} [f(x) \cdot g(x)] \overset{\text{Theorem 2.3(c)}}{\overset{\downarrow}{=}} \lim_{x \to c} f(x) \lim_{x \to c} g(x)$$

$$\text{continuity of } f \text{ and } g\text{:} \quad = f(c)g(c) = (fg)(c)$$

CHECK YOUR UNDERSTANDING 2.10

Prove Theorem 2.4 (a).

Answer: See page A-9.

In the exercises you are asked to show that all polynomial and rational functions are continuous. The sine and cosine functions are also continuous. That being the case, the composite functions $\sin(x^2 + 2x - 5)$ and $\cos\left(\frac{3x}{x^2 + 7}\right)$ are also continuous; for:.

Recall that:
$(g \circ f)(x) = g[f(x)]$
(See Definition 1.4, page 7)

THEOREM 2.5
COMPOSITION THEOREM

If f and g are continuous functions with the range of f contained in the domain of g, then the composite function $g \circ f$ is also continuous.

PROOF: Let c be in the domain of $g \circ f$, and let $\varepsilon > 0$ be given. As is suggested in Figure 2.4(a), we need to find a $\delta > 0$ such that:

$$|x - c| < \delta \Rightarrow |g[f(x)] - g[f(c)]| < \varepsilon \qquad (*)$$

Let's do it:

Since g is continuous at $f(c)$, we can find a $\bar{\delta}$ such that:

$$|y - f(c)| < \bar{\delta} \Rightarrow |g(y) - g[f(c)]| < \varepsilon \text{ [Figure 2.4(b)].}$$

(In particular: $|f(x) - f(c)| < \bar{\delta} \Rightarrow |g[f(x)] - g[f(c)]| < \varepsilon$)

Now, think of $\bar{\delta}$ as being an "ε-challenge" for the function f. By the continuity of f, we can find a δ such that:

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \bar{\delta} \text{ [Figure 2.4(c)].}$$

Merging Figures (b) and (c) we see that (*) holds [Figure 2.4(d)]:

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \bar{\delta} \Rightarrow |g[f(x)] - g[f(c)]| < \varepsilon$$

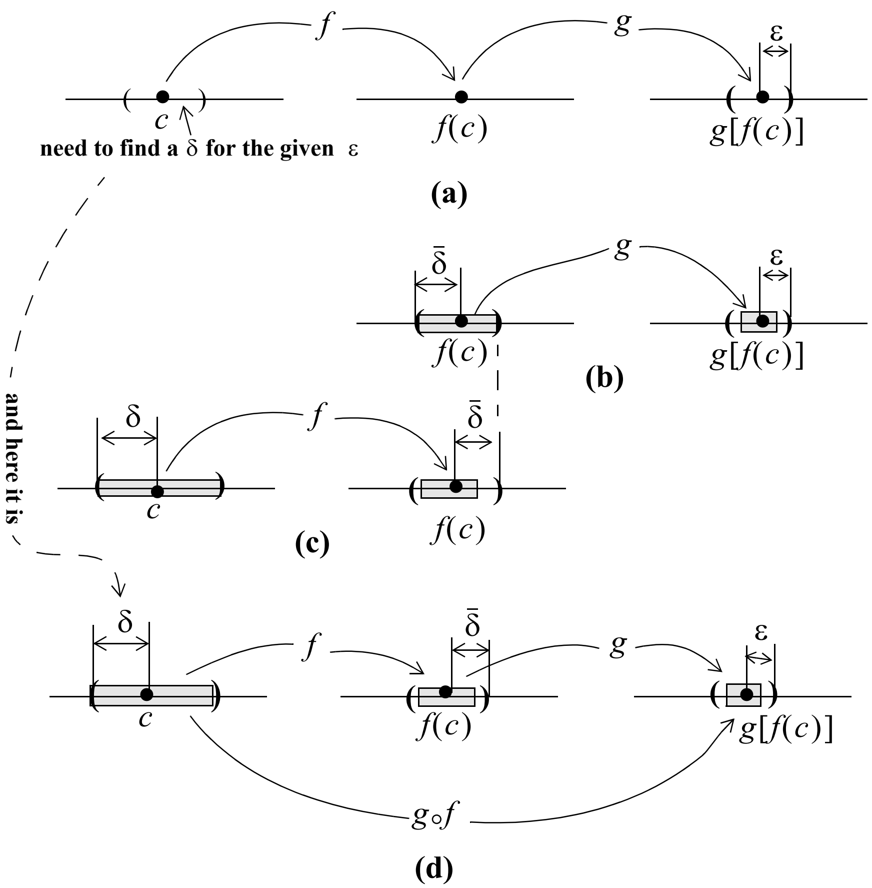


Figure 2.4

CHECK YOUR UNDERSTANDING 2.11

Prove: If f is continuous at b and if $\lim\limits_{x \to a} g(x) = b$, then:

$$\lim_{x \to a} f[g(x)] = f[\lim_{x \to a} g(x)]$$

Answer: See page A-9.

We now extend the limit concept to accommodate the concept of infinity:

DEFINITION 2.3

$\lim\limits_{x \to c} f(x) = \infty$ if for any given number M there exists $\delta > 0$ such that: $0 < |x - c| < \delta \Rightarrow f(x) > M$.

$\lim\limits_{x \to c} f(x) = -\infty$ if for any given number M there exists $\delta > 0$ such that: $0 < |x - c| < \delta \Rightarrow f(x) < M$.

$\lim\limits_{x \to \infty} f(x) = c$ if for any given $\varepsilon > 0$ there exists a number N such that: $x > N \Rightarrow |f(x) - c| < \varepsilon$.

$\lim\limits_{x \to \infty} f(x) = \infty$ if for any given number M there exists a number N such that: $x > N \Rightarrow f(x) > M$.

CHECK YOUR UNDERSTANDING 2.12

Formulate a definition for:

(a) $\lim\limits_{x \to c^-} f(x) = \infty$ (b) $\lim\limits_{x \to c^+} f(x) = \infty$ (c) $\lim\limits_{x \to -\infty} f(x) = c$

(d) $\lim\limits_{x \to \infty} f(x) = -\infty$ (e) $\lim\limits_{x \to -\infty} f(x) = \infty$ (f) $\lim\limits_{x \to -\infty} f(x) = -\infty$

Answers: See page A-10.

EXERCISES

Exercises 1-6. Determine the limit L. Find, for the given $\varepsilon > 0$, the largest $\delta > 0$ for which $0 < |x - c| < \delta$ implies that the function values fall within ε units of L.

1. $\lim\limits_{x \to 1} (2x)$, $\varepsilon = 3$

2. $\lim\limits_{x \to 1} (2x)$, $\varepsilon = \frac{1}{3}$

3. $\lim\limits_{x \to 2} (x - 5)$, $\varepsilon = 1$

4. $\lim\limits_{x \to 2} (x - 5)$, $\varepsilon = \frac{1}{10}$

5. $\lim\limits_{x \to 2} (x^2 + 1)$, $\varepsilon = 1$

6. $\lim\limits_{x \to 2} (x^2 + 1)$, $\varepsilon = \frac{1}{2}$

Exercises 7-15. Establish the following claim.

7. $\lim\limits_{x \to 1} (5x - 3) = 2$

8. $\lim\limits_{x \to 1} (3x - 5) = -2$

9. $\lim\limits_{x \to -2} (-x - 1) = 1$

10. $\lim\limits_{x \to 1} \left(\frac{2}{3}x + 3\right) = \frac{11}{3}$

11. $\lim\limits_{x \to \frac{1}{2}} \left(\frac{1}{2}x + 1\right) = \frac{5}{4}$

12. $\lim\limits_{x \to -\frac{1}{2}} \left(x + \frac{1}{2}\right) = 0$

13. $\lim\limits_{x \to 2} x^2 = 4$

14. $\lim\limits_{x \to -2} x^2 = 4$

15. $\lim\limits_{x \to 3} (x^2 - 1) = 8$

Exercises 16-17. **(Theory)** Prove:

16. $\lim\limits_{x \to c} x = c$ for any number c.

17. $\lim\limits_{x \to c} d = d$ for any numbers c and $f(x) = d$.

Exercises 18-23. Use Exercises 16-17, and Theorem 2.3 to establish the claim.

18. $\lim\limits_{x \to 1} x^2 = 1$

19. $\lim\limits_{x \to 2} 3x^2 = 12$

20. $\lim\limits_{x \to 5} \frac{x + 1}{x^2 - x} = \frac{3}{10}$

21. $\lim\limits_{x \to -2} (2x + 1)^3 = -27$

22. $\lim\limits_{x \to -2} \frac{2x^3 - 7x - 1}{3x + 5} = 3$

23. $\lim\limits_{x \to 3} (x^3 - 25)^3 = 8$

24. For what values of a and b is $f(x) = \begin{cases} ax^3 + bx + 1 & \text{if } x < 2 \\ bx^2 + a & \text{if } x \geq 2 \end{cases}$ continuous at 2?

25. For what values of a and b is $f(x) = \begin{cases} ax^2 - b & \text{if } x < 1 \\ bx^3 + ax + 3 & \text{if } x \geq 1 \end{cases}$ continuous at 1?

Exercises 26-29. **(Theory)** Give examples of functions f and g such that neither f nor g is continuous at c but:

26. $f + g$ is continuous at c.

27. gf is continuous at c.

28. $\frac{f}{g}$ is continuous at c.

29. $g \circ f$ is continuous at c.

Exercises 30-40. (Theory) Prove:

30. Theorem 2.1
31. Theorem 2.2
32. Theorem 2.3(b)
33. Theorem 2.3(d)
34. Theorem 2.4(b)
35. Theorem 2.4(d)
36. Theorem 2.4(e)
37. $\lim\limits_{x \to c} f(x) = 0$ if and only if $\lim\limits_{x \to c} |f(x)| = 0$. That is:

$$\lim_{x \to c} f(x) = 0 \Rightarrow \lim_{x \to c} |f(x)| = 0 \text{ and } \lim_{x \to c} |f(x)| = 0 \Rightarrow \lim_{x \to c} f(x) = 0$$

38. Every polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is a continuous function.
39. Every rational function $r(x) = \dfrac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}$ is a continuous function.
40. Prove Theorem 2.1, page 46.
41. If f and g are continuous functions and if $\lim\limits_{x \to \infty} g(x) = b$, then: $\lim\limits_{x \to \infty} f[g(x)] = f(b)$.

CHAPTER SUMMARY

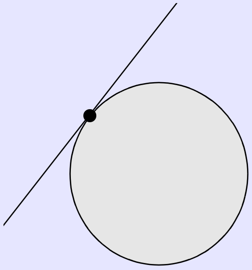
THE LIMIT	$\lim_{x \to c} f(x) = L$
INTUITIVE:	"As x approaches c, the function values $f(x)$ approach L."
RIGOROUS DEFINITION:	For any given $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \lvert x - c \rvert < \delta$ then $\lvert f(x) - L \rvert < \varepsilon$
LIMIT THEOREMS:	If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$ The limit of a sum (difference) is the sum (difference) of the limits. $\lim_{x \to c} [f(x) \cdot g(x)] = LM$ and $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$ The limit of a product (quotient) is the product (quotient) of the limits. $\lim_{x \to c} [af(x)] = aL$ The limit of a constant times a function is the constant times the limit of the function.
CONTINUITY AT A POINT:	A function f is **continuous** at c if: $\lim_{x \to c} f(x) = f(c)$ In other words: The limit exists and is equal to the function value.
CONTINUITY THEOREMS:	If f and g are continuous at c then so are the functions: $f \pm g$ $\quad fg \quad$ $\frac{f}{g}$ [providing $g(c) \neq 0$] $\quad af$
CONTINUOUS FUNCTION:	A function that is continuous throughout its domain is said to be a **continuous function**.
COMPOSITION THEOREM:	If f and g are continuous functions with the range of f contained in the domain of g, then the composite function $g \circ f$ is also continuous.

CHAPTER 3
THE DERIVATIVE

§1. TANGENT LINES AND THE DERIVATIVE

Consider the two lines of Figure 3.1. Which do you feel better represents the tangent line to the curve at the indicated point $(c, f(c))$? Chances are that you chose the dashed line, and might have based that decision on the concept of a tangent line to a circle (see margin). Our goal in this section is to define (find?) the "tangent line" of Figure 3.1, so that it conforms with our predisposed notion of tangency.

The tangent line to a point on the circle is that line which touches the circle only at that point:



This will not do for more general curves. The "tangent line" in Figure 3.1, for example, touches the curve at more than one point.

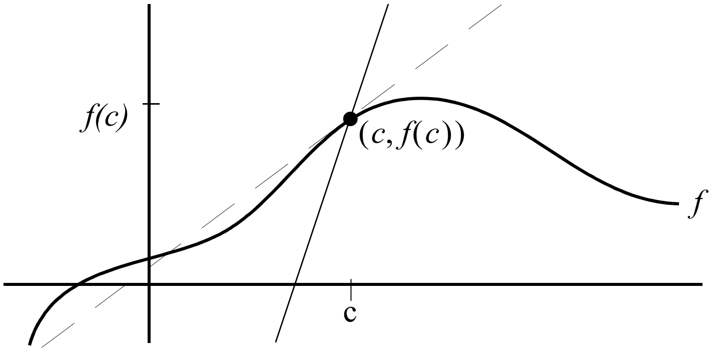


Figure 3.1

But why bother? What's so special about tangent lines? For one thing, near the point of interest a tangent line offers a nice approximation for the given function [see Figure 3.2(a)]. For another, tangent lines can be used to find where maxima and minima occur [see Figure 3.2(b)].

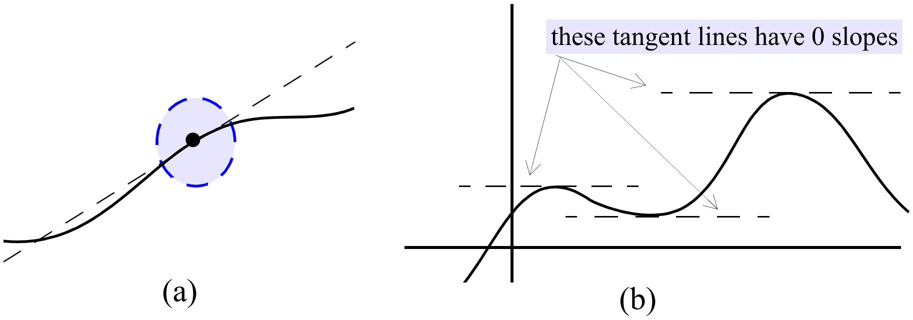


Figure 3.2

Returning to the our goal of defining the tangent line to the graph of a function f at a point $(c, f(c))$, we reasonably demand that it must contain the point $(c, f(c))$. That being the case, we can now focus our attention on "finding" the slope of the line in question.

But there is a problem. We need 2 points to find the slope, and we have but one: $(c, f(c))$. And so we turn our attention to the situation in Figure 3.3, where the would-be tangent line T is represented in dotted form (it really doesn't exist, until we define it). A solid line W_h also appears in the figure, and it is the line passing through the two points on the curve: $(c, f(c))$ and $(c+h, f(c+h))$.

We call the line W_h to remind us that we got it by moving h units from c along the x-axis. (If h were negative, then $c+h$ would lie to the left of c.

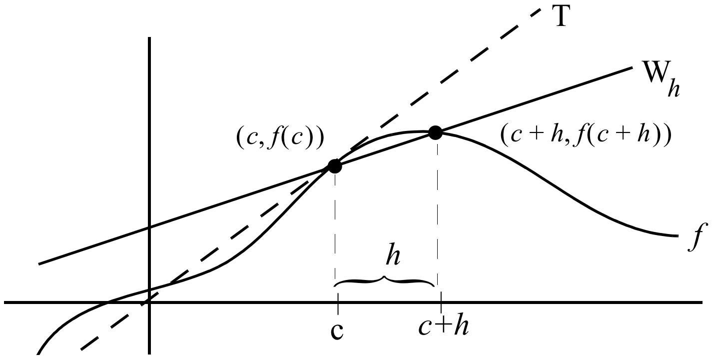


Figure 3.3

The line W_h is not the tangent line we seek. But we can do something with W_h which we were not able to do with our phantom line T; we can calculate its slope:

$$m = \frac{f(c+h)-f(c)}{(c+h)-c} = \frac{f(c+h)-f(c)}{h} \quad \left(\frac{\text{change in y}}{\text{change in x}}\right)$$

It is easy to see that the "wrong" lines W_h will pivot closer and closer to T as h gets smaller and smaller! It is therefore totally natural to define:

$$\text{slope of T} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

Yes, the above limit (when it exists) is the slope of the tangent line to the graph of the function f at the point $(c, f(c))$, but it is also called the derivative of f at c, and is denoted by $f'(c)$:

The definition of left- and right-hand limits (page 46) gives rise to that of left- and right-hand derivatives:

$$\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}$$

and

$$\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}.$$

DEFINITION 3.1
DERIVATIVE OF A FUNCTION AT A POINT

The **derivative of a function f at c** is the number $f'(c)$ given by:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

providing the limit exists. If it does, then the function is said to be **differentiable at c**.

EXAMPLE 3.1

Determine $f'(0)$ and $f'(1)$ for the function $f(x) = -3x^2+6x-1$.

SOLUTION: Turning to Definition 3.1 with $c = 0$ we have:

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \overset{c\ =\ 0}{=} \lim_{h\to 0}\frac{f(h)-f(0)}{\underset{\uparrow\ \text{undetermined}}{h}}$$

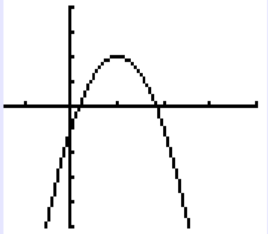
$$= \lim_{h\to 0}\frac{(-3h^2+6h-1)-(-1)}{h} = \lim_{h\to 0}\frac{-3h^2+6h}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(-3h+6)}{\cancel{h}} = \lim_{h\to 0}(-3h+6) = 6$$

The graph of:

$f(x) = -3x^2+6x-1$

appears below.



From the figure, we can anticipate that $f'(0)$ will be a positive number (tangent line climbs, and rather rapidly), and that $f'(1) = 0$.

Repeating the process with $c = 1$, we have:

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[-3(1+h)^2+6(1+h)-1]-2}{h}$$

$$= \lim_{h\to 0}\frac{[-3(1+2h+h^2)+6+6h-1]-2}{h}$$

$$= \lim_{h\to 0}\frac{-3-6h-3h^2+6+6h-1-2}{h} = \lim_{h\to 0}\frac{-3h^2}{h}$$

$$= \lim_{h\to 0}-3h = 0$$

We did some work in Example 3.1 to find $f'(1)$, and repeated the same process to find $f'(0)$. We could save some time by finding the derivative function, $f'(x)$, and then evaluating it at 0 and at 1; where:

Note that the $f'(c)$ of Definition 3.1 is a **number**: the slope of the tangent line at $(c, f(c))$. On the other hand, $f'(x)$ is a **function** whose value at x is the slope of the tangent line at the point $(x, f(x))$.

DEFINITION 3.2

DERIVATIVE FUNCTION

The **derivative of a function *f*** is the function $f'(x)$ given by:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

providing the limit exists.

EXAMPLE 3.2

Find the derivative, $f'(x)$, of the function $f(x) = \frac{x}{2x+1}$, and then use it to determine $f'(0)$, $f'(1)$, and $f'(-5)$.

SOLUTION: Turning to Definition 3.2, we have:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{\boldsymbol{h}}\\
&= \lim_{h\to 0}\frac{\dfrac{x+h}{2(x+h)+1}-\dfrac{x}{2x+1}}{\boldsymbol{h}}\\
\text{common denominator:}\quad &= \lim_{h\to 0}\frac{\dfrac{(x+h)(2x+1)-x[2(x+h)+1]}{[2(x+h)+1](2x+1)}}{\boldsymbol{h}}\\
\text{expand the numerator:}\quad &= \lim_{h\to 0}\frac{2x^2+x+2xh+h-2x^2-2xh-x}{\boldsymbol{h}[2(x+h)+1](2x+1)}\\
\text{simplify:}\quad &= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{\boldsymbol{h}}[2(x+h)+1](2x+1)}\quad \boxed{\text{Ah!}}\\
&= \lim_{h\to 0}\frac{1}{[2(x+h)+1](2x+1)}\\
\text{Take the limit:}\quad &= \frac{1}{[2(x+0)+1](2x+1)} = \frac{1}{(2x+1)^2}
\end{aligned}
$$

If the derivative exists, then the ***h*** in the denominator **has** to eventually cancel with an *h*-factor in the numerator. For this to happen, all terms in the numerator that do not contain an ***h*** must drop out.

We found the derivative of $f(x) = \dfrac{x}{2x+1}$:

$$f'(x) = \frac{1}{(2x+1)^2}$$

In particular:,

$$
\begin{aligned}
f'(0) &= \frac{1}{(2\cdot 0+1)^2} = 1\\
f'(1) &= \frac{1}{(2\cdot 1+1)^2} = \frac{1}{9}\\
f'(-5) &= \frac{1}{[2(-5)+1]^2} = \frac{1}{81}
\end{aligned}
$$

We see that the tangent line to the graph at $x = 0$, $x = 1$, $x = -5$ has a positive slope. Since the tangent line approximates the graph of the function at the indicated point, the graph must be climbing at those points. It climbs faster at $x = 0$ than at $x = 1$, and is nearly flat at $x = -5$,

EXAMPLE 3.3 Determine the equation of the tangent line to the graph of the function $f(x) = \sqrt{x+2}$, at $x = 7$.

SOLUTION: Whenever you see the word “line” you should think of:

$$y = \underset{\text{slope}\,\uparrow}{m}x + \underset{\uparrow\,\text{y-intercept}}{b}$$

The first step is to find the slope *m*, which is to say: $f'(7)$.

You can also determine $f'(x)$ and then evaluate it at 7.

$$\begin{aligned}
f'(7) &= \lim_{h\to 0}\frac{f(7+h)-f(7)}{\boldsymbol{h}}\\
&= \lim_{h\to 0}\frac{\sqrt{(7+h)+2}-3}{\boldsymbol{h}}\\
&= \lim_{h\to 0}\frac{\sqrt{9+h}-3}{\boldsymbol{h}}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\\
&= \lim_{h\to 0}\frac{(9+h)-9}{\boldsymbol{h}(\sqrt{9+h}+3)}\\
&= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{\boldsymbol{h}}(\sqrt{9+h}+3)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3} \underset{\substack{\uparrow\\ \text{take the limit}}}{=} \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}
\end{aligned}$$

At this point we know that the tangent line is of the form:

$$y = \frac{1}{6}x + b \qquad (*)$$

To determine b we use the fact that the tangent line must pass through the point **on the curve** whose x-coordinate is 7, namely, the point:

$$(\mathbf{7}, f(7)) = (\mathbf{7}, \mathbf{3})$$

$$f(7) = \sqrt{7+2} = \mathbf{3}$$

Substituting **7** for x and **3** for y in (*) we solve for b:

$$\mathbf{3} = \frac{1}{6}\cdot\mathbf{7} + b$$

$$b = 3 - \frac{7}{6} = \frac{11}{6}$$

Tangent line: $y = \dfrac{x}{6} + \dfrac{11}{6}$

CHECK YOUR UNDERSTANDING 3.1

Find the tangent line to the graph of the function $f(x) = \dfrac{x}{2x+1}$ at $x = -1$. (See Example 3.2.)

Answer: $y = x + 2$

ALTERNATE FORM FOR THE DERIVATIVE

The Greek letter "Δ" (called "delta") is often used to denote a "change in." Replacing h with the symbol Δx (for change in x) in the expression $\lim\limits_{h \to 0} \frac{f(c+h)-f(c)}{h}$ we have:

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} \text{ or } \lim_{\Delta x \to 0} \frac{\Delta y \leftarrow \text{change in } y}{\Delta x \leftarrow \text{change in } x}$$

The "double-*d*" notation for the derivative, $\frac{dy}{dx}$, is attributed to Gottfried Leibnitz (1646-1716)

When using the above form, one typically replaces the derivative symbol $f'(x)$ with the symbol $\frac{dy}{dx}$ or $\frac{d}{dx}f(x)$, and the symbol $f'(c)$ with $\left.\frac{dy}{dx}\right|_{x=c}$.

TO ILLUSTRATE:

For $f(x) = \frac{x}{2x+1}$:	For $y = f(x) = \frac{x}{2x+1}$:
Example 3.2: $f'(x) = \frac{1}{(2x+1)^2}$	$\frac{dy}{dx} = \frac{1}{(2x+1)^2}$
Also: $\left(\frac{x}{2x+1}\right)' = \frac{1}{(2x+1)^2}$	Also: $\frac{d}{dx}\left(\frac{x}{2x+1}\right) = \frac{1}{(2x+1)^2}$
In particular: $f'(2) = \frac{1}{25}$	In particular: $\left.\frac{dy}{dx}\right\|_{x=2} = \frac{1}{25}$

It is important to note that we are simply acknowledging different notations for one and the same thing.

The ratio $\frac{\Delta y}{\Delta x}$ in the expression $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ denotes the **average rate of change** of y with respect to x over the interval Δx, and one calls the derivative $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ the **instantaneous rate of change** of y with respect to x, or simply the **rate of change** of y with respect to x.

In particular, if the volume V of a balloon varies with respect to the temperature t, then the derivative $V'(t)$ or $\frac{dV}{dt}$ denotes the rate of change of volume with respect to temperature.

As you know, the rate of change of position with respect to time is called velocity, and the rate of change of velocity with respect to time also has a special name: acceleration. We will have occasions to focus on these two important rate of change functions (derivatives) later in the text.

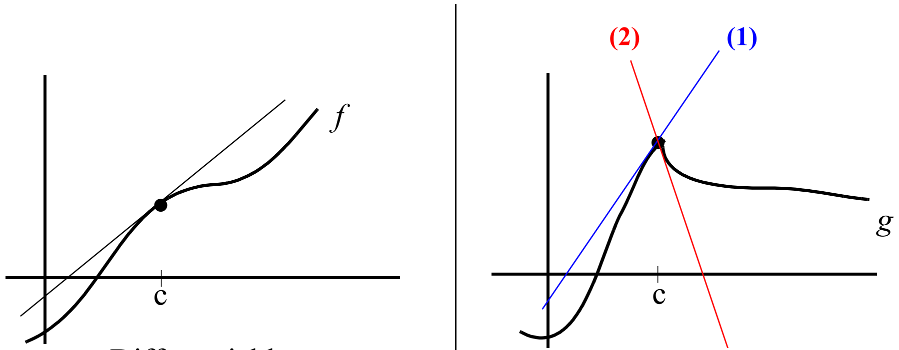
CHECK YOUR UNDERSTANDING 3.2

Determine $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{x=2}$ for the function $y = f(x) = 3x^2 - x + 1$.

Answer: $\frac{dy}{dx} = 6x - 1$, $\left.\frac{dy}{dx}\right|_{x=2} = 11$

GEOMETRICAL INSIGHTS INTO THE DERIVATIVE

Consider the two graphs of Figure 3.4. The function f in (a) has a (positive) derivative at $x = c$ (tangent line exists and has positive slope). The function g in (b) is not differentiable at $x = c$. Why not?



Differentiable at c
(a)

Not differentiable at c
(b)

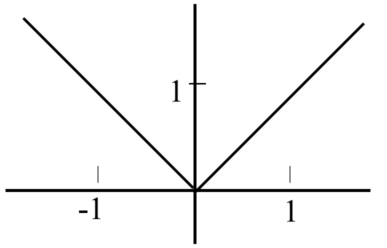
Figure 3.4

BECAUSE: Lines are notoriously straight, and the graph of g has a sharp bend at c [the "would-be tangent line **(1)**" is of positive slope, while the "would-be tangent line **(2)**" is of negative slope]. Since no line can approximate the graph of g at c, there cannot be a tangent line at c [in other words: $f'(c)$ does not exist]. To be more specific, we call your attention to the graph of the absolute value function:

The function g in Figure 3.4(b) does appear to have a left- and right-hand derivative at c; the left-hand derivative being positive and the right-hand derivative negative.

Geometrically speaking, if a function is differentiable at c, then the graph has to change direction gradually at that point [as in Figure 3.4(a)]. If the graph abruptly changes direction at that point [as in Figure 3.4(b)], then the function is not differentiable at c.

$$\text{Abs}(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



From our previous discussion, we can anticipate that the absolute value function is not differentiable at $x = 0$; a fact which we now verify:

EXAMPLE 3.4 Show that the absolute value function $f(x) = |x|$ is not differentiable at 0.

SOLUTION: For $f(x) = |x|$, and $c = 0$ the derivative formula:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

takes the form:

$$\lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Letting x approach 0 from the left, we have:.

$$\lim_{h \to 0^-} \frac{|h|}{h} = \overset{\text{since } h \text{ is negative}}{\frac{-h}{h}} = -1$$

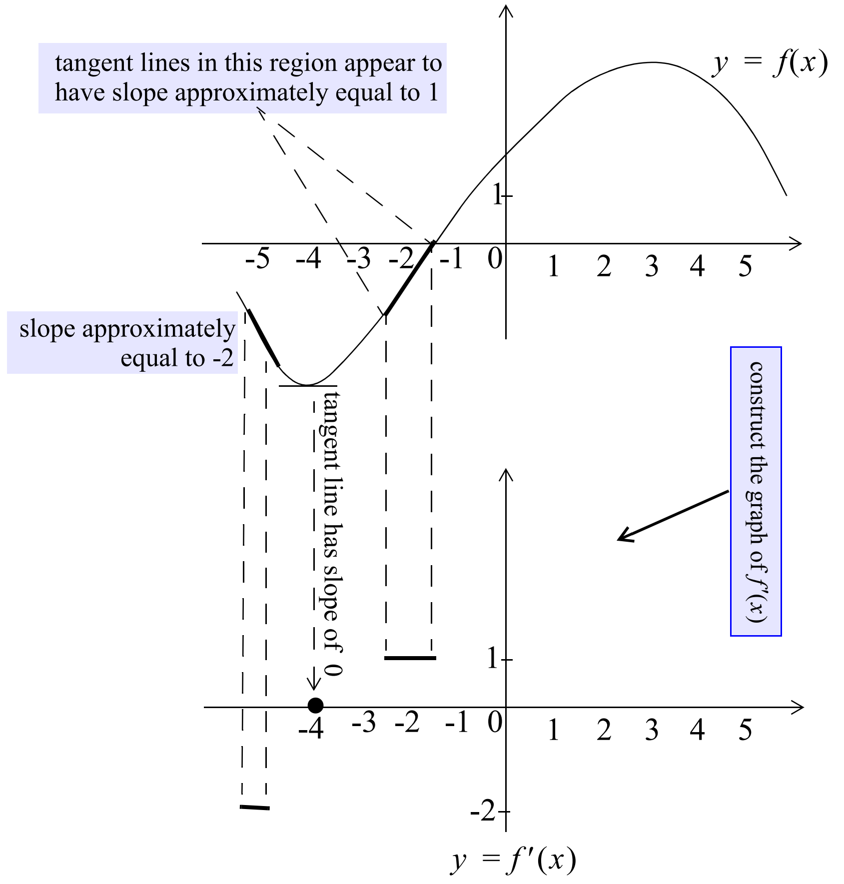
From the right

$$\lim_{h \to 0^+} \frac{|h|}{h} = \overset{\text{since } h \text{ is positive}}{\frac{h}{h}} = 1$$

Since the left-hand limit is different than the right-hand limit, the limit (derivative) does not exist. It follows that the absolute value function is not differentiable at 0.

CHECK YOUR UNDERSTANDING 3.3

Complete the construction of the graph of $f'(x)$ from the given graph of $y = f(x)$ at the top of the figure.



Answer: See page A-11

CONTINUITY AND THE DERIVATIVE

The following result asserts that differentiability implies continuity:

THEOREM 3.1 If a function f is differentiable at c, then f is continuous at c.

It follows that if a function is not continuous at c, then it is not differentiable at c.

PROOF: Let f be differentiable at c. We are to show that:

$$\lim_{x \to c} f(x) = f(c)$$

or, equivalently (see margin) that:

$$\lim_{h \to 0} f(c+h) = f(c)$$

$$\lim_{h \to 0} [f(c+h) - f(c)] = 0$$

In the expression:

$$\lim_{x \to c} f(x) = f(c)$$

make the substitution:

$$x = c + h$$

to arrive at:

$$\lim_{h \to 0} f(c+h) = f(c)$$

$x = c + h \Rightarrow h = x - c$
and to say that $x \to c$
is to say that $h \to 0$

Let's do it:

$$\lim_{h \to 0} [f(c+h) - f(c)] = \lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} \cdot h \right]$$

Theorem 2.3(c), page 55:

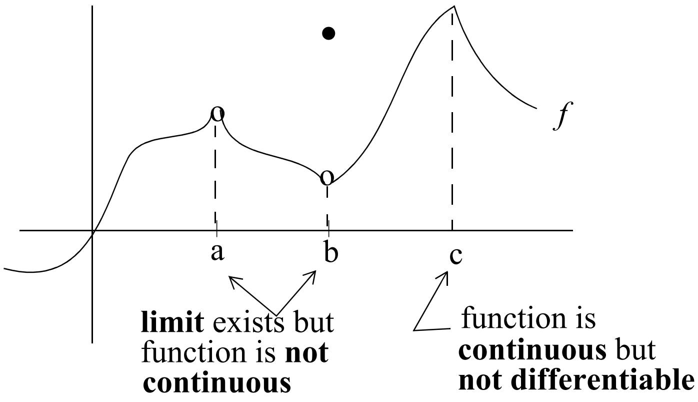
$$= \lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} \right] \lim_{h \to 0} h$$

$$= f'(c) \cdot 0 = 0$$

We've established the following pecking order:

Differentiability $\Rightarrow$ Continuity $\Rightarrow$ Limit Exists

Figure 3.5 illustrates that neither of the above two implications is reversible.



limit exists but function is **not continuous**

function is **continuous** but **not differentiable**

Figure 3.5

CHECK YOUR UNDERSTANDING 3.4

Sketch the graph of a function f satisfying the following three conditions:

(i) f is differentiable at $x = 1$.

(ii) f is continuous but not differentiable at $x = 2$.

(iii) f has a limit but is not continuous at $x = 3$.

Answer: See page A-11

EXERCISES

Exercises 1-9. (Derivative at a Point) Determine $f'(2)$ for the given function.

1. $f(x) = 4x^2$
2. $f(x) = 3x^2 + x$
3. $f(x) = -x^2 + 3x - 1$
4. $f(x) = x^3$
5. $f(x) = 55$
6. $f(x) = \frac{x}{x+1}$
7. $f(x) = \frac{x^2 + x}{x}$
8. $f(x) = x(x-3)$
9. $f(x) = \sqrt{3x+1}$

Exercises 10-12. (Derivative at a Point) Determine $\left.\frac{dy}{dx}\right|_{x=2}$ for the given function.

10. $y = 2x^2 - x + 1$
11. $y = \frac{3x}{x+1}$
12. $y = \sqrt{x+2}$

Exercises 13-24. (Derivative Function) Determine $f'(x)$ for the given function.

13. $f(x) = x$
14. $f(x) = 5$
15. $f(x) = 3x^2$
16. $f(x) = 3x^2 + 3$
17. $f(x) = -2x^2 + x - 2$
18. $f(x) = \frac{x-5}{x}$
19. $f(x) = \frac{2x+3}{x+1}$
20. $f(x) = -\frac{2}{2x+1}$
21. $f(x) = \sqrt{x+3}$
22. $f(x) = \frac{1}{\sqrt{x+3}}$
23. $f(x) = \frac{x}{x^2+1}$
24. $f(x) = -\frac{2x}{\sqrt{x+1}}$

Exercises 25-27. (Derivative Function) Determine $\frac{dy}{dx}$ for the given function.

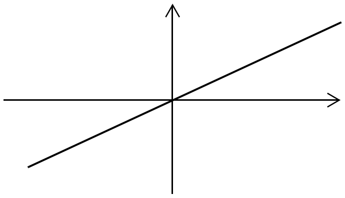
25. $y = -x^2 - x$
26. $y = \frac{x+1}{2x}$
27. $y = \frac{1}{\sqrt{2x+3}}$

Exercises 28-29. (Tangent Line) Find the equation of the tangent line to the graph of the given function at the indicated point.

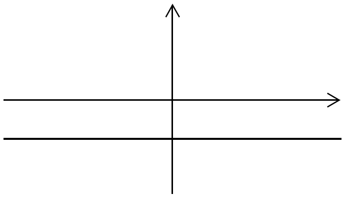
28. $f(x) = x^2 + 2x$ at $x = 0$
29. $f(x) = x^2 + 2x$ at $x = 1$

30. **(Graphs of Functions and their Derivatives)** Pair off each function [A] through [F] with its corresponding derivative function [1] through [6].

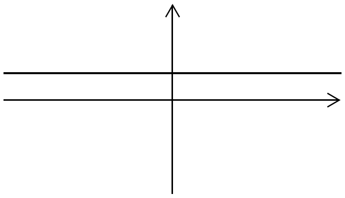
[A]



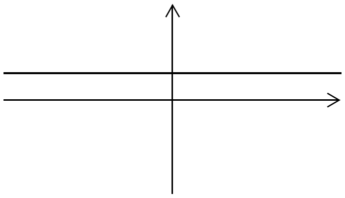
[1]



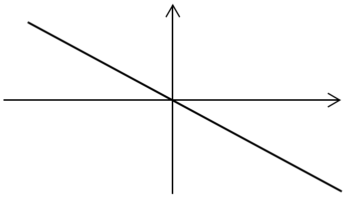
[B]



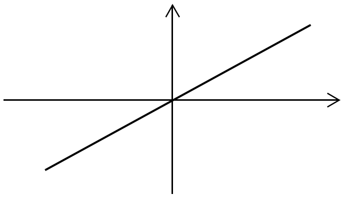
[2]



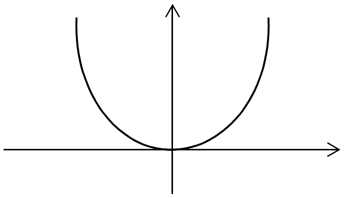
[C]



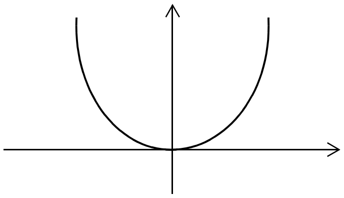
[3]



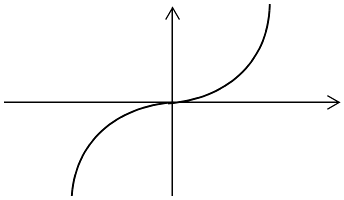
[D]



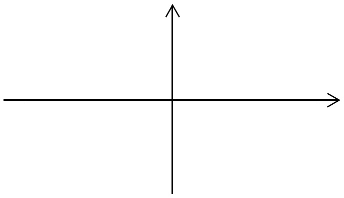
[4]



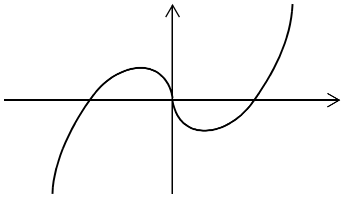
[E]



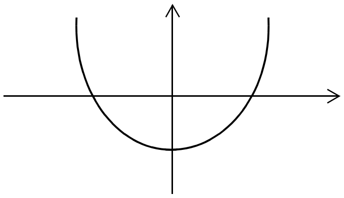
[5]



[F]

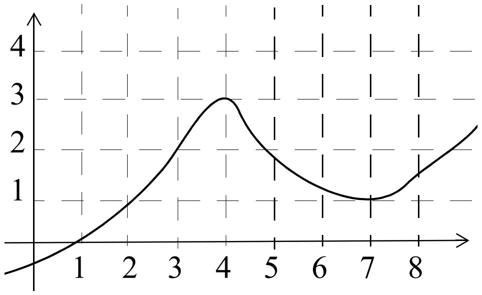


[6]

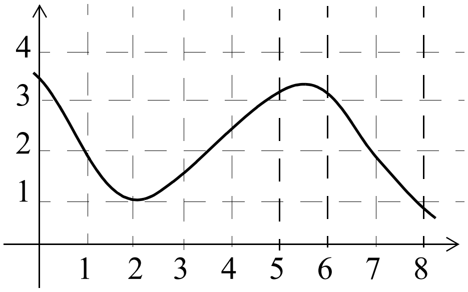


Exercises 31-32. **(Geometrical Interpretation)** By positioning a tangent line to the graph of the function f, at the indicated point, estimate the value of $f'(2)$, $f'(4)$, and $f'(7)$.

31.

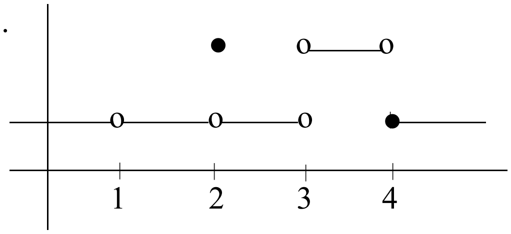


32.

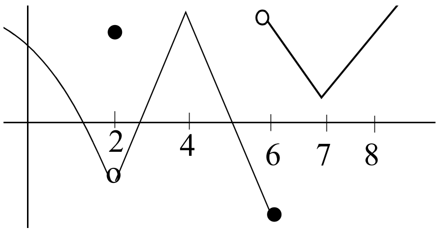


Exercises 33-34. (Geometrical Interpretation) Consider the given graph of the function *f*. Where does *f* fail: (a) to have a limit? (b) to be continuous? (c) to be differentiable?

33.

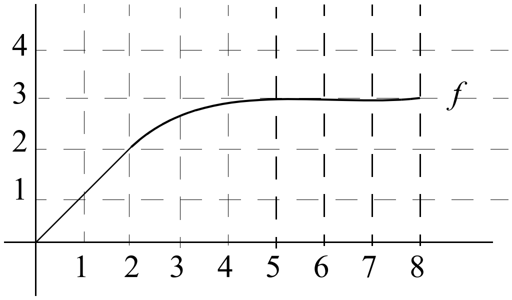


34.

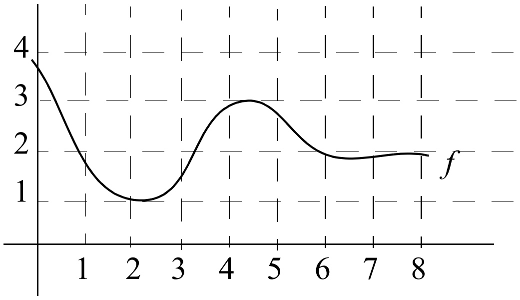


Exercises 35-36. (Geometrical Insight) Sketch the graph of $y = f'(x)$ from the given graph of the function $y = f(x)$.

35.



36.



Exercises 37-39. (Geometrical Insight) Sketch the graph of a function *f* satisfying the following conditions:

37. *f* does not have a limit at 0; it has a limit at 1 but is not continuous at 1; it is continuous at 2 but not differentiable at 2.

38. *f* is not defined at 0 but has a limit at 0; it is defined at 1 but does not have a limit at 1; it has a limit of 5 at 2, but is not continuous at 2; it is continuous at 3 with function value 6, but is not differentiable at 3.

39. Where *f* is differentiable in $(0, 2)$, *f* has a positive derivative. *f* has a negative derivative between 2 and 4. *f* is not continuous at 1. *f* is continuous at 2 but not differentiable at that point.

Exercises 40-41. (Theory) Sketch the graph of the given function. Verify that the function is not differentiable at $x = 2$.

40. $f(x) = \begin{cases} 2x+2 & \text{if } x \le 2 \\ 3x & \text{if } x > 2 \end{cases}$

41. $f(x) = \begin{cases} x & \text{if } x < 2 \\ x^2-2 & \text{if } x \ge 2 \end{cases}$

Exercises 42-43. (Theory) Sketch the graph of the given function. Verify that the function is differentiable at $x = 1$.

42. $f(x) = \begin{cases} x & \text{if } x \le 1 \\ \dfrac{x^2}{2} & \text{if } x > 1 \end{cases}$

43. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x-1 & \text{if } x \ge 1 \end{cases}$

§2. DIFFERENTIATION FORMULAS

We begin by listing some derivative formulas which, with a bit of practice, will enable you to quickly and easily determine the derivative of numerous functions.

THEOREM 3.2

(a) The derivative of any constant function is **0**.

For example: $(17)' = 0$ and $(-375\pi)' = 0$

(b) For any real number r:

$$(x^r)' = rx^{r-1}$$

For example: $(x^5)' = 5x^4$ and $(x^{-2})' = -2x^{-3}$

(c) For any real number r and any differentiable function f:

$$[rf(x)]' = rf'(x)$$

For example: $(7x^5)' = 7(x^5)' = 7(5x^4) = 35x^4$ and $(4x^{-2})' = -8x^{-3}$

(d) If f and g are differentiable, then so are $f+g$ and $f-g$; and:

$$[f(x)+g(x)]' = f'(x)+g'(x) \text{ and } [f(x)-g(x)]' = f'(x)-g'(x).$$

For example: $(7x^5+x^3)' = 35x^4+3x^2$ and $(2x+3-2x^{-4})' = 2+8x^{-5}$

(e) If f and g are differentiable, then so is fg, and:

$$[f(x)g(x)]' = f(x)g'(x)+g(x)f'(x).$$

For example: $[(5x^3-x)(x^7)]' = (5x^3-x)(x^7)' + x^7(5x^3-x)'$

$= (5x^3-x)(7x^6)+x^7(15x^2-1) = 50x^9-8x^7$

(f) If f and g are differentiable, then so is $\dfrac{f}{g}$, and:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2} \text{ [for } g(x) \neq 0\text{]}.$$

For example: $\left(\dfrac{5x-4}{3x+2}\right)' = \dfrac{(3x+2)(5x-4)'-(5x-4)(3x+2)'}{(3x+2)^2}$

$= \dfrac{(3x+2)(5)-(5x-4)(3)}{(3x+2)^2} = \dfrac{15x+10-15x+12}{(3x+2)^2} = \dfrac{22}{(3x+2)^2}$

We are not currently in a position to establish (b) in all of its splendor. A proof that $(x^n)' = nx^{n-1}$ holds for any positive integer n is offered at the end of the section. (A general proof appears in Section 6.3)

PROOF: We offer a proof of (a), the sum part of (d), and (e). You are invited to establish (c) and (f) in the exercises.

(a) Let $f(x) = c$ (the function that assigns the number c to every x). Then:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

A geometrical argument: The graph of the function $f(x) = c$ is a horizontal line. At each point on that line, the tangent line is the horizontal line itself, which is of slope 0.

(d) (Sum part):

$$[\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})]' = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

regroup: $$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]$$

Theorem 2.3(a), page 55: $$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \boldsymbol{f}'(\boldsymbol{x})+\boldsymbol{g}'(\boldsymbol{x})$$

(e) $$[\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})]' = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-\boldsymbol{f(x+h)g(x)}+\boldsymbol{f(x+h)g(x)}}^{\text{a clever zero}}-f(x)g(x)}{h}$$

regroup: $$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+\frac{[f(x+h)-f(x)]}{h}g(x)\right]$$

Theorem 2.3, page 55 $$= \lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}\lim_{h\to 0}g(x)$$

$$= \boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x})+\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})$$

Differentiability implies continuity (Theorem 3.1, page 73).
Consequently: $\lim_{h\to 0} f(x+h) = f(x)$

CHECK YOUR UNDERSTANDING 3.5

(a) Appeal to a geometrical argument, similar to that offered in the proof of Theorem 3.2(a), to show that $x' = 1$.

(b) Use Definition 3.2, page 67 to prove that $x' = 1$.

Answers: See page A-11.

EXAMPLE 3.5

(a) Determine $\left(\dfrac{3x^2+2x-4}{2x+1}\right)'$

(b) For $y = \dfrac{3x^3-2x^2+1}{x^2}$, determine $\dfrac{dy}{dx}$.

(c) For $Z(y) = 3y^3+2y-4$, find the rate of change of Z with respect to y.

SOLUTION: (a)

$$\left(\frac{3x^2+2x-4}{2x+1}\right)' = \frac{(2x+1)(\mathbf{3x^2+2x-4})'-(3x^2+2x-4)(\mathbf{2x+1})'}{(2x+1)^2}$$

$$= \frac{(2x+1)(\mathbf{6x+2})-(3x^2+2x-4)(\mathbf{2})}{(2x+1)^2} = \frac{6x^2+6x+10}{(2x+1)^2}$$

You could use the quotient rule, but going with powers of x is the better choice [a choice that was not available in (a)].

(b) $\frac{d}{dx}\left(\frac{3x^3-2x^2+1}{x^2}\right) = \frac{d}{dx}(3x-2+x^{-2}) = 3-2x^{-3} = 3-\frac{2}{x^3}$

(c) $\frac{dZ}{dy} = \frac{d}{dy}(3y^3+2y-4) = 9y^2+2$

CHECK YOUR UNDERSTANDING 3.6

Differentiate the given function.

(a) $f(x) = 3x^4-2x+5-5x^{-4}$ (b) $y = \frac{4x^2-525}{x+3}$

Answers:

(a) $12x^3-2+\frac{20}{x^5}$

(b) $\frac{4x^2+24x+525}{(x+3)^2}$

EXAMPLE 3.6

Find the tangent line to the graph of the function $f(x) = \frac{x^2-x+1}{3x^3+2}$ at $x = 1$

SOLUTION: We first set our sites on determining the slope of the tangent line [namely $f'(1)$]:

$$f'(x) = \left(\frac{x^2-x+1}{3x^3+2}\right)'$$

Theorem 3.2(f) $$= \frac{(3x^3+2)(x^2-x+1)'-(x^2-x+1)(3x^3+2)'}{(3x^3+2)^2}$$

Theorem 3.2(a)-(e): $$= \frac{(3x^3+2)(2x-1)-(x^2-x+1)(9x^2)}{(3x^3+2)^2}$$

In particular:

$$f'(\mathbf{1}) = \frac{(3\cdot\mathbf{1}^3+2)(2\cdot\mathbf{1}-1)-(\mathbf{1}^2-\mathbf{1}+1)(9\cdot\mathbf{1}^2)}{(3\cdot\mathbf{1}^3+2)^2} = -\frac{\mathbf{4}}{\mathbf{25}}$$

At this point, we know that our tangent line is of the form $y = -\frac{\mathbf{4}}{\mathbf{25}}x+\boldsymbol{b}$. Knowing that the point $(1,\boldsymbol{f}(\mathbf{1})) = \left(1,\frac{\mathbf{1}}{\mathbf{5}}\right)$ lies on the line enables us to determine b:

$$\frac{1}{5} = -\frac{4}{25}\cdot 1+b$$

$$\boldsymbol{b} = \frac{1}{5}+\frac{4}{25} = \frac{\mathbf{9}}{\mathbf{25}}$$

Tangent line: $y = -\frac{\mathbf{4}}{\mathbf{25}}x+\frac{\mathbf{9}}{\mathbf{25}}$.

EXAMPLE 3.7 Determine the points on the graph of the function $f(x) = x^3 - x - 1$ at which the tangent line is parallel to the tangent line to the graph of the function $g(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$ at $x = 2$.

SOLUTION: Reading the problem carefully we see that we need to solve the equation $f'(x) = g'(2)$ to find the x-coordinates of the points in question. Lets do it:

$$f(x) = x^3 - x - 1 \qquad\qquad g(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$$

$$f'(x) = 3x^2 - 1 \qquad\qquad g'(x) = x^3 - 3x \Rightarrow g'(2) = 2$$

$$\begin{aligned} f'(x) &= g'(2) \\ 3x^2 - 1 &= 2 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Evaluating the function $f(x) = x^3 - x - 1$ at $x = \pm 1$ will yield the corresponding y-coordinates of the two points:

$$f(1) = 1^3 - 1 - 1 = -1 \text{ and } f(-1) = (-1)^3 - (-1) - 1 = -1$$

CONCLUSION: At $(1, -1)$ and $(-1, -1)$, the graph of $f(x) = x^3 - x - 1$ has tangent lines parallel to that of $g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$ at $x = 2$.

CHECK YOUR UNDERSTANDING 3.7

Determine the points on the graph of the function $f(x) = 2x^4 - 4x^2 + 1$ where the tangent line is horizontal.

Answer:
$(0, 1)$, $(1, -1)$, $(-1, -1)$

APPROXIMATING FUNCTION VALUES

Consider the function *f* in Figure 3.6 along with the tangent line at the point $(c, f(c))$. As is depicted in the figure:

$$\Delta y = f(c+\Delta x) - f(c) \text{, or: } f(c+\Delta x) = f(c) + \Delta y .$$

We are trying our best to say that, which Figure 3.6 so aptly displays.

Moreover, since the tangent line *T* has slope $f'(\mathrm{c})$, and since it hovers close to the graph of the function near *c*:

$$\boldsymbol{\Delta y \approx f'(c)\Delta x}\text{; so that: } \boldsymbol{f(c+\Delta x) = f(c) + \Delta y \approx f(c) + f'(c)\Delta x} \quad (*)$$

The tangent line *T* at the point $(c, f(c))$ is sometimes said to be the **linearization of *f* at *c***; the symbol Δx is at times replaced by the symbol dx (called the **differential of *x***); and yet another symbol, the symbol dy (called the **differential of *y***), is used to represent the expression $f'(c)dx$; leading one to the so called **differential** form:

$$dy = f'(x)dx$$

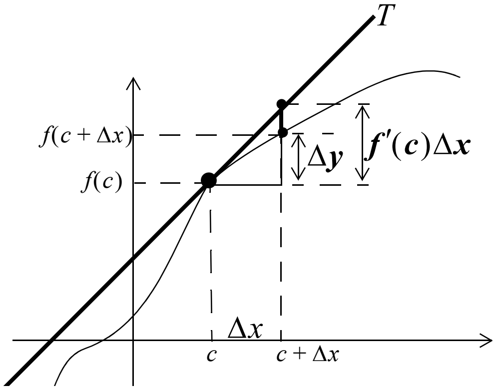


Figure 3.6

EXAMPLE 3.8 Approximate the value of $\sqrt{25.3}$.

SOLUTION: For $f(x) = \sqrt{x}$: $f'(x) = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Turning to (*), with $c = 25$, $\Delta x = 0.3$ we have:

$$\sqrt{25.3} = f(25+0.3) \approx f(25) + f'(25)(0.3) = \sqrt{25} + \frac{0.3}{2\sqrt{25}} = 5.03$$

CHECK YOUR UNDERSTANDING 3.8

Proceed as in Example 3.8 to approximate the value of $\sqrt[3]{8.1}$.

Answer:
$2 + \frac{0.1}{12} \approx 2.008$

EXAMPLE 3.9 The edge of a cube is measured as 10 inches with a possible error in measurement of at most 0.05 inches. Estimate the corresponding largest possible error in calculating the volume of the cube. Estimate the relative volume error (error divided by volume) stemming from the calculation.

SOLUTION: We find an approximation for the volume error ΔV resulting from a change of an edge measurement from 10 to $10 + \Delta x$ inches, where $\Delta x = 0.05$ inches:

For $V(x) = x^3$: $V'(x) = 3x^2$. Thus:

$$\Delta V \approx V'(10)\Delta x = 3(10)^2(0.05) = 15\text{in}^3$$

Conclusion: Maximum Possible Volume Error: $\Delta V \approx 15\text{in}^3$.

Relative Volume Error: $\frac{\Delta V}{V} \approx \frac{15}{10^3} = 0.015$.

CHECK YOUR UNDERSTANDING 3.9

The radius of a circle is measured to be 50 cm with a possible error in measurement of 0.5 cm. Estimate the maximum possible error in using that measurement to calculate the area of the circle. Estimate the relative error in the area calculation.

Answer:
$50\pi\ \text{cm}^2$, $\frac{1}{50} = 0.02$

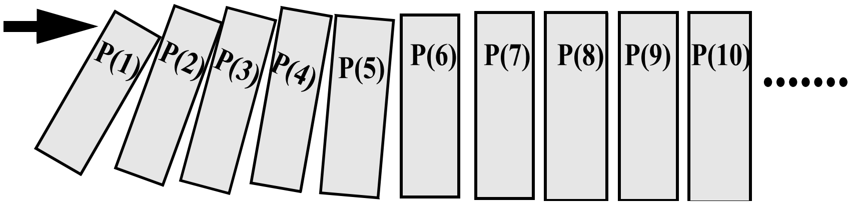
MATHEMATICAL INDUCTION

Roughly speaking, axioms are "dictated truths" upon which, with the cement of logic, mathematical theories are constructed.

The following axiom, called the Principle of Mathematical Induction, will be used to show that $(\boldsymbol{x^n})' = \boldsymbol{nx^{n-1}}$ for any positive integer n. Here is how that all-important principle works:

<table>
<tr><td colspan="2">Let $P(n)$ denote a proposition that is either true or false, depending on the value of the integer n.</td></tr>
<tr><td>If:</td><td>I. $P(1)$ is True.</td></tr>
<tr><td>And if, from the assumption that:</td><td>II. $P(k)$ is True</td></tr>
<tr><td>one can show that:</td><td>III. $P(k+1)$ is also True</td></tr>
<tr><td colspan="2">then the proposition $P(n)$ is valid for all integers $n \geq 1$.</td></tr>
</table>

Step II of the induction procedure may strike you as being a bit strange. After all, if one can assume that the proposition is valid at $n = k$, why not just assume that it is valid at $n = k + 1$ and be done with it? Well, you can assume whatever you want in Step II, but if the proposition is not valid for all n you simply are not going to be able to demonstrate, in Step III, that the proposition holds at the next value of n. It's sort of like the domino theory. Just imagine that the propositions $P(1), P(2), P(3), \ldots, P(k), P(k+1), \ldots$ are lined up, as if they were an infinite set of dominoes:



If you knock over the first domino (Step I), and if when a domino falls (Step II) it knocks down the next one (Step III), then all of the dominoes will surely fall. But if the falling k^{th} domino fails to knock over the next one, then all the dominoes will not fall.

To illustrate how the process works, we ask you to consider the sum of the first n odd integers, for $n = 1$ through $n = 5$:

The Principle of Mathematical Induction might have been better named the Principle of Mathematical Deduction, for inductive reasoning is used to formulate a conjecture, while deductive reasoning is used to rigorously establish whether or not the conjecture is valid.

n	Sum of the first n odd integers	Sum
1	1	1
2	1 + 3	4
3	1 + 3 + 5	9
4	1 + 3 + 5 + 7	16
5	1 + 3 + 5 + 7 + 9	25

n	Sum
1	1
2	4
3	9
4	16
5	25
6	?

Figure 3.7

Looking at the pattern of the table on the right in Figure 3.7, you can probably anticipate that the sum of the first 6 odd integers will turn out to be $6^2 = 36$, which is indeed the case. In general, the pattern certainly suggests that the sum of the first n odd integers is n^2; a fact that we now establish using the Principle of Mathematical Induction:

Let $P(n)$ be the proposition that the sum of the first n odd integers equals n^2.

I. Since the sum of the first 1 odd integers is 1^2, $P(1)$ is true.

II. **Assume** $P(k)$ is true; that is: $\mathbf{1+3+5+\cdots+(2k-1) = k^2}$

see margin

III. We show that $P(k+1)$ is true, thereby completing the proof:

$$\overbrace{\underbrace{[\mathbf{1+3+5+\cdots+(2k-1)}]}_{\text{induction hypothesis: Step II}}+(\mathbf{2k+1})}^{\text{the sum of the first } k+1 \text{ odd integers}} = \mathbf{k^2}+(\mathbf{2k+1}) = (k+1)^2$$

The last integer in:
The sum of the first **3** odd integers is:

$1+3+5 \longleftarrow \boxed{2\cdot\mathbf{3}-1}$

The sum of the first **4** odd integers is:

$1+3+5+7 \longleftarrow \boxed{2\cdot\mathbf{4}-1}$

Suggesting that the last integer in the sum of the first $\boldsymbol{k}$ odd integers is:

$1+3+\ldots+\boxed{(2k-1)}$

EXAMPLE 3.10 Use the Principle of Mathematical Induction to verify that $(\boldsymbol{x}^n)' = \boldsymbol{n}\boldsymbol{x}^{n-1}$ for any positive integer n.

SOLUTION:

I. Since $x' = 1$ (CYU 3.5), and since $1x^{1-1} = x^0 = 1$, the claim holds at $n = 1$.

II. Assume the claim holds at $n = k$: $(x^k)' = kx^{k-1}$.

III. We show the claim holds at $n = k+1$; which is to say, that $(x^{k+1})' = (k+1)x^{(k+1)-1} = (k+1)x^k$:

$$\begin{aligned} (x^{k+1})' &= (x \cdot x^k)' \\ \text{Theorem 3.2(e):} \quad &= x\boldsymbol{(x^k)'} + x^k \cdot x' \\ \text{II:} \quad &= x\boldsymbol{kx^{k-1}} + x^k \\ &= kx^k + x^k = (k+1)x^k \end{aligned}$$

CHECK YOUR UNDERSTANDING 3.10

Using Theorem 3.2(f), extend the result of the previous example to show that: $\boldsymbol{(x^n)' = nx^{n-1}}$ holds for all integers n. (For $n \le 0$, assume that $x \ne 0$).

Answer: See page A-12.

HIGHER ORDER DERIVATIVES

The second derivative of a function *f*, denoted by $f''(x)$, is simply the derivative of $f'(x)$ (if it exists) — the third derivative, $f'''(x)$, is the derivative of the second derivative. We note that the symbol $f^{(n)}(x)$ can also be used to denote the n^{th} derivative of the function *f*. For example $f^{(3)}(x) = f'''(x)$

In the Leibniz notation, the second derivative of $y = f(x)$ is denoted $\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}f(x)$ — the third derivative $\frac{d^3y}{dx^3}$, and so on.

For example: $(5x^3 + 3x^2 - x + 1)'' = (15x^2 + 6x - 1)' = 30x + 6$

$$\text{and: } \frac{d^2}{dx^2}(-x^6 + 2x) = \frac{d}{dx}(-6x^5 + 2) = -30x^4$$

CHECK YOUR UNDERSTANDING 3.11

(a) Find the third derivative of the function $f(x) = x^5$.

(b) Find an expression for the n^{th} derivative of the function $f(x) = x^{-1}$.

(c) Establish the validity of your claim in (b) using the Principle of Mathematical Induction.

Answer: (a) $60x^2$
(b) $(-1)^n n! x^{-(n+1)}$
(c) See page A-12.

EXERCISES

Exercises 1-18. Find the derivative of the given function.

1. $f(x) = 3x^5 + 4x^3 - 7$
2. $f(x) = 4x^4 + 7x^3 - 3x - 2$
3. $g(x) = 7x^3 + 5x^2 - 4x + x^{-4} + 1$
4. $f(x) = \frac{1}{3}x^3 + \frac{1}{5}x^2 - x - 1$
5. $g(x) = -x^7 + 2x^2 - x^{-1} - x^{-2} + 101$
6. $h(x) = \frac{4x^4 + x^3 - 2x^2}{x^2}$
7. $h(x) = \frac{x^5 + 3x^4 - 5x^2}{x^2}$
8. $f(x) = x^2 + x^{\frac{1}{2}} - 5x^{\frac{2}{5}}$
9. $f(x) = \sqrt{x} + 2$
10. $g(x) = 3x^{\frac{1}{3}} + x^{-\frac{1}{2}} + 2x + 1$
11. $F(x) = \frac{3x^2 + 2x - 5}{x + 4}$
12. $F(x) = \frac{-x^5 + 3x - 4}{x^2 + 2x}$
13. $f(x) = \frac{5}{3x^2 + 1}$
14. $g(x) = \frac{1}{(x - 2)^2}$
15. $K(x) = (4x^4 + 2x^3 + x^2)(x^3 + x + 1)$
16. $K(x) = \left(x^4 + \frac{1}{x} - 2x^{-3}\right)\left(x^2 + x + \frac{1}{x^2}\right)$
17. $h(x) = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}$
18. $f(x) = \frac{5\sqrt{x}}{3x^2 + 1}$

Exercises 19-20. (Second Derivative) Determine $f''(2)$ for the given function.

19. $f(x) = x^5 - 3x^2 - x + 1$
20. $f(x) = \frac{x}{x + 1}$

Exercise 21-22. (Second Derivative) Determine $\left.\frac{d^2y}{dx^2}\right|_{x = 2}$ for the given function.

21. $y = 2x^4 - x - 1$
22. $y = \frac{x - 1}{2x}$

Exercises 23-34. (Derivative Rules) Evaluate the given expression at the indicated point, if:

$$f(0) = 1, f(1) = 3, f(2) = 6, f'(0) = 2, f'(1) = 6, f'(2) = 0$$
$$g(0) = 3, g(1) = 2, g(2) = 5, g'(0) = 1, g'(1) = 2, g'(2) = 2$$
$$h(0) = 0, h(1) = 6, h(2) = 2, h'(0) = 3, h'(1) = 1, h'(2) = 1$$

23. $[f(x) + g(x)]'$ at $x = 1$
24. $[f(x) \cdot g(x)]'$ at $x = 1$
25. $\left[\frac{f(x)}{g(x)}\right]'$ at $x = 1$
26. $\left[\frac{g(x)}{f(x)}\right]'$ at $x = 1$
27. $[f(x) + g(x) + h(x)]'$ at $x = 2$
28. $[f(x) + g(x) \cdot h(x)]'$ at $x = 2$
29. $[f(x) \cdot g(x) + h(x)]'$ at $x = 2$
30. $\left[\frac{f(x) + g(x)}{h(x)}\right]'$ at $x = 1$

31. $\left[\frac{f(x)\cdot g(x)}{h(x)}\right]'$ at $x = 1$

32. $\left[\frac{f(t)-g(t)}{h(t)+1}\right]'$ at $t = 0$

33. $\left(\frac{g(t)}{h(t)}+g(t)\right)'$ at $t = g(1)$

34. $\frac{f(s)}{g'(s)}+g(2)\cdot h'(s)$ at $s = g'(1)$

Exercises 35-38. (Tangent Line) Determine the tangent line to the graph of the given function at the indicated point.

35. $f(x) = 3x^2-x-1$ at $x = 1$

36. $f(x) = -x^3-2x+2$ at $x = 0$

37. $f(x) = \frac{x^5+2x}{x^4}$ at $x = -1$

38. $f(x) = \frac{2x+3}{x^2+1}$ at $x = 1$

Exercises 39-40. (Horizontal Tangent Lines) Determine all points on the graph of the given function at which the tangent line is horizontal (derivative is zero).

39. $f(x) = \frac{2}{3}x^3-\frac{1}{2}x^2-x+1$

40. $f(x) = \frac{3x}{x^2+1}$

Exercises 41-42. (Tangent Lines of a Given Slope) Determine all points on the graph of the given function at which the tangent line has the indicated slope.

41. $f(x) = x^3-\frac{5}{2}x^2+1$; slope: 2.

42. $f(x) = x^3-x^2+1$; slope: 1.

Exercises 43-51. (Tangent Line Problems)

43. Show that no tangent line to the graph of the function $f(x) = x^3+x^2-100$ has a slope equal to -4.

44. Show that there is but one tangent line to the graph of the function $f(x) = \sqrt{x}+2$ with y-intercept equal to 4. Determine the equation of the tangent line.

45. Show that there does not exist a tangent line to the graph of the function $f(x) = \sqrt{x}+2$ with y-intercept equal to -4.

46. Show that for any $b>2$ there exists a unique tangent line to the graph of the function $f(x) = \sqrt{x}+2$ with y-intercept equal to b.

47. Find the point(s) on the graph of the function $f(x) = \frac{x^3}{3}+x^2+x$ which have $y = 4x+9$ as tangent line.

48. Show that the line $y = \frac{x}{4}+4$ is tangent to the graph of the function $f(x) = \sqrt{x}+2$ at some point. Determine the point of tangency.

49. Find a second degree polynomial $p(x) = ax^2+bx+c$ such that $p(1) = -4$, $p'(1) = 11$, and $p''(1) = 6$

50. Find a second degree polynomial $p(x) = ax^2+bx+c$ such that its graph passes through the point $(1, 3)$, the tangent line at $x = 3$ has slope 1, and the tangent line at $x = 1$ has slope 3.

51. Determine a, b, c, d such that $y = 3x-3$ and $y = -2x+1$ are the tangent lines to the graph of the polynomial function $p(x) = ax^3+bx^2+cx+d$ at $(1, 0)$ and $(0, 1)$, respectively.

Exercises 52-53. (Normal Line) The **normal line** to the graph of a function f at the point $(c, f(c))$ is the line passing through that point that is perpendicular (or orthogonal) to the tangent line at that point. Using the fact that a line of slope m_1 is perpendicular to a line of slope m_2 if and only if $m_1 = -\frac{1}{m_2}$, determine the equation of the normal line to the graph of the given function at the indicated point.

52. $f(x) = 3x^4 + x^2 - 2x + 1$ at $x = 1$

53. $f(x) = \frac{2x^3 - x^2}{x}$ at $x = 2$

Exercises 54-58. (Theory) Prove:

54. Theorem 3.2(c)

55. Theorem 3.2(f)

56. Use Theorem 3.2(f) to establish the following reciprocal rule:

$$\text{If } f \text{ is differentiable then } \left[\frac{1}{f(x)}\right]' = -\frac{f'(x)}{[f(x)]^2} \text{ (providing } f(x) \neq 0\text{)}$$

57. Show that if f, g, and h are differentiable, then:

$$(fgh)'(x) = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

58. Show that if $(x-a)^2$ is a factor of a polynomial $p(x)$, then $(x-a)$ is a factor of $p'(x)$. Is the converse true? Justify your answer.

Exercises 59-64. (Mathematical Induction)

59. Prove that for every integer $n \geq 1$, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

60. Prove that for every integer $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

61. Prove that the sum of n differentiable functions is again differentiable.

62. Prove that the product of n differentiable functions is again differentiable.

63. Prove that the n^{th} derivative of x^n equals $n!$ for any positive integer n.

64. What is wrong with the following "Proof" that any two positive integers are equal:

    Let $\max(a, b)$ denote the larger of the two integers a and b.

    Let $P(n)$ be the proposition: *If a and b are any two positive integers such that $\max(a, b) = n$, then $a = b$.*

    I. $P(1)$ is true: If $\max(a, b) = 1$, then both a and b must equal 1.

    II. Assume $P(k)$ is true: If $\max(a, b) = k$, then $a = b$.

    III. We show $P(k+1)$ is true: If $\max(a, b) = k+1$ then $\max(a-1, b-1) = k$.

    By II, $a - 1 = b - 1 \Rightarrow a = b$.

§3. DERIVATIVES OF TRIGONOMETRIC FUNCTIONS AND THE CHAIN RULE.

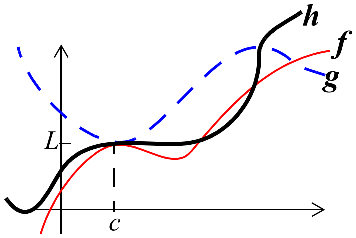
It is not difficult to convince oneself of the validity of the following result, a proof of which is relegated to the exercises:

This theorem is also called: *The Sandwich Theorem* or *The Squeeze Theorem.* Aptly named, since it maintains that if both $f(x)$ and $g(x)$ tend to L as x approaches c, and if h is pinched (or sandwiched, or squeezed) between f and g, then, $h(x)$ must also tend to L as x approaches c.

THEOREM 3.3
THE PINCHING THEOREM

Let f, g, and h be such that within an open interval about c:

$$f(x) \le h(x) \le g(x)$$



If: $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = L$ then $\lim\limits_{x \to c} h(x) = L$

EXAMPLE 3.11

Show that $\lim\limits_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$.

SOLUTION: We cannot simply substitute 0 for x in the given expression, nor can we hope to get rid of the bothersome x in $\frac{1}{x}$ by some algebraic means. What we can do is observe that, for any $x \ne 0$:

$$-1 \le \sin\frac{1}{x} \le 1 \text{ and that therefore: } -x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

Noting that $\lim\limits_{x \to 0}(-x^2) = \lim\limits_{x \to 0}(x^2) = 0$, we apply the Pinching Theorem to conclude that $\lim\limits_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$.

CHECK YOUR UNDERSTANDING 3.12

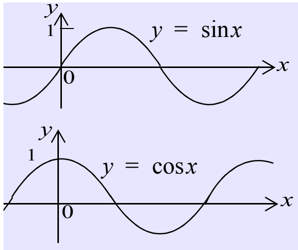
Given that $1 - \frac{x^2}{4} \le h(x) \le 1 + \frac{x^2}{2}$ for all $x \ne 0$, find $\lim\limits_{x \to 0} h(x)$.

Answer: See page A-12.

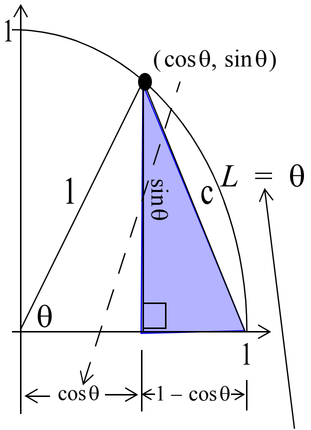
THEOREM 3.4

As is suggested by the graphs of the sine and cosine functions (margin):

$$\lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1$$



PROOF: Letting θ represent the angle with radian measure x, we first show that $\lim\limits_{\theta \to 0^+} \sin\theta = 0$ and $\lim\limits_{\theta \to 0^+} \cos\theta = 1$:



L is to θ as the perimeter of the circle (2π) is to a complete revolution (2π): $\frac{L}{\theta} = \frac{2\pi}{2\pi} = 1$

Or: $L = \theta$

Applying the Pythagorean Theorem to the shaded right triangle in the margin, we have:

$$\sin^2\theta + (1 - \cos\theta)^2 = c^2$$

Since the length c is less than the represented arc length $L = \theta$ (see boxed region in margin):

$$\sin^2\theta + (1 - \cos\theta)^2 < \theta^2$$

It follows that:

$$\begin{array}{ccc} \sin^2\theta < \theta^2 & \text{and} & (1-\cos\theta)^2 < \theta^2 \\ |\sin\theta| < |\theta| & \text{and} & |1-\cos\theta| < |\theta| \\ -|\theta| < \sin\theta < |\theta| & \text{and} & -|\theta| < 1-\cos\theta < |\theta| \end{array}$$

Applying the Pinching Theorem (note that $\lim\limits_{\theta \to 0} -|\theta| = 0$ and $\lim\limits_{\theta \to 0} |\theta| = 0$) we conclude that:

$$\lim_{\theta \to 0^+} \sin\theta = 0 \quad \text{and} \quad \lim_{\theta \to 0^+} (1 - \cos\theta) = 0 \Rightarrow \lim_{\theta \to 0^+} \cos\theta = 1$$

A similar argument can be used to show that the above limits also hold if we allow θ to approach 0 from below.

CHECK YOUR UNDERSTANDING 3.13

Prove that: $\lim\limits_{x \to 0} \tan x = 0$.

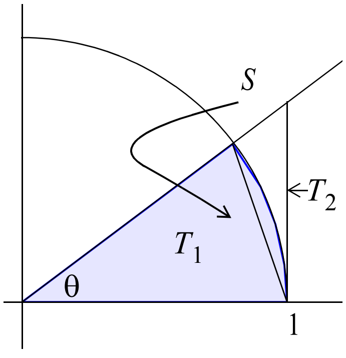
Answer: See page A-12.

The limits of Theorem 3.4 might have been anticipated. The same cannot be said for the following important result:

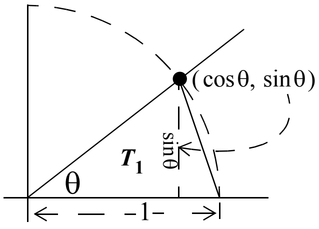
THEOREM 3.5

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

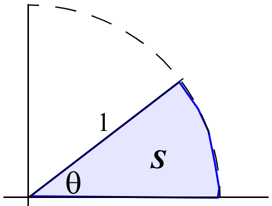
PROOF: Letting θ represent the angle with radian measure x, we show that $\lim\limits_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$, leaving it for you to verify that $\lim\limits_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$:



The (positive) angle θ in the adjacent figure intersects the unit circle; giving rise to three regions: Triangle T_1, Sector S, and Triangle T_2. Letting A_1, A_S, A_2 denote the area of those three regions, respectively, we observe below that $A_1 = \frac{1}{2}\sin\theta$, $A_S = \frac{\theta}{2}$, and $A_2 = \frac{1}{2}\tan\theta$:

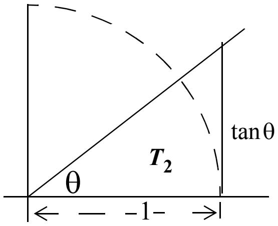


T_1 is a triangle of base 1 and height $\sin\theta$, so: $\boldsymbol{A_1 = \frac{1}{2}\sin\theta}$



$$\frac{A_S}{\text{Area of circle}} = \frac{\theta}{\text{circumference}}$$

$$\frac{A_s}{\pi} = \frac{\theta}{2\pi} \Rightarrow \boldsymbol{A_S = \frac{\theta}{2}}$$



T_2 is a triangle of base 1 and height $\tan\theta$, so: $\boldsymbol{A_2 = \frac{1}{2}\tan\theta}$

By virtue of inclusion:

$$A_1 \le A_S \le A_2 \Rightarrow \frac{1}{2}\sin\theta \le \frac{\theta}{2} \le \frac{1}{2}\tan\theta$$

$$\Rightarrow \sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$$

since $\sin\theta > 0$:

$$\Rightarrow 1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

$$\Rightarrow 1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

Noting that $\lim\limits_{\theta \to 0} 1 = 1$ and $\lim\limits_{\theta \to 0} \cos\theta = 1$ (Theorem 3.4), we apply the Pinching Theorem and conclude that $\lim\limits_{\theta^+ \to 0} \frac{\sin\theta}{\theta} = 1$.

CHECK YOUR UNDERSTANDING 3.14

Prove: $\lim\limits_{x \to 0} \frac{\cos x - 1}{x} = 0$

Suggestion. Start with: $\lim\limits_{x \to 0} \frac{\cos x - 1}{x} = \lim\limits_{x \to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$

Answer: See page A-13.

We are now in a position to establish the following important derivative formulas:

THEOREM 3.6 $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$

PROOF: Turning our attention to the derivative of $\sin x$ we have:

Definition 3.2, page 67

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\sin(\alpha+\beta)$
$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$
(Theorem 1.5(ii), page 37)

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0}\left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right]$$

$$= \lim_{h \to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]$$

$$= \lim_{h \to 0} \sin x \lim_{h \to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \cos x \lim_{h \to 0}\left(\frac{\sin h}{h}\right)$$

h does not appear in either $\sin x$ or $\cos x$

$$= \sin x \boldsymbol{\lim_{h \to 0}\left(\frac{\cos h - 1}{h}\right)} + \cos x \boldsymbol{\lim_{h \to 0}\left(\frac{\sin h}{h}\right)}$$

CYU 3.14

Theorem 3.5

$$= \sin x \cdot \mathbf{0} + \cos x \cdot \mathbf{1} = \boldsymbol{\cos x}$$

In the CYU below you are invited to offer a proof like the one above to show that $(\cos x)' = -\sin x$. Here, we will cheat a bit by showing that **if** the cosine function is differentiable (and it is), then $(\cos x)' = -\sin x$:

Theorem 3.2(e), page 78 (the product theorem)

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1\\
\cos^2 x &= 1 - \sin^2 x\\
(\cos x \cdot \cos x)' &= (1 - \sin x \cdot \sin x)'\\
\longleftarrow \cos x(\cos x)' + (\cos x)'\cos x &= 0 - [\sin x(\sin x)' + (\sin x)'\sin x]\\
2\cos x(\cos x)' &= -[\sin x\cos x + \cos x\sin x]\\
2\cos x(\cos x)' &= -2\sin x\cos x\\
(\cos x)' &= -\sin x
\end{aligned}$$

CHECK YOUR UNDERSTANDING 3.15

Fill in the "…" in: $(\cos x)' = \lim\limits_{h \to 0} \dfrac{\cos(x+h) - \cos x}{h} = \dots = -\sin x$

Answer: See page A-13.

Parts (a) and (b) of the following theorem have already been established. We prove (c) and invite you to verify the rest in CYU 3.16 below.

THEOREM 3.7

(a) $\dfrac{d}{dx}(\sin x) = \cos x$ (b) $\dfrac{d}{dx}(\cos x) = -\sin x$

(c) $\dfrac{d}{dx}(\tan x) = \sec^2 x$ (d) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

(e) $\dfrac{d}{dx}(\csc x) = -\csc x\cot x$ (f) $\dfrac{d}{dx}(\sec x) = \sec x\tan x$

PROOF:

(c) $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

CHECK YOUR UNDERSTANDING 3.16

Prove Theorem 3.7(d), (e), and (f).

Answer: See page A-13.

EXAMPLE 3.12 Differentiate the given function.

(a) $f(x) = x^2\sin x$ (b) $y = \dfrac{\sec x}{1+\tan x}$

SOLUTION:

(a) $f'(x) = (x^2\sin x)' = x^2(\sin x)' + \sin x\cdot(x^2)'$

$= x^2\cos x + 2x\sin x$

(b) $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sec x}{1+\tan x}\right) = \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

Theorem 3.7(f) and (c): $$= \frac{(1+\tan x)(\sec x\tan x) - \sec x(0+\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x\tan x + \sec x\tan^2 x - \sec^3 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$\sin^2 x + \cos^2 x = 1$$
$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\mathbf{\tan^2 x + 1 = \sec^2 x}$$

⟵ **$\sec^2 x = \tan^2 x + 1$**: $$= \frac{\sec x[\tan x + \tan^2 x - (\tan^2 x + 1)]}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

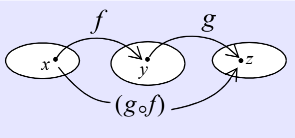
CHECK YOUR UNDERSTANDING 3.17

Differentiate:

(a) $f(x) = x\sec x + \tan x$ (b) $y = \dfrac{\sin x\cos x}{x^2}$

Answers:
(a) $(\sec x)(x\tan x + \sec x + 1)$
(b) $\dfrac{x\cos 2x - \sin 2x}{x^3}$

THE CHAIN RULE



Consider the functions $y = f(x)$, $z = g(y)$, along with the composite function $z = (g\circ f)(x) = g[f(x)]$.

Suppose that the following derivatives exist:

$$f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \text{ and } g'[f(x)] = \frac{dz}{dy} \approx \frac{\Delta z}{\Delta y}$$

Then, algebraically speaking:

$$\boldsymbol{(g\circ f)'(x)} = \frac{dz}{dx} \approx \frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \approx \boldsymbol{g'[f(x)]f'(x)}$$

Suggesting that:

For the sake of remembering the chain rule:

$$\frac{dz}{dx} = \frac{dz}{\cancel{dy}}\frac{\cancel{dy}}{dx}$$

But **only** for the sake of remembering.

dy is but part of the mathematical word $\frac{dy}{dx}$. Canceling a dy makes just as much sense as canceling the word ***at*** from the word ***cat***.

THEOREM 3.8
THE CHAIN RULE

If f is differentiable at x and g is differentiable at $f(x)$, then the composite function $g \circ f$ is differentiable at x, and:

$$(g \circ f)'(x) = g'[f(x)]f'(x)$$

In Leibniz noation: If $y = f(x)$ and $z = g(y)$ then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

In words: The derivative of a composite is the product of the derivatives.

A proof of the above theorem is offered at the end of the section. Our priority here is to make sure you know how to use it.

First of all, note that in applying the chain rule, you take the derivative of the outermost function in the chain first. For example:

derivative of sine evaluated at stuff

times the derivative of stuff

$$(\sin \boxed{\text{stuff}})' = \cos \boxed{\text{stuff}} \cdot \boxed{\text{stuff}}'$$

For example:

$$[\mathbf{sin}(\boldsymbol{x^2 + 2x})]' = \cos(x^2 + 2x) \cdot (x^2 + 2x)' = \mathbf{cos}(\boldsymbol{x^2 + 2x}) \cdot (\boldsymbol{2x + 2})$$

derivative of sine evaluated at $(x^2 + 2x)$ times the derivative of $(x^2 + 2x)$

EXAMPLE 3.13

Differentiate

(a) $(x^3 - 2x)\sin(x^2 + 2x)$

(b) $\cos\left(\sin\frac{x^2}{2x + 3}\right)$

SOLUTION: (a) The first thing you should see when you look at:

$$(x^3 - 2x)\sin(x^2 + 2x)$$

is that it is the product of two functions. So, that's what you do first:

$$\begin{aligned}
&[(x^3 - 2x)\sin(x^2 + 2x)]' \\
&\quad = (x^3 - 2x)[\mathbf{sin}(\boldsymbol{x^2 + 2x})]' + \sin(x^2 + 2x)(x^3 - 2x)' \\
&\quad = (x^3 - 2x)\mathbf{cos}(\boldsymbol{x^2 + 2x})(\boldsymbol{2x + 2}) + \sin(x^2 + 2x)(3x^2 + 2) \\
&\quad = (x^3 - 2x)(2x + 2)\cos(x^2 + 2x) + (3x^2 + 2)\sin(x^2 + 2x) \\
&\quad = (2x^4 + 2x^3 - 4x^2 - 4x)\cos(x^2 + 2x) + (3x^2 + 2)\sin(x^2 + 2x)
\end{aligned}$$

(b) $\cos\left(\sin\frac{x^2}{2x+3}\right)$ is the composite of three functions. So:

$$\begin{aligned}\frac{d}{dx}\cos\left(\sin\frac{x^2}{2x+3}\right) &= -\sin\left(\sin\frac{x^2}{2x+3}\right)\cdot\frac{d}{dx}\left(\sin\frac{x^2}{2x+3}\right)\\ &= -\sin\left(\sin\frac{x^2}{2x+3}\right)\cdot\cos\left(\frac{x^2}{2x+3}\right)\cdot\frac{d}{dx}\left(\frac{x^2}{2x+3}\right)\\ &= -\sin\left(\sin\frac{x^2}{2x+3}\right)\cdot\cos\left(\frac{x^2}{2x+3}\right)\cdot\frac{(2x+3)(2x)-x^2\cdot 2}{(2x+3)^2}\\ &= -\sin\left(\sin\frac{x^2}{2x+3}\right)\cos\left(\frac{x^2}{2x+3}\right)\cdot\frac{2x^2+6x}{(2x+3)^2}\\ &= -\frac{2x^2+6x}{(2x+3)^2}\sin\left(\sin\frac{x^2}{2x+3}\right)\cos\left(\frac{x^2}{2x+3}\right)\end{aligned}$$

CHECK YOUR UNDERSTANDING 3.18

Differentiate the given function:

(a) $f(x) = x\tan\left(\frac{x}{x+1}\right)$ (b) $g(x) = \frac{\sin x^2}{\cos x}$

Answers:

(a) $\frac{x}{(x+1)^2}\cdot\sec^2\left(\frac{x}{x+1}\right) + \tan\left(\frac{x}{x+1}\right)$

(b) $\frac{2x\cos x\cos x^2 + \sin x\sin x^2}{\cos^2 x}$

Here is an important consequence of the chain rule:

THEOREM 3.9
GENERALIZED POWER RULE

If f is differentiable at x then so is the function $[f(x)]^r$ for any real number r, and:

$$\frac{d}{dx}[f(x)]^r = r[f(x)]^{r-1}\frac{d}{dx}[f(x)]$$

Alternate notation: $([f(x)]^r)' = r[f(x)]^{r-1}f'(x)$

If $f(x) = x$, then:

$$\begin{aligned}\frac{d}{dx}[f(x)]^r &= (x^r)'\\ &= rx^{r-1}(x')\\ &= rx^{r-1}\end{aligned}$$

and we're back to Theorem 3.2(b), page 78.

PROOF: Let $g(x) = x^r$. Then:

$$(g\circ f)(x) = g[f(x)] = [f(x)]^r$$
$$\uparrow\_\_\_ (*) \_\_\_\lrcorner$$

Applying the Chain Rule Theorem:

$$([f(x)]^r)' \underset{(*)}{=} (g[f(x)])' = \mathbf{g'[f(x)]}f'(x) \underset{\text{see margin}}{=} \mathbf{r[f(x)]^{r-1}}f'(x)$$

From Theorem 3.2(b):

$g'(x) = (x^r)' = rx^{r-1}$

Consequently:

$\mathbf{g'[f(x)] = r[f(x)]^{r-1}}$

In the spirit of full-disclosure we point out that while Theorem 3.2(b) does indeed hold for all r, we have (up to now) only established its validity for integer exponents (see Example 3.10 and CYU 3.10 of the previous section).

EXAMPLE 3.14

Differentiate

(a) $f(x) = (x^4-3x^2)^{23}$ (b) $g(x) = \cos^5 x^3$

(c) $h(x) = \frac{4}{x^2+1}$ (d) $k(x) = \sqrt{\tan x^2}$

SOLUTION: (a)
$$\begin{aligned}\frac{d}{dx}(x^4-3x^2)^{23} &= 23(x^4-3x^2)^{23-1}\frac{d}{dx}(x^4-3x^2)\\ &= 23(x^4-3x^2)^{22}(4x^3-6x)\\ &= 23(4x^3-6x)(x^4-3x^2)^{22}\end{aligned}$$

(b) You may find it safer to rewrite the function $h(x) = \cos^5 x^3$ in its more revealing form: $h(x) = (\cos x^3)^5$. Then:

$$\begin{aligned}\frac{d}{dx}(\cos x^3)^5 &= 5(\cos x^3)^{5-1}\cdot\frac{d}{dx}(\cos x^3)\\ &= 5(\cos x^3)^4(-\sin x^3)\frac{d}{dx}x^3\\ &= 5\cos^4 x^3(-\sin x^3)\cdot 3x^2 = -15x^2(\cos^4 x^3)\sin x^3\end{aligned}$$

(c)
$$\begin{aligned}\left(\frac{4}{x^2+1}\right)' &= [4(x^2+1)^{-1}]' = 4\boldsymbol{[(x^2+1)^{-1}]'}\\ &= 4\boldsymbol{[-1(x^2+1)^{-1-1}](x^2+1)'}\\ &= -4(x^2+1)^{-2}\cdot 2x = -\frac{8x}{(x^2+1)^2}\end{aligned}$$

(d)
$$\begin{aligned}(\sqrt{\tan x^2})' &= \left[(\tan x^2)^{\frac{1}{2}}\right]' = \frac{1}{2}(\tan x^2)^{\frac{1}{2}-1}\cdot(\tan x^2)'\\ &= \frac{1}{2}(\tan x^2)^{-\frac{1}{2}}(\tan x^2)'\\ \text{Chain Rule Theorem:}\quad &= \frac{1}{2\sqrt{\tan x^2}}\cdot\sec^2 x^2\cdot(x^2)'\\ &= \frac{1}{2\sqrt{\tan x^2}}\cdot\sec^2 x^2\cdot(2x) = \frac{x\sec^2 x^2}{\sqrt{\tan x^2}}\end{aligned}$$

CHECK YOUR UNDERSTANDING 3.19

Differentiate the given function:

(a) $f(x) = (x^4+\sin x)^3$ (b) $g(x) = \sin^3 x^3$

(c) $f(x) = \sqrt{\sin x^2}$

Answers:

(a) $3(x^4+\sin x)^2(4x^3+\cos x)$

(b) $9x^2\cos x^3\sin^2 x^3$

(c) $\dfrac{x\cos x^2}{\sqrt{\sin x^2}}$

Now that you are comfortable applying the chain rule, we want to make sure that you truly understand what it is saying. With this in mind we ask you to consider the following situation:

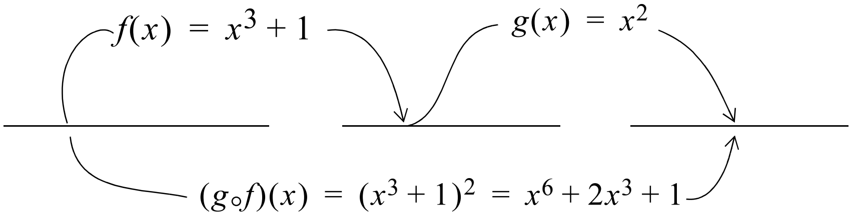


Figure 3.8

Three functions are depicted in Figure 3.8:
$f(x) = x^3 + 1$, $g(x) = x^2$, and the composite function:

$$(g\circ f)(x) = g[f(x)] = g(x^3+1) = (x^3+1)^2 = x^6 + 2x^3 + 1$$

The chain rule $(g\circ f)'(x) = g'[f(x)]f'(x)$ asserts that the derivative of the composite function equals the derivatives of the function g evaluated at $f(x)$ times the derivative of $f(x)$. Let's check it out:

$$(g\circ f)'(x) = (x^6 + 2x^3 + 1)' = \mathbf{6x^5 + 6x^2}$$

Let's show that $g'[f(x)]f'(x)$ leads to the same result:

From $g(x) = x^2$: $g'(x) = 2x$. In particular:

$$g'[f(x)] = g'(x^3+1) = 2(x^3+1) = 2x^3 + 2$$

Also: $f'(x) = (x^3+1)' = 3x^2$

Bringing us to: $g'[f(x)]f'(x) = (2x^3+2)(3x^2) = \mathbf{6x^5 + 6x^2}$

EXAMPLE 3.15

For $f(x) = 4x + 1$ and $g(x) = x^2 + 2x$, determine $(g\circ f)'(x)$, both with, and without using the chain rule.

SOLUTION: With the chain rule:

since $f(x) = 4x + 1, f'(x) = 4$

$$(g\circ f)'(x) = g'[f(x)] \cdot f'(x) = g'(4x+1) \cdot 4$$

$g(x) = x^2 + 2x \Rightarrow g'(x) = 2x + 2$
$\Rightarrow \mathbf{g'(4x+1)} = 2(4x+1) + 2 = \mathbf{8x} + \mathbf{4}$

$$= (\mathbf{8x + 4}) \cdot 4 = 32x + 16$$

Without the chain rule:

$$(g\circ f)(x) = g[f(x)] = g(4x+1) = (4x+1)^2 + 2(4x+1)$$

$$= (16x^2 + 8x + 1) + (8x + 2) = 16x^2 + 16x + 3$$

Differentiating:

$$(g\circ f)'(x) = (16x^2 + 16x + 3)' = 32x + 16$$

CHECK YOUR UNDERSTANDING 3.20

For $f(x) = 2x^2 - 5$ and $g(x) = x^2 + 2$ determine $(g \circ f)'(x)$, both with and without using the chain rule.

Answer: $16x^3 - 40x$

PROOF OF THE CHAIN RULE:

Here is a seductively simple "proof" for the Chain Rule theorem:

If $y = f(x)$ and $z = g(y)$ are differentiable (which is to say that $\frac{dy}{dx}$ and $\frac{dz}{dy}$ exist) then so is $z = (g \circ f)(x)$ differentiable:

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \frac{\Delta z}{\Delta y} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} = \frac{dy}{dx} \frac{dz}{dy}$$

Alas, there is a flaw in the above argument:

While it is true that as Δx goes to zero so must Δy, there is nothing preventing Δy from assuming the value of 0 along the way, in which case the expression $\frac{\Delta z}{\Delta y}$ is undefined! We have to be more careful, and to make sure we are not tempted to do **silly things** like canceling the "dy" in the Leibnitz form of the chain rule $\left(\frac{dz}{dx} = \frac{\cancel{d}y}{dx} \frac{dz}{d\cancel{y}} \text{: nonsense!} \right)$ we shall use the prime notation in the statement and proof of the Chain Rule Theorem:

Chain Rule Theorem

If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c, and:

$$(g \circ f)'(c) = g'[f(c)]f'(c)$$

PROOF: Let f be differentiable at c, and let g be differentiable at $f(c)$. Consider the function

$$F[f(x)] = \begin{cases} \dfrac{g[f(x)] - g[f(c)]}{f(x) - f(c)} & \text{if } f(x) \neq f(c) \\ g'[f(c)] & \text{if } f(x) = f(c) \end{cases} \quad (*)$$

Since $\lim\limits_{f(x) \to f(c)} \dfrac{g[f(x)] - g[f(c)]}{f(x) - f(c)} = g'[f(c)]$, F is continuous at $f(c)$. In the event that $x \neq c$:

$$\frac{g[f(x)] - g[f(c)]}{x - c} = F[f(x)]\left[\frac{f(x) - f(c)}{x - c}\right] \quad (**)$$

(see margin)

If $f(x) \neq f(c)$:

$$\boldsymbol{F[f(x)]}\left[\frac{f(x)-f(c)}{x-c}\right]$$

$$= \frac{\boldsymbol{g[f(x)] - g[f(c)]}}{\boldsymbol{f(x) - f(c)}} \cdot \frac{f(x)-f(c)}{x-c}$$

$$= \frac{g[f(x)] - g[f(c)]}{x-c}$$

If $f(x) = f(c)$:
The right side of (**) is zero, as is the let side:
Since g is differentiable at $f(c)$, *it* is continuous at $f(c)$.

Since f is continuous at c, and F is continuous at $f(c)$, the composite function $F \circ f$ is also continuous (Theorem 2.5, page 58); bringing us to:

$$\lim_{x \to c} F[f(x)] = F[f(c)] \underset{\underset{(*)}{\uparrow}}{=} g'[f(c)] \quad (***)$$

Finally:

$$\lim_{x \to c} \frac{g[f(x)] - g[f(c)]}{x - c} \overset{\overset{(**)}{\downarrow}}{=} \lim_{x \to c} F[f(x)]\left[\frac{f(x) - f(c)}{x - c}\right]$$

$$= \lim_{x \to c} F[f(x)] \lim_{x \to c}\left[\frac{f(x) - f(c)}{x - c}\right]$$

$$(***)\!: = g'[f(c)]f'(c)$$

Conclusion: $(g \circ f)'(c) = \lim\limits_{x \to c} \dfrac{g[f(x)] - g[f(c)]}{x - c} = g'[f(c)]f'(c)$

EXERCISES

Exercises 1-24. Differentiate the given function.

1. $f(x) = (x^2+3x-10)^{15}$
2. $f(x) = (3x^4+2x-5)^{-7}$
3. $f(x) = \sqrt{x^3+2x}$
4. $f(x) = \frac{1}{\sqrt{3x^4+x^2}}$
5. $f(x) = \frac{3x}{\sqrt{x+1}}$
6. $f(x) = \sqrt{\frac{1}{x+1}}$
7. $f(x) = \sin(2x^2+1)$
8. $f(x) = x\sin(2x^2+1)$
9. $f(x) = \sin(\cos x)$
10. $f(x) = \sin^2 x\cos x$
11. $f(x) = x\cos(\sin x)$
12. $f(x) = \tan(\sec x)$
13. $f(x) = \frac{\sin^2 x}{\cos x}$
14. $f(x) = \sec x\tan x^2$
15. $f(x) = \tan[\sin(x^2+x-1)]$
16. $f(x) = \sin x^2\cos x^2$
17. $f(x) = \sqrt{\sec(2x+3)}$
18. $f(x) = \cot\sqrt{2x+3}$
19. $f(x) = \sin(\cos^2 x)$
20. $f(x) = \sin^2(\cos x)$
21. $f(x) = \cot^2(\cos x^2)$
22. $f(x) = \sin^2(\cos^2 x^2)$
23. $f(x) = [\csc(\cos x^2)]^{\frac{2}{3}}$
24. $f(x) = \sqrt{\tan(\cos^2 x)}$

Exercises 25-26. (Rate of Change) Determine the rate of change of the given function, at the indicated point.

25. $f(x) = \frac{1}{(3x-5)^3}$, at $x = 2$
26. $f(x) = \frac{x}{\sin x}$, at $x = \frac{\pi}{3}$

Exercise 27-28. (Composite Functions) Determine the derivative of $(g\circ f)(x)$ both with and without using the chain rule (as in Example 3.15), for:

27. $f(x) = 3x^2+x$ and $g(x) = \frac{1}{x+1}$
28. $f(x) = \sin x$ and $g(x) = x^2$

Exercises 29-34. (Chain Rule) Evaluate the given function at the given point, if:

$$f(0) = 1,\ f(1) = 3,\ f(2) = 2,\ f'(0) = 2,\ f'(1) = 6,\ f'(2) = 0,\ f'(3) = 3$$
$$g(0) = 2,\ g(1) = 2,\ g(2) = 5,\ g'(0) = 1,\ g'(1) = 2,\ g'(2) = 2,\ g'(3) = 2$$

29. $(g\circ f)'(0)$
30. $(f\circ g)'(0)$
31. $(g\circ g)'(0)$
32. $(g\circ f)'(1)$
33. $(f\circ f)'(1)$
34. $(g\circ f)'(2)$

Exercises 35-38. (Tangent Line) Determine the tangent line to the graph of the given function, at the indicated point.

35. $f(x) = \left(\frac{x}{2x+5}\right)^2$, at $x = -2$

36. $f(x) = 3x + \sqrt{\frac{1}{3x+1}}$, at $x = 0$

37. $f(x) = \sin^2 x$, at $x = \frac{\pi}{2}$

38. $f(x) = x\cos x$, at $x = \pi$

Exercises 39-41. (Point of Tangency)

39. Determine the numbers $0 \le x < 2\pi$ where the tangent line to the graph of the function $f(x) = \sin x + \sqrt{3}\cos x$ is horizontal.

40. For what values of x is the slope of the tangent line to the graph of $\sin x$ parallel to that of $\cos x$?

41. Show that the line $y = -4x + 9$ is tangent to the graph of the function $f(x) = \frac{1}{(2x-3)^2}$ at some point. Determine the point of tangency.

Exercises 42-43. (Normal Line) Determine the normal line to the graph of the given function at the given point (See Exercises 52-53, page 88).

42. $f(x) = (x^2 - 2x + 1)^5$ at $x = 1$

43. $f(x) = x^2 \sin x$ at $x = \frac{\pi}{2}$

Exercises 44-46. (Pinching Theorem) Evaluate:

44. $\lim_{x \to 2} f(x)$, given that $3 \le f(x) \le (x-3) + 4$ for $|x-2| < 1$.

45. $\lim_{x \to 2} f(x)$, given that $-(x-2)^2 \le f(x) \le 0$ for $x \ne 2$.

46. $\lim_{x \to 0} f(x)$, given that $5 - |x+2|^2 \le f(x) \le \frac{1}{x^2+1}$ for $x \ne 0$.

Exercises 47-49. (Pinching Theorem) Show that:

47. $\lim_{x \to 0}\left(x\sin\frac{1}{x}\right) = 0$

48. $\lim_{x \to 0}\left(x^2\cos\frac{5}{x}\right) = 0$

49. $\lim_{x \to 1}\left[(x-1)^2\sin\frac{100}{x-1}\right] = 0$

50. **(Theory)** Prove that $\lim_{x \to 0}\frac{\sin cx}{cx} = 1$ for any $c \ne 0$. (Suggestion: Make the substitution $y = cx$)

Exercises 51-61. Evaluate (You may need to use the result of Exercise 50):

51. $\lim_{x \to 0}\frac{3\sin x}{x}$

52. $\lim_{x \to 0}\frac{\sin 3x}{3x}$

53. $\lim_{x \to 0}\frac{\sin 3x}{x}$

54. $\lim_{x \to 0}\frac{\sin 3x}{7x}$

55. $\lim_{x \to 0}\frac{\sin 7x}{3x}$

56. $\lim_{x \to 0}\frac{\sin^2 x}{x^2}$

57. $\lim_{x \to 0}\frac{\sin^2(2x)}{x^2}$

58. $\lim_{x \to 0}\frac{\sin x}{x + \tan x}$

59. $\lim_{x \to 0}\cos\left[\frac{\cos x - 1}{x}\right]$

60. $\lim_{x \to 0}\tan\left[\frac{\cos x - 1}{x}\right]$

61. $\lim_{x \to 0}\sin\left(\frac{\sin \pi x}{x}\right)$

62. **(Investment)** If \$100 is invested at an annual interest rate r compounded quarterly, then the future value FV (in dollars) accumulated after 10 years is given by: $FV = 100\left(1+\frac{r}{4}\right)^{40}$. Find the rate of change of the future value with respect to r.

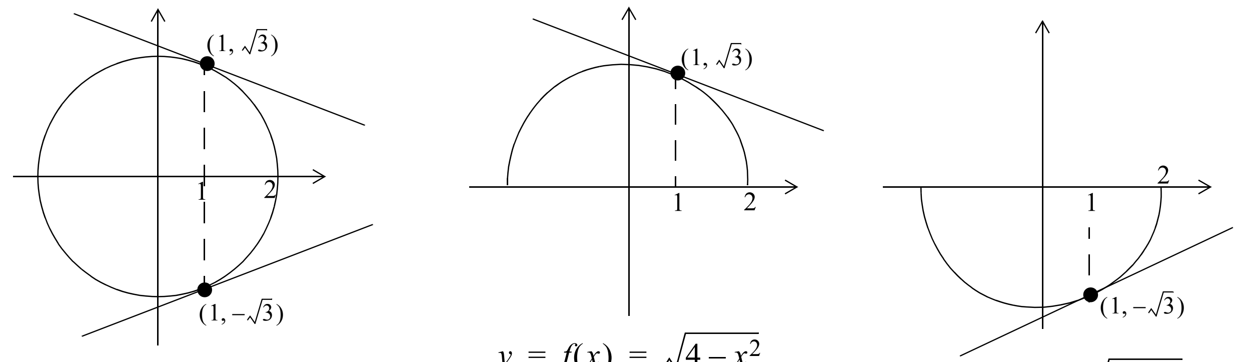
63. **(Investment)** The effective rate r_e of an annual nominal rate r compounded monthly is given by: $r_e = \left(1+\frac{r}{12}\right)^{12} - 1$. Find the rate of change of the effective rate with respect to the nominal rate.

64. **(Sales)** A baseball stadium has a capacity of 35,000 fans. Attendance starts falling off when the temperature rises above 90 degree Fahrenheit, in accordance with the formula $A(x) = 35000 - 500x$, where x is the (average) number of degrees above 90 during the game. The number of sodas sold during a baseball game at the stadium to a **capacity** crowd of fans is given by $N(T) = 35T^{3/5}$, where T is the average temperature at the stadium during of the game. A quarter profit is made on each can sold.
    (a) How many cans of soda are sold during a game, when the temperature is $90°$ Fahrenheit? $95°$? $100°$?
    (b) Express the soda-profit for a game as a function of temperature, for $90 \le T \le 100$.
    (c) Use the function in (b) to find the rate of change of profit with respect to temperature.
    (d) Use the Chain rule to find the rate of change of profit with respect to temperature.

65. **(Theory)** Derive the chain rule formula for three differentiable functions f, g, and h: $[(h \circ g \circ f)(x)]' = \ldots$

§4. IMPLICIT DIFFERENTIATION

While the circle in Figure 3.9(a) is **NOT** the graph of a function, the curve does possess tangent lines at $(1, \sqrt{3})$, $(1, -\sqrt{3})$.



$x^2 + y^2 = 4$ or $y = \pm\sqrt{4 - x^2}$
(a)

$y = f(x) = \sqrt{4 - x^2}$
(b)

$y = g(x) = -\sqrt{4 - x^2}$
(c)

Figure 3.9

We exhibit two differentiation methods which can be used to determine the slopes of those tangent lines.

EXPLICIT DIFFERENTIATION METHOD:

In this method, a function with graph coinciding with the given curve at the point of interest is explicitly displayed.

From Figures 3.9(b) and (c) we see that the slope at $(1, \sqrt{3})$ is $f'(1)$ where f is **the function** $f(x) = \sqrt{4 - x^2}$, and that the slope at $(1, -\sqrt{3})$ is $g'(1)$ where g is **the function** $g(x) = -\sqrt{4 - x^2}$. Specifically:

$$f'(x) = \left[(4 - x^2)^{\frac{1}{2}}\right]' = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{4 - x^2}}$$

and

$$g'(x) = \left[-(4 - x^2)^{\frac{1}{2}}\right]' = -\frac{1}{2}(4 - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{x}{\sqrt{4 - x^2}}$$

It follows that the slope of the tangent line to the curve of Figure 3.9(a) at $(1, \sqrt{3})$ is $f'(1) = -\frac{1}{\sqrt{3}}$ and that the slope at $(1, -\sqrt{3})$ is $g'(1) = \frac{1}{\sqrt{3}}$.

IMPLICIT DIFFERENTIATION METHOD:

Assume that there exists a function $y = f(x)$ whose graph coincides with that of the curve $x^2 + y^2 = 4$ at the point $(1, \sqrt{3})$, and a function $y = g(x)$ with graph coinciding with the curve about the point $(1, -\sqrt{3})$. Differentiating both sides of $x^2 + y^2 = 4$ with respect to x, we have:

While $\frac{d}{dx}x^2 = 2x$:

$\frac{d}{dx}y^2$ is $2y\frac{d}{dx}y = 2y\boldsymbol{y'}$ for y **is a function of** $\boldsymbol{x}$!

After all: $\frac{d}{dx}[f(x)]^2$ is not simply $2f(x)$ — it is $2f(x)\boldsymbol{f'(x)}$, right?

see margin

$$\begin{aligned}(x^2)' + (\boldsymbol{y^2})' &= 4' \\ 2x + \boldsymbol{2yy'} &= 0 \\ 2yy' &= -2x \\ y' &= -\frac{x}{y}\end{aligned}$$

In particular, to find the slope of the tangent line to the curve $x^2 + y^2 = 4$ at the point $(1, \sqrt{3})$, we simply substitute 1 for x and $\sqrt{3}$ for y in the slope equation $y' = -\frac{x}{y}$: $y' = -\frac{1}{\sqrt{3}}$. By the same token, the slope of the tangent line at the point $(x, y) = (1, -\sqrt{3})$ is: $y' = -\frac{x}{y} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$.

Before moving on to other implicit differentiation examples, we call your attention to the curve of Figure 3.10. It is not the graph of a function. Still, at just about every point on the curve there does exist a function whose graph coincides with the curve about that point. In particular, the graph of the depicted function g coincides with the curve near the point (b, y_1) while the function f does the same near (b, y_2). The function h coincides with the curve near (c, y_3), but h is not differentiable at that point (why not?). Note that no function (of x) can approximate the curve near the points (a, y) or (d, y_4) (why not?).

Note: To determine the slope of a tangent line to the graph of a function at a given point, only the x-coordinate of that point need be supplied, for there can be but one y associated with that x. That may not be the case when it comes to a general curve. There are, for example, two points on the adjacent curve with x-coordinate b. The slope of the tangent line at (b, y_1) appears to be a bit negative, while that at (b, y_2) looks to be slightly positive.

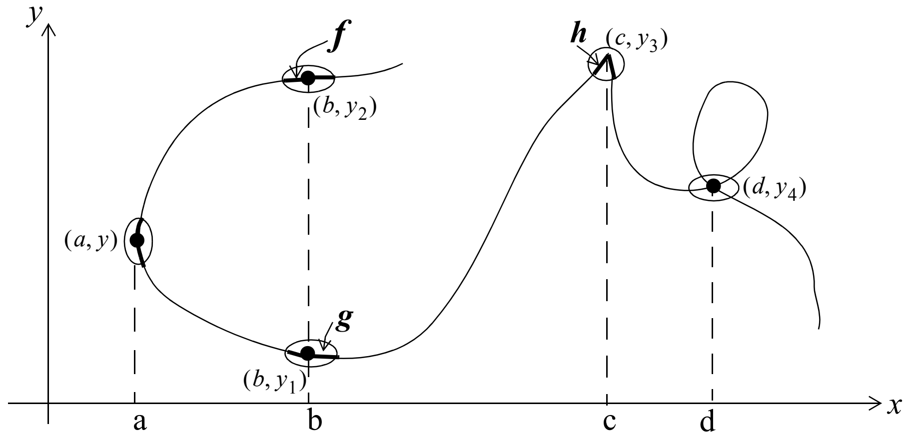


Figure 3.10

Is the explicit differentiation option viable in this example?

EXAMPLE 3.16 Find the slope of the tangent line to the curve $x^3 - 2x^2y^2 + y^3 = 1$ at the point $(1, 2)$ and at the point $(0, 1)$.

SOLUTION: Assume that the curve near each of those two points coincides with the graph of some function (not necessarily the same function for both points). Differentiating we have:

$$(x^3 - 2x^2\mathbf{y^2} + \mathbf{y^3})' = (1)'$$

$$\leftarrow\boxed{\text{margin}}\text{—}\quad 3x^2 - 2(x^2\mathbf{2yy'} + y^2 2x) + \mathbf{3y^2y'} = 0$$

Note the **big difference** between taking the derivative with respect to x of x^3 and of $\mathbf{y^3}$. Why so? Because the chain rule tells us that:

$$([\underbrace{f(x)}_{y}]^3)' = 3[f(x)]^2\underbrace{f'(x)}_{y'}$$

So, while

$$\frac{d}{dx}(x^3) = 3x^2\frac{dx}{dx} = 3x^2$$

$$\mathbf{\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}}$$

By the same token, when applying the product rule to $(x^2\mathbf{y^2})'$ we have:

$$\begin{aligned} & x^2(\mathbf{y^2})' + y^2(x^2)' \\ &= x^2\mathbf{2yy'} + y^2 2xx'\text{—}\boxed{x' = \frac{dx}{dx} = 1} \\ &= x^2\mathbf{2yy'} + y^2 2x \leftarrow \end{aligned}$$

To find the slope of the tangent line at $(1, 2)$ we substitute 1 for x and 2 for y in the above equation, and then solve for y'. Substituting 0 for x and 1 for y leads us to the slope of the tangent line at $(0, 1)$:

$$3x^2 - 2(x^2 2yy' + y^2 2x) + 3y^2y' = 0$$

$$\begin{array}{r|c} x = 1, y = 2: & x = 0, y = 1: \\ 3 - 2(4y' + 8) + 12y' = 0 & 3y' = 0 \\ 4y' = 13 & \mathbf{y' = 0} \\ \mathbf{y' = \frac{13}{4}} & \end{array}$$

So, even though we may not have a nice picture of the curve $x^3 - 2x^2y^2 + y^3 = 1$, we do know that the tangent line at $(1, 2)$ has slope $\frac{13}{4}$, which tells us that the curve is climbing at that point (move to the right a bit, and the y values will increase by about $\frac{13}{4}$ times that bit). We also know that the curve has a horizontal tangent line at $(0, 1)$.

CHECK YOUR UNDERSTANDING 3.21

Determine an equation for the tangent line to the curve $x^2 + xy + 2y^2 = 8$ at $(2, 1)$ and at $(2, -2)$.

Answer: $y = -\frac{5}{6}x + \frac{8}{3}$,
$y = \frac{1}{3}x - \frac{8}{3}$.

EXAMPLE 3.17 Find y' and y'' in terms of x and y, given that $y^2 = xy + 1$

SOLUTION:

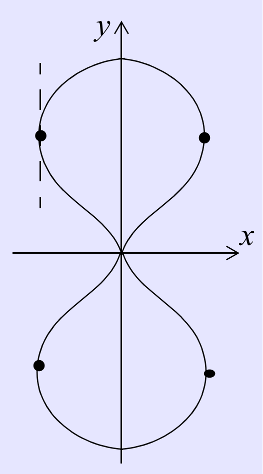
$$\begin{aligned}(y^2)' &= (xy+1)'\\ 2yy' &= xy'+y\\ (2y-x)y' &= y\\ y' &= \frac{\mathbf{y}}{\mathbf{2y-x}} \longrightarrow y'' = \frac{(2y-x)y'-y(2y'-1)}{(2y-x)^2}\end{aligned}$$

$$\begin{aligned}&= \frac{(2y-x)\left(\frac{\mathbf{y}}{\mathbf{2y-x}}\right)-y\left(\frac{\mathbf{2y}}{\mathbf{2y-x}}-1\right)}{(2y-x)^2}\\ &= \frac{2y^2-xy-y(\cancel{2y}-\cancel{2y}+x)}{(2y-x)^3}\\ &= \frac{2y^2-2xy}{(2y-x)^3} = \frac{2(y^2-xy)}{(2y-x)^3}\\ \text{since } y^2 = xy+1: \quad &= \frac{2\cdot 1}{(2y-x)^3} = \frac{2}{(2y-x)^3}\end{aligned}$$

Answer: $-\frac{24y}{(3y^2-1)^3}$

CHECK YOUR UNDERSTANDING 3.22

Find $\frac{d^2y}{dx^2}$ in terms of x and y, given that $y^3+2x = y$.



EXAMPLE 3.18 The curve $y^4-y^2+x^2 = 0$ (called a *lemniscate*) appears in the margin. Determine the four indicated points on the curve where vertical tangent lines occur.

SOLUTION: Using implicit differentiation, we find y':

$$\begin{aligned}4y^3y'-2yy'+2x &= 0\\ 2y^3y'-yy'+x &= 0\\ y' &= \frac{-x}{2y^3-y}\\ &= \frac{-x}{y(2y^2-1)} = \frac{-x}{y(\sqrt{2}y+1)(\sqrt{2}y-1)}\end{aligned}$$

We see that the derivative y' fails to exist at $y = 0$ and at $y = \pm\frac{1}{\sqrt{2}}$.

Returning to the equation $y^4 - y^2 + x^2 = 0$ we find the points on the curve with y-coordinate 0 or $\pm\frac{1}{\sqrt{2}}$:

For $y = \mathbf{0}: \mathbf{0}^4 - \mathbf{0}^2 + x^2 = 0 \Rightarrow x = 0$. Point: $(\mathbf{0}, 0)$.

$$\text{For } y = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}: \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^4 - \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^2 + x^2 = 0$$

$$\frac{1}{4} - \frac{1}{2} + x^2 = 0 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

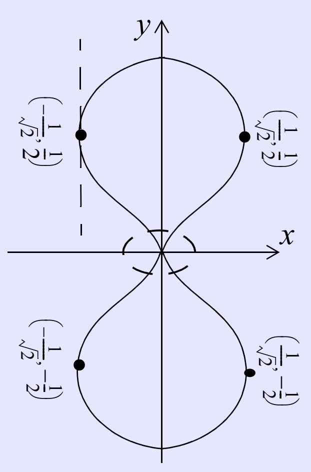
$$\text{Points: } \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, \frac{1}{2}\right) \text{ and } \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, -\frac{1}{2}\right)$$

$$\text{For } y = -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}: \left(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^4 - \left(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^2 + x^2 = 0$$

$$\frac{1}{4} - \frac{1}{2} + x^2 = 0 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

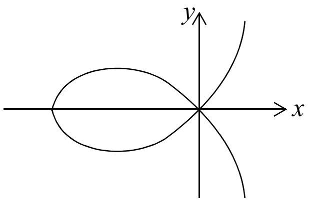
$$\text{Points: } \left(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, \frac{1}{2}\right) \text{ and } \left(-\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, -\frac{1}{2}\right)$$

Note that the curve cannot be approximated by the graph of a function at $(0, 0)$ (see margin) which means that implicit differentiation can not be used there. Why is y' not defined at $y = \pm\frac{1}{\sqrt{2}}$? Because there are vertical tangent lines at the four points $\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ and $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ (see margin).



CHECK YOUR UNDERSTANDING 3.23

It appears that the adjacent curve $2y^2 - x^3 - x^2 = 0$ contains but one point where it has a vertical tangent line and two points where it has a horizontal tangent line. Find those points.



Answer: Vertical tangent line at $(-1, 0)$ and horizontal tangent lines at $\left(-\frac{2}{3}, \pm\sqrt{\frac{2}{27}}\right)$.

EXERCISES

Exercises 1-8. (Explicit vs. Implicit) Sketch the given curve and determine the slope of the tangent line at the given points, both by means of explicit and implicit differentiation.

1.The points $(1, 1)$ and $(1, -1)$ on the parabola $x = y^2$.

2.The points $\left(2, -1+\frac{1}{\sqrt{2}}\right)$ and $\left(2, -1-\frac{1}{\sqrt{2}}\right)$ on the parabola $x-2 = 2(y+1)^2$.

3. The points $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the circle $x^2+y^2 = 1$.

4. The points $(1+\sqrt{3}, 0)$ and $(1-\sqrt{3}, 0)$ on the circle $(x-1)^2+(y-1)^2 = 4$.

5. The points $\left(1, \frac{\sqrt{3}}{2}\right)$ and $\left(1, -\frac{\sqrt{3}}{2}\right)$ on the ellipse $\frac{x^2}{4}+y^2 = 1$.

6. The points $\left(1, -3+\frac{4\sqrt{2}}{3}\right)$ and $\left(1, -3-\frac{4\sqrt{2}}{3}\right)$ on the ellipse $\frac{(x-2)^2}{9}+\frac{(y+3)^2}{4} = 1$.

7. The points $(4, \sqrt{3})$, $(4, -\sqrt{3})$, $(-4, \sqrt{3})$ and $(-4, -\sqrt{3})$ on the hyperbola $\frac{x^2}{4}-y^2 = 1$.

8.The points $(1, 2\sqrt{2})$, $(1, -2\sqrt{2})$, $(-1, 2\sqrt{2})$ and $(-1, -2\sqrt{2})$ on the hyperbola $-x^2+\frac{y^2}{4} = 1$.

Exercises 9-24. (Tangent Line) Determine the tangent line to the given curve at the given point.

9. $x^2y^2 = 4$ at $(1, 2)$

10. $x^2-4y^2 = 4$ at $(2, 0)$

11. $x^2+xy^2-y = 4$ at $(2, 0)$

12. $x^2+xy-y^2 = 1$ at $(2, 3)$

13. $x^3+y^3 = 6xy$ at $(3, 3)$

14. $x^3y^4 = 4$ at $(1, \sqrt{2})$

15. $y^4-xy = x^2-1$ at $(1, 0)$

16. $x^2+y^2 = (2x^2+2y^2-x)^2$ at $\left(0, \frac{1}{2}\right)$

17. $x^2+y^3 = 2y+3$ at $(2, 1)$

18. $4x^4+8x^2y^2 = 25x^2y-4y^4$ at $(2, 1)$

19. $x+\cos y = xy$ at $\left(0, \frac{3\pi}{2}\right)$

20. $(x^2+y^2)^2 = (x-y)^2$ at $(1, -1)$

21. $x\cos^2 y = \sin y$ at $(0, 0)$

22. $2\sin(\pi x-y) = y$ at $(1, 0)$

23. $x\sin y-y\cos 2x = 2x$ at $\left(\frac{\pi}{2}, \pi\right)$

24. $\sin(x+y) = y^2\cos x$ at $(0, 0)$

Exercises 25-26. (Horizontal Tangent Lines) Find the points on the given curve at which a horizontal tangent line occurs.

25. $x^2+y^3-3y-2 = 0$

26. $xy^2 = 2y+2$

Exercises 27-28. (Normal Line) Determine the normal line to the given curve at the given point (See Exercises 52-53, page 88).

27. $\sqrt{x^2+y} = y^2-7x$ at $(1, 3)$
28. $xy+x^2-y^2 = 1$ at $(1, 1)$

Exercises 29-34. (Leibnitz Notation) Determine $\frac{dy}{dx}$.

29. $xy^2 = yx^2+3x-2y$
30. $xy^2 = yx^2+3$
31. $y^2 = \frac{xy}{x+1}$
32. $x^{2/3}+y = y^{2/3}+x$
33. $y^2\sin y = x+y$
34. $\frac{x+1}{y} = 2x+y^2$

Exercises 35-38. (Second Derivative) Determine y'' at the given point.

35. $x^2+y^2 = 25$ at $(3, -4)$
36. $x^3+3xy+y^2 = 5$ at $(1, 1)$
37. $\sqrt{x}+\sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$
38. $\sin(xy)+y^2 = 4$ at $(0, 2)$

Exercises 39-42. (Second Derivative) Express y'' in terms of x and y.

39. $x^3-y^3 = 1$
40. $x^3-2y^2 = 0$
41. $xy^3 = 12$
42. $\cos y^2 = xy+1$

Exercise 43-46. (Orthogonal Curves) Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given curves are orthogonal.

43. $xy = 2$ and $x^2-y^2 = 3$
44. $y^2-4x-4 = 0$ and $y^2-64+6x = 0$
45. $x^2+y^2 = 4$ and $2x+3y = 0$
46. $x^2+y^2 = x$ and $x^2+y^2 = 2y$

47. **(Theory)** Prove that the tangent line at any point on the circle $(x-x_0)^2+(y-y_0)^2 = r^2$ is perpendicular to the radius of the circle at that point.

48. **(Theory)** (a) Prove that the tangent line at any point $(\bar{x}, \bar{y})$ on the circle $x^2+y^2 = r^2$ is given by $y = -\frac{\bar{x}}{\bar{y}}x+\frac{r^2}{\bar{y}}$.

    (b) Verify directly that the formula in (a) holds at the points $(1, \sqrt{3})$ and $(1-\sqrt{3})$ on the circle $x^2+y^2 = 4$.

    (c) Generalize the result of (a) for any point $(\bar{x}, \bar{y})$ on the circle $(x-x_0)^2+(y-y_0)^2 = r^2$.

    (d) Find all points on the circle $x^2+y^2 = 4$ with tangent line passing through the point $(0, 2)$.

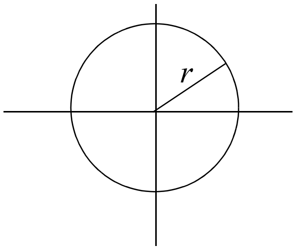
§5. RELATED RATES

Our concern in this section is to determine how the rates of change with respect to time, of certain quantities affect the rates of change of other quantities. The Leibniz notation (rather than the prime notation) will be utilized to underline the fact that the quantities are varying as a function of time.

> Throughout this section we will use the following geometrically plausible fact — roughly stated here, and formally established in the next chapter:
>
> $\boldsymbol{\frac{df}{dt} > 0 \Leftrightarrow f}$ **is increasing** $\boldsymbol{\frac{df}{dt} < 0 \Leftrightarrow f}$ **is decreasing**

EXAMPLE 3.19 The radius of a circle is increasing at the rate of 3 centimeters per minute. Determine the rate of change of its area when the radius is 4 cm.

SOLUTION: As it is with any word problem, the first step is to display the problem in a compact visual form. You want to "see the problem," and to have no further need to return to its initial verbal representation:



$$\frac{dr}{dt} = 3\frac{\text{cm}}{\text{min}}$$

$$\left.\frac{dA}{dt}\right|_{r=4} = ?$$

As you can see we are given the rate $\frac{dr}{dt}$ and want to find the rate $\frac{dA}{dt}$. The next step is to find a relation between the quantities in the numerators of those expressions [the $\boldsymbol{r}$ of $\frac{d\boldsymbol{r}}{dt}$ and $\boldsymbol{A}$ of $\frac{d\boldsymbol{A}}{dt}$]:

$$\boldsymbol{A} = \pi \boldsymbol{r}^2$$

Differentiating both sides of the above equation with respect to t, we have:

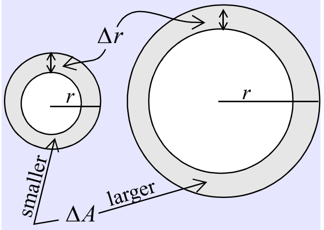
$$\frac{dA}{dt} = \pi \underset{\uparrow}{\boldsymbol{2r\frac{dr}{dt}}}$$

Since r is a function of time, $\frac{d}{dt}r^2 = \boldsymbol{2r\frac{dr}{dt}}$ (Chain Rule!)

Substituting 4 cm for r and $3\frac{\text{cm}}{\text{min}}$ for $\frac{dr}{dt}$ in the above equation we find that:

$$\frac{dA}{dt} = \pi 2 \cdot 4 \cdot 3 = 24\pi\frac{\text{cm}^2}{\text{min}}$$

Note that the rate of change of area with respect to time increases as r increases. This makes perfectly good geometrical sense:



Conclusion: At the instance of time when the radius is 4cm the area in increasing ($\frac{dA}{dt}$ is positive) at a rate of $24\pi\frac{\text{cm}^2}{\text{min}}$.

CHECK YOUR UNDERSTANDING 3.24

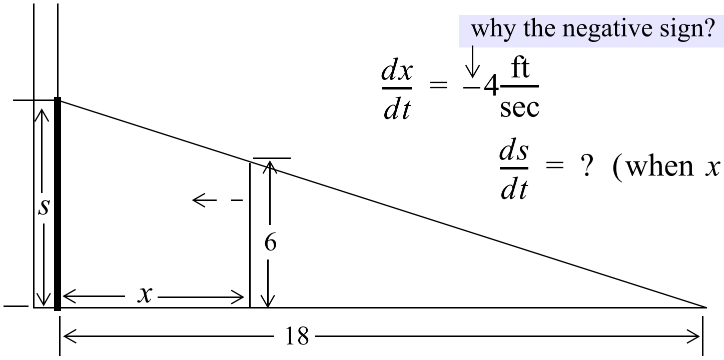
With reference to Example 3.19, determine the rate of change of the circumference of the circle with respect to time when $r = 12\text{cm}$.

Answer: $6\pi\frac{\text{cm}}{\text{min}}$

EXAMPLE 3.20

A spotlight on the ground shines on a wall 18 feet away. If a man 6 feet tall walks toward the wall at a speed of 4 feet per second, how fast is the length of his shadow changing when he is 5 feet from the wall?

Solution: Step 1. See the Problem:



why the negative sign?

$$\frac{dx}{dt} = -4\frac{\text{ft}}{\text{sec}}$$

$$\frac{ds}{dt} = ? \ (\text{when } x = 5\text{ft})$$

Step 2. Find a relation between the quantities in the numerators, x and s.

From the two similar triangles in the above figure:

$$\frac{s}{18} = \frac{6}{18-x}$$

$$s = \frac{108}{18-x} = 108(18-x)^{-1}$$

Two triangles are similar if they have the same set of angles; and while similar triangles have the same shape, they need not be of the same size. However: **The ratios of corresponding sides of similar triangles are equal.**

Step 3. Differentiate:

from where?

$$\frac{ds}{dt} = -108(18-x)^{-2}\left(-\frac{dx}{dt}\right)$$

Step 4. Evaluate at the specified instant of time. When $x = 5$:

$$\frac{ds}{dt} = -108(18-5)^{-2}[-(-4)] = -\frac{432}{13^2} \approx -2.6\frac{\text{ft}}{\text{sec}}$$

(His shadow is decreasing at the rate of 2.6 ft/sec)

CHECK YOUR UNDERSTANDING 3.25

A ladder 10 feet long is leaning against the side of a building, and its foot is being pulled away from the building at the rate of 2 feet per second. Determine the rate of change of the distance from the top of the ladder to the ground when it is 6 feet from the wall.

Answer: $\frac{3}{2}$ feet per second.

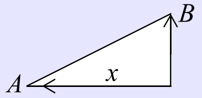
EXAMPLE 3.21

At noon ship *A* is 120 km east of ship *B*. Ship *A* is sailing west at 35 km/hr and ship *B* is sailing north at 25 km/hr. How fast is the distance between the ships changing at 3 PM?

SOLUTION: Typically, the "snap-shot" of interest (in this case the situation at 3 PM) only comes into play at the end of the solution process. In this example, however, we need to know if ship A is still east of ship B, and it is (see margin):

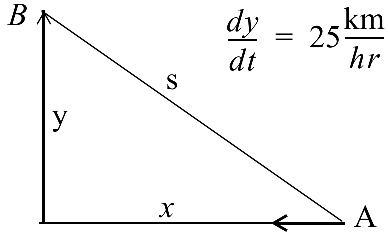
In 3 hours time ship *A* traveled $3 \cdot 35 = 105$ km. Since initially it was 120 km east of B, *A* will still be east of B at 3 PM. As such, x decreases with time.

Were *A* west of *B*:



then $\frac{dx}{dt}$ would be positive.

Step1.



$$\frac{dy}{dt} = 25\frac{\text{km}}{hr}$$

$$\frac{dx}{dt} = -35\frac{\text{km}}{hr}$$

$$\frac{ds}{dt} = ? \quad \text{(at 3 PM)}$$

Step2. $s^2 = x^2 + y^2$

Step 3.

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s}$$

Step 4. At 3 PM, $x = 120 - (3 \cdot 35) = 15\text{km}$, $y = 3 \cdot 25 = 75\text{km}$, and $s = \sqrt{15^2 + 75^2} = 15\sqrt{26}$.

So, at 3 PM:

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s} = \frac{15(-35) + 75(25)}{15\sqrt{26}} = \frac{90}{\sqrt{26}} \approx 17.7\frac{\text{km}}{\text{hr}}$$

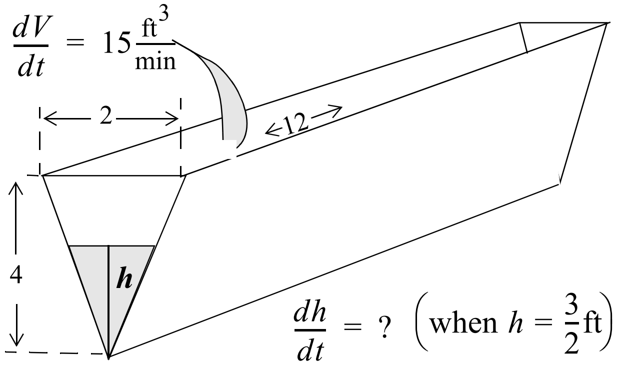
CHECK YOUR UNDERSTANDING 3.26

Referring to Example 3.21, find the rate of change of the distance between the two ships at 4 PM.

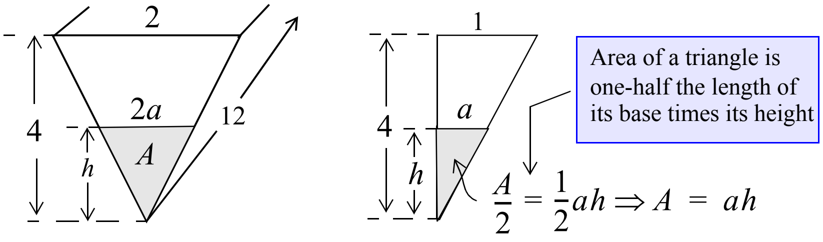
Answer: $\frac{160}{\sqrt{26}} \approx 31.4\frac{\text{km}}{\text{hr}}$

EXAMPLE 3.22 Water is flowing into a 12 foot trough at a rate of $15\frac{\text{ft}^3}{\text{min}}$. The cross sections of the trough are inverted isosceles triangles with base of length 2 feet and height of length 4 feet. How fast is the water level rising when the water level is 18 inches deep?

SOLUTION:



We need to relate the water volume V and the water height h. The water volume V is given by the area A of the shaded triangle in Figure 3.11(a) times the length of the trough (12 ft): $V = 12A$. Cutting that triangle in half, brings us to Figure 3.11(b).



(a) (b)

Figure 3.11

At this point we have the relation $V = 12A = 12ah$. From the similar triangles of Figure 3.11(b) we see that $\frac{a}{h} = \frac{1}{4} \Rightarrow a = \frac{h}{4}$. Thus:

$$V = 12 \cdot \frac{h}{4} \cdot h = 3h^2$$

Differentiating:

$$\frac{dV}{dt} = 6h\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{1}{6h} = \frac{15}{6h} = \frac{5}{2h}$$

In particular, when $h = 18 \text{ in.} = \frac{3}{2}\text{ft}$: $\frac{dh}{dt} = \frac{5}{2 \cdot \frac{3}{2}} = \frac{5}{3}\,\frac{\text{ft}}{\text{min}}$

Conclusion: The water level is rising ($\frac{dh}{dt} > 0$) at the rate of $\frac{5}{3}\,\frac{\text{ft}}{\text{min}}$.

CHECK YOUR UNDERSTANDING 3.27

Water is leaking out from the bottom of a cone-shaped cup at a constant rate of $c\,\frac{\text{in}^3}{\text{min}}$. The cup is 16 inches across the top and 32 inches deep. Determine c, given that the depth of water is decreasing at a rate of 2 inches per minute at the instant of time when the water depth is 4 inches.

(Volume of a cone of radius r and height h: $V = \frac{1}{3}\pi r^2 h$)

Answer: $2\pi\frac{\text{in}^3}{\text{min}}$

EXERCISES

1. **(Cube)** The edge x of a cube is increasing at the rate of $1\frac{\text{cm}}{\text{min}}$. Determine:
   (a) The rate of change of the volume of the cube when $x = 50 \text{ cm}$.
   (b) The rate of change of the surface area of the cube when $x = 50 \text{ cm}$.
   (c) The rate of change of the volume of the cube when its surface area is 2400 cm^2.

2. **(Circle)** The radius r of a circle is increasing at the rate of $1\frac{\text{cm}}{\text{min}}$. Determine:
   (a) The rate of change of the area of the circle when $r = 50 \text{ cm}$.
   (b) The rate of change of the circumference of the circle when $r = 50 \text{ cm}$.
   (c) The rate of change of the area of the circle when its circumference is $40\pi \text{ cm}$.

3. **(Sphere)** The radius r of a sphere is decreasing at the rate of $1\frac{\text{cm}}{\text{min}}$. Determine:
   (a) The rate of change of the volume of the sphere when $r = 50 \text{ cm}$. Note: $V = \frac{4}{3}\pi r^3$.
   (b) The rate of change of the surface area of the sphere when $r = 50 \text{ cm}$. Note: $S = 4\pi r^2$.
   (c) The rate of change of the volume of the sphere when its surface area is $1600\pi \text{ cm}^2$.

4. **(Cone)** The radius r of a cone is increasing at a rate of 2 inches per second while its height is decreasing in such a way that the volume remains constant at $12\pi \text{ in}^3$. At what rate is the height decreasing when the radius is 1 inch? Note: $V = \frac{1}{3}\pi r^2 h$.

5. **(Cylinder)** The radius r of a cylinder is decreasing at the rate of $4\frac{\text{ft}}{\text{min}}$ and its height is increasing at the rate of $2\frac{\text{ft}}{\text{min}}$? Find the rate of change of the volume when the radius of the cylinder is 2 feet and its height is 3 feet.

6. **(Cylinder)** The radius r of a cylinder is increasing at 2 inches per second. When the radius is 4 inches, the volume is 400 in^3 and is increasing at $24\pi\frac{\text{in}^3}{\text{sec}}$. How fast is the height of the cylinder increasing at that instant?

7. **(Circular Ripples)** A stone is dropped into a pool of water creating a series of concentric circular ripples.
   (a) At what rate is the area of the outer circle changing when its diameter is 6 feet, if the diameter of that outer ripple is changing at a constant rate of 4 feet per second?
   (b) At what rate is the diameter of the outer circle changing when its diameter is 3 feet, if the area of the outer circle is changing at a constant rate of 30 square feet per second?

8. **(Rectangle)** One side or a rectangle is 5 cm longer than the other side. Both sides are increasing at a rate of $10\frac{\text{cm}}{\text{min}}$.
   (a) How fast is the area (A) of the rectangle increasing when the length of the shorter side is 50 cm?
   (b) How fast is the perimeter (P) of the rectangle increasing when the length of the shorter side is 50 cm?
   (c) How fast is the diagonal (S) of the rectangle increasing when the length of the shorter side is 50 cm?

9. **(Rectangle)** The length l of a rectangle is increasing at a rate of $25\frac{\text{cm}}{\text{min}}$ and its width w is decreasing at a constant rate $c\frac{\text{cm}}{\text{min}}$. Determine c if its area is increasing at a rate of $250\frac{\text{cm}^2}{\text{min}}$ when $l = 25 \text{ cm}$ and $w = 20 \text{ cm}$.

10. **(Rectangle)** The length l of a rectangle is increasing at a rate of $1\frac{\text{cm}}{\text{sec}}$. Find the value of l at which the area of the rectangle starts to decrease if the perimeter of the rectangle is held fixed at 20 cm.

11. **(Rectangle)** The length of a rectangle is increasing at 2 inches per second. Determine the rate of change of the area when the rectangle is a square, if the perimeter remains constant at 42 inches.

12. **(Rectangle)** The length of a rectangle is increasing at 2 inches per second. How fast is the perimeter increasing when the length is 6 inches if its width decreases in such a way that the area remains constant at 24 square inches?

13. **(Equilateral Triangle)** At a certain instant of time the sides of an equilateral triangle are 1 inch long and increasing at a rate of $1\ \frac{\text{in}}{\text{min}}$. Determine:
    (a) The rate of change of the area of the triangle.
    (b) The rate of change of the perimeter of the triangle.
    (c) The rate of change of an angle of the triangle.

14. **(Shadow)** A man 6 feet tall is walking away from a 24 foot lamppost at a rate of 3 feet per second. At what rate is the end of his shadow moving away from him?

15. **(Ladder)** A ladder 12 feet long is leaning against the side of a building, and its foot is being pulled away from the building at the rate of 1 foot per second. Determine the rate of change of the angle formed by the ladder and the ground when the top of the ladder is 9 feet from the ground.

16. **(Equilateral Triangle)** The area of an equilateral triangle is 5 in.^2 and it is increasing at the rate of $5\frac{\text{in.}^2}{\text{min}}$. At what rate is the side of the triangle increasing at that time?

17. **(Triangle)** The base of a triangle is increasing at the rate of 3 inches per minute, while the altitude is decreasing at the same rate. At what rate is the area changing when:
    (a) The base is 7 inches and the altitude is 6 inches?
    (b) The altitude is 7 inches and the base is 6 inches?

18. **(Triangle)** The altitude of a triangle is increasing at a rate of $1\dfrac{\text{cm}}{\text{min}}$ and its base is increasing at a rate of $2\dfrac{\text{cm}}{\text{min}}$. At what rate is the area of the triangle increasing when its height 15 cm and its area is 90 cm^2?

19. **(Isosceles Triangle)** The base of an isosceles triangle is held constant at 24 inches. At what rate is the vertex angle changing at the instant of time when the altitude is 12 inches and is increasing at the rate of 1 inch per minute?

20. **(Balloon)** A spherical balloon is expanding in such a way that its radius is increasing at a rate proportional to its surface area. Show that the surface area is increasing at a rate proportional to its volume.

21. **(Mothball)** A spherical mothball evaporates in such a way that its volume decreases at a rate proportional to its surface area. Show that the radius decreases at a constant rate.

    Note: $V = \dfrac{4}{3}\pi r^3, S = 4\pi r^2$.

22. **(Balloon)** A balloon rises vertically at a rate of 200 feet per minute, from a point on the ground that is 500 feet from an observer. Determine:
    (a) The rate of change of the distance between the observer and the balloon at the instant when the balloon is 600 feet above the ground.
    (b) The rate of change of the angle of inclination of the observer's line of sight when the balloon is 500 feet above the ground.

23. **(Sand)** The volume of a cone is increasing at a constant rate of 2 cubic feet per minute in such a way that the height of the cone is always equals to its diameter. (Note: $V = \dfrac{1}{3}\pi r^2 h$.)
    (a) At what rate is the height of the cone changing when the height is 2 feet?
    (b) At what rate is the radius of the cone changing when the height is 2 feet?
    (c) At what rates are the radius and the height of the cone changing when the volume is 35 cubic feet?

24. **(Boat)** A boat is pulled toward a dock by a rope attached to the bow of the boat and passing through a ring on the dock that is 6 feet higher than the bow of the boat.
    (a) How fast is the boat approaching the dock when it is 12 feet from the dock, if the rope is pulled in at a rate of 1 foot per second?
    (b) At what constant rate must the rope be pulled for the boat to approach the dock at 1 foot per second when it is 12 feet from the dock?

25. **(Boyle's Law)** A gas occupies a volume of 1000 in^3 and is subjected to a pressure of $1\dfrac{\text{lb}}{\text{in.}^2}$

    Find the rate at which the pressure is changing at the instant when the volume is 800in.^3 if the gas is being compressed at a rate of $4\dfrac{\text{in.}^3}{\text{min}}$.

    Use Boyle's Law: $\text{pressure} \times \text{volume} = \text{constant}$.

26. **(Two Ships)** At noon ship A is 200 km west of ship B. Ship A is sailing south at $25\frac{\text{km}}{\text{hr}}$ and ship B is sailing north at $35\frac{\text{km}}{\text{hr}}$. How fast is the distance between the ships changing at 2:00 PM?

27. **(Walking)** At 1 PM a man starts walking north at a rate of $300\ \frac{\text{ft}}{\text{min}}$ from a point P. Five minutes later, a woman starts walking east at a rate of $250\ \frac{\text{ft}}{\text{min}}$ from a point Q that is 1000 feet west of P. How fast is the distance between the two individuals changing at:
    (a)1:08 PM? (b) 1:10PM?

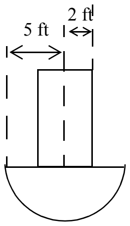
28. **(Particle)** A particle moves along the curve $x^3 - 3y + 2 = 0$. Find the points on the curve at which the x-coordinate is increasing 9 times faster than its y-coordinate.

29. **(Water)** Water is leaking out of an inverted conical tank of height 120 inches and radius 10 inches at a rate of $3\frac{\text{in}^3}{\text{sec}}$, while water is being pumped into the tank at a constant rate. (Note: $V = \frac{1}{3}\pi r^2 h$.) Find that constant rate if:

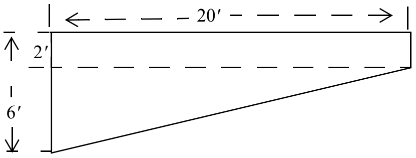
    (a) The water level is rising at a rate of $6\frac{\text{in}}{\text{sec}}$ when the height of the water is 40 inches.

    (b) The volume of water is decreasing at a rate of $1\frac{\text{in}^3}{\text{sec}}$.

30. **(Water)** Water is pumped into a tank at the rate of 75 cubic feet per minute. The tank consists of a cylinder of radius 2 feet, centered at the top of a hemisphere of radius 5 feet. (Volume of sphere: $\frac{4}{3}\pi r^3$). How fast is the water level rising when the water is:
    (a) 3 feet deep? (b) 7 feet deep? (c) 5 feet deep?

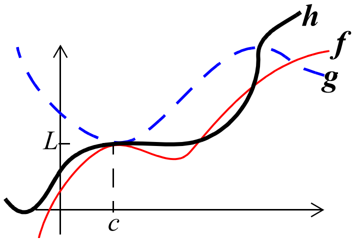


31. **(Swimming Pool)** A rectangular swimming pool 20 feet long and 10 feet wide is 6 feet deep at one end and 2 feet deep at the other. Water is pumped into the empty pool at the rate of $500\frac{\text{ft}^3}{\text{min}}$. At what rate is the water level rising when it is:
    (a) 2 feet deep at the deep end? (b) 5 feet deep at the deep end?



CHAPTER SUMMARY

DERIVATIVE AT A POINT:	The **derivative of a function $y = f(x)$ at c** is the number: $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ or: $\left.\frac{dy}{dx}\right\rvert_{x=c} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}$
GEOMETRICAL INTERPRETATION:	The slope of the tangent line to the graph of the function f at the point $(c, f(c))$.
DERIVATIVE FUNCTION:	The **derivative of a function $y = f(x)$** is the function: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
ALTERNATE NOTATION:	or: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
THEOREM:	If a function is differentiable at c, then it is continuous at c.
DERIVATIVE FORMULAS:	The derivative of any constant function is 0. For any real number r: $(x^r)' = rx^{r-1}$. For any real number r and any differentiable function f: $[rf(x)]' = rf'(x)$ If f and g are differentiable then: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$ $[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$ $\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:	$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\frac{d}{dx}(\sec x) = \sec x \tan x$
THE CHAIN RULE:	If f is differentiable at x and g is differentiable at $f(x)$, then the composite function $g \circ f$ is differentiable at x, and: $(g \circ f)'(x) = g'[f(x)]f'(x)$

GENERALIZED POWER RULE:	If f is differentiable at x then so is the function $[f(x)]^r$, and: $([f(x)]^r)' = r[f(x)]^{r-1}f'(x)$
PINCHING THEOREM:	Let f, g, and h be such that within an interval about c: $f(x) \le h(x) \le g(x)$ If: $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = L$ then $\lim\limits_{x \to c} h(x) = L$ 
IMPLICIT DIFFERENTIATION:	AN ILLUSTRATION: Differentiating both sides of the equation $2x + 3y = x^2y^3$ with respect to x, we have: $2 + 3y' = x^2(3y^2y') + y^3 2x$
RELATED RATES PROCEDURE:	Step 1. See the problem: Draw a diagram which includes variables representing the quantities that vary. Specify the given rate(s) of change, and the rate of change you are looking for. Step 2. Find an equation involving the variables in Step 1. Step 3. Differentiate both sides of the equation with respect to time t. Step 4. Calculate the desired rate of change at the specified instance of time.

CHAPTER 4

THE MEAN VALUE THEOREM AND APPLICATIONS

§1. THE MEAN VALUE THEOREM

We begin by introducing a main character of this section, one that plays an essential role in the development of the calculus:

THEOREM 4.1

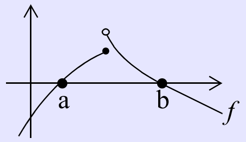
ROLLE'S THEOREM

Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) for which $f'(c) = 0$.

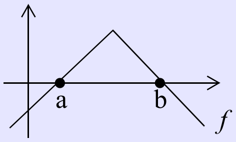
A slightly weaker version: If f is differentiable on $[a, b]$ and if $f(a) = f(b) = 0$ then there is at least one number c in (a, b) for which: $f'(c) = 0$ [Recall that differentiability implies continuity]

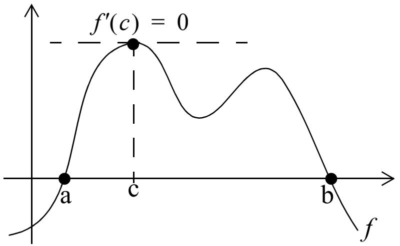
A (partial) proof of the above result is offered at the end of the section. For now, we suggest that Rolle's Theorem is geometrically "believable." It is saying that the graph of any "nice" function [continuous on $[a, b]$ and differentiable on (a, b)] whose graph passes through the points $(a, 0)$ and $(b, 0)$ must encounter at least one horizontal tangent line along the way (see Figure 4.1).

continuity is needed:



differentiability is needed:





Rolle's Theorem
Figure 4.1

As a consequence of Rolle's Theorem, we have:

THEOREM 4.2

MEAN VALUE THEOREM

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) for which:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem is a special case of this theorem, in that if $f(a) = f(b) = 0$ then:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

for some c in (a, b).

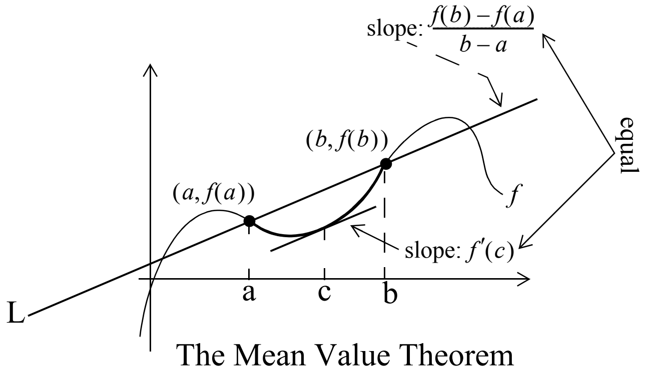
Indeed if $f(a) = f(b)$ (not necessarily 0), then we still must have that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

for some c in (a, b).

Before turning to a proof of the above theorem, lets acknowledge what it is saying:

> If f satisfies the conditions of the theorem, then there exists at least one number $a < c < b$ such that the slope of the tangent line at the point $(c, f(c))$ is parallel to the slope of the line passing through the points $(a, f(a))$ and $(b, f(b))$ [see Figure 4.2].



The Mean Value Theorem

Figure 4.2

PROOF: Let $L(x)$ denote the linear function whose graph is the line passing through the points $(a, f(a))$, $(b, f(b))$.

Consider the function $H(x) = f(x) - L(x)$. Since L is differentiable everywhere, and since f is continuous on $[a, b]$ and differentiable on (a, b), H is again continuous on $[a, b]$ and differentiable on (a, b). Moreover, since $f(a) = L(a)$ and $f(b) = L(b)$, $H(a) = H(b) = 0$. Applying Rolle's Theorem to the function H, we choose $c \in (a, b)$ for which $H'(c) = 0$. Since $H(x) = f(x) - L(x)$:

$$H'(c) = f'(c) - L'(c) = 0 \Rightarrow f'(c) = L'(c)$$

Noting that $L'(c)$ is the slope of the line L (why?), we have:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Figure 4.2 displays the geometrical interpretation of the Mean Value Theorem. Here is an analytical interpretation:

In particular: On any trip, there will be at least one instant of time at which your instantaneous velocity matches the average velocity of the trip.

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number $c \in (a, b)$ where the instantaneous rate of change of the function at c, $[f'(c)]$, equals the average rate of change of the function over the interval $[a, b]$: $\frac{f(b)-f(a)}{b-a}$.

CHECK YOUR UNDERSTANDING 4.1

(a) Rolle's Theorem assures us that the graph of the function $f(x) = x^4 - 2x^2$ has at least one horizontal tangent line in the interval $(-\sqrt{2}, \sqrt{2})$ (how so?). Find all $c \in (-\sqrt{2}, \sqrt{2})$ such that $f'(c) = 0$.

(b) The Mean Value Theorem assures us that in any given interval $[a, b]$ there will exist at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Find all such c for the function $f(x) = x^3 - x^2$ within the interval $[-2, 3]$.

Answers: (a) $-1, 0, 1$.
(b) $\frac{1 \pm \sqrt{19}}{3}$

The Mean Value Theorem contains two conditions:

IF **(1)** f is continuous on $[a, b]$ and
(2) f is differentiable on (a, b)
THEN: $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some $a < c < b$.

There are, however, functions which are neither continuous on $[a, b]$ nor differentiable on (a, b) for which the conclusion of the Mean Value Theorem still holds. Consider the following CYU:

CHECK YOUR UNDERSTANDING 4.2

(a) Show that the function $f(x) = \begin{cases} |x| & \text{if } x \neq 1 \\ -1 & \text{if } x = 1 \end{cases}$ is not continuous on $[-1, 1]$ and is not differentiable on $(-1, 1)$.

(b) Find a c in the interval $(-1, 1)$ for which $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$.

Answer: See page A-17.

SOME PARTICULARLY IMPORTANT CONSEQUENCES OF THE MEAN VALUE THEOREM

Geometrically speaking, a function is increasing where its graph is climbing, and it is decreasing where its graph is falling. To be more precise:

DEFINITION 4.1

INCREASING AND DECREASING FUNCTIONS

A function f is:

(a) **Increasing** on an interval I if for every x_1, x_2 in I: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

(b) **Decreasing** on an interval I if for every x_1, x_2 in I: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Since the derivative at a point on the graph of a function corresponds to the slope of the tangent line at that point, it should come as no surprise to find that:

THEOREM 4.3 Let f be differentiable on the open interval $I = (a, b)$ [or (a, ∞) or $(-\infty, b)$].

(a) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.

(b) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

(c) If $f'(x) = 0$ for all $x \in I$, then f is constant on I.

PROOF: The Mean Value Theorem assures us that for any $a < x_1 < x_2 < b$, there exists a number $c \in (x_1, x_2)$ such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Noting that the denominator $x_2 - x_1$ is positive, we can conclude that:

$$\text{[For (a)]: } f'(c) > 0 \Rightarrow f(x_2) - f(x_1) > 0$$

$$\text{[For (b)]: } f'(c) < 0 \Rightarrow f(x_2) - f(x_1) < 0$$

$$\text{[For (c)]: } f'(c) = 0 \Rightarrow f(x_2) = f(x_1)$$

CHECK YOUR UNDERSTANDING 4.3

Prove that if $f'(x) = g'(x)$ for every x in an open interval I, then f and g differ by a constant on that interval.

Suggestion: Consider the function $h(x) = f(x) - g(x)$

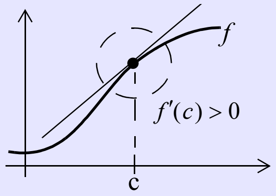
Answer: See age A-18.

The following result may also be anticipated (see margin):

THEOREM 4.4 Let f be differentiable at c.

(a) If $f'(c) > 0$, then there exists $\delta > 0$ such that $f(x) > f(c)$ for all $x \in (c, c + \delta)$, and $f(x) < f(c)$ for all $x \in (c - \delta, c)$.

(b) If $f'(c) < 0$, then there exists $\delta > 0$ such that $f(x) < f(c)$ for all $x \in (c, c + \delta)$ and $f(x) > f(c)$ for all $x \in (c - \delta, c)$.



If the tangent line has a positive slope at $(c, f(c))$, then near that point the graph is climbing. [Similarly for (b)]

PROOF: (a) Let $f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} > 0$. Letting the positive number $f'(c)$ play the role of ε in the definition of the limit (see margin), we can find $\delta > 0$ such that:

$\lim_{x \to c} f(x) = L$:
Given $\varepsilon > 0$ there exist $\delta > 0$ such that:
$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$$0 < |h| < \delta \Rightarrow \left|\frac{f(c + h) - f(c)}{h} - f'(c)\right| < f'(c)$$

$$\Rightarrow -f'(c) < \frac{f(c + h) - f(c)}{h} - f'(c) < f'(c)$$

$$\Rightarrow 0 < \frac{f(c + h) - f(c)}{h} < 2f'(c)$$

In particular, $f(c + h) - f(c)$ must be positive for any $0 < h < \delta$, while $f(c + h) - f(c)$ must be negative for any $-\delta < h < 0$.

Now it's your turn:

CHECK YOUR UNDERSTANDING 4.4

Prove Theorem 4.4(b).

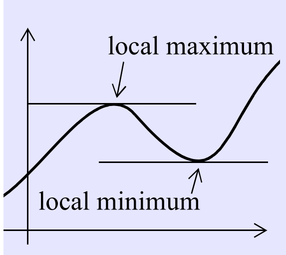
Answer: See page A-18.

More precisely
A local maximum occurs at c if there exist $\varepsilon > 0$ such that $f(c) \geq f(x)$ for every point $x \in (c - \varepsilon, c + \varepsilon)$ that lie in the domain of f.

DEFINITION 4.2
LOCAL EXTREMES

A function f has a **local** (or relative) **maximum** at a point c in its domain if $f(c) \geq f(x)$ for all x in its domain that are sufficiently close to c.

A function f has a **local** (or relative) **minimum** at c if $f(c) \leq f(x)$ for all x sufficiently close to c.



Another totally believable result (see margin):

THEOREM 4.5 Let f be differentiable in some open interval containing c. If f has a local maximum or a local minimum at c, then $f'(c) = 0$.

PROOF: Suppose that f has a local maximum at c.

Can $f'(c)$ be positive? No, for if it were positive then there would be x's immediately to the right of c with function values greater than $f(c)$ [Theorem 4.4(a)].

Can $f'(c)$ be negative? No, for if it were negative then there would be x's immediately to the left of c with function values greater than $f(c)$ [Theorem 4.4(b)].

Since $f'(c)$ exists and cannot be positive or negative, it must be 0.

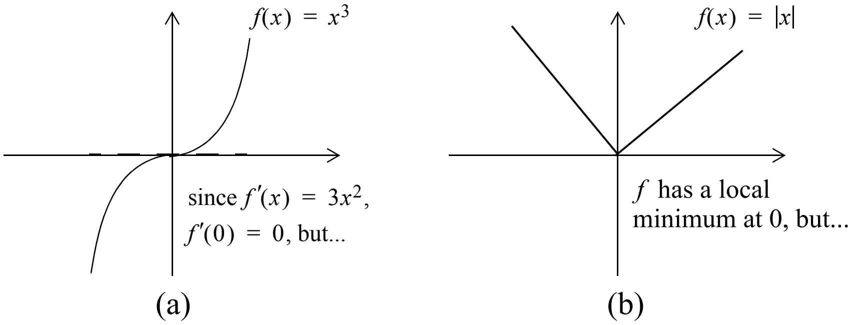
CHECK YOUR UNDERSTANDING 4.5

Establish the "minimum part" of Theorem 4.5.

Answer: See page A-18.

It is important that you do not read into Theorem 4.5 more than that which it is saying. In particular:

- There can be a zero derivative at c **without** either a local maximum or local minimum occurring at that point [see Figure 4.3(a)].
- A maximum or minimum can occur at c **without** the derivative assuming a value of zero at c [see Figure 4.3(b)]



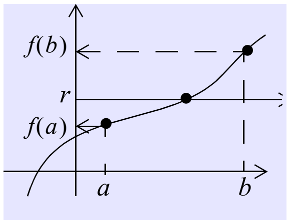
(a) (b)

Figure 4.3

The end-point situation is discussed in the next section.

To summarize: If c is an interior point in the domain of a function f at which a local maximum or minimum occurs, then either $f'(c) = 0$ or $f'(c)$ does not exist. The points in the domain of f where the derivative is zero or does not exist are called **critical points**. Local maxima and minima occur **among** those points.

SOME ADDITIONAL POINTS OF INTEREST



Intuitively speaking, if a function f is continuous on the closed interval $[a, b]$, then you can sketch its graph from $(a, f(a))$ to $(b, f(b))$ without lifting the writing utensil. Continuing along the path of intuition, one can then anticipate that the graph of f must certainly cross the horizontal line $f(x) = r$ for every r lying between $f(a)$ and $f(b)$ (see margin). Intuition is right on target:

A proof of this believable theorem lies outside the scope of this text.

THEOREM 4.6
INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and if r is a number lying between $f(a)$ and $f(b)$, then there exists at least one c between a and b such that $f(c) = r$.

Both the Intermediate Value Theorem and Rolle's Theorem come into play in the solution of the following Example.

EXAMPLE 4.1 Show that the equation $2x^5 + x^3 + 7x - 2 = 0$ has exactly one solution.

SOLUTION: We show that the graph of the function $f(x) = 2x^5 + x^3 + 7x - 2$ has **one and only one** x-intercept; which is to say, that $f(x) = 0$ for one and only one value of x:

f has at least one x-intercept:

Since f is continuous and since $f(0) < 0$ and $f(1) > 0$, the Intermediate Value theorem assures us that $f(r) = 0$ for some $r \in (0, 1)$.

f cannot have more that one x-intercept:

We show that the assumption that there are two or more x-intercepts leads to a contradiction (see margin):

One can prove that a proposition P is True by demonstrating that the assumption that P is False leads to a false conclusion (something like $1 = 2$).
For:
Logic dictates that from Truth, only Truth can follow. So, if the assumption that P is False leads to a false conclusion, then the assumption that P is false must itself be false. In other words:
P must be True.

Assume that for some $a < b$: $f(a) = f(b) = 0$.

Rolle's Theorem tells us that there exists some $c \in (a, b)$ for which $f'(c) = 0$.

But $f'(x) = (2x^5 + x^3 + 7x - 2)' = 10x^4 + 3x^2 + 7 \geq 7$ for all x — a **contradiction.**

CHECK YOUR UNDERSTANDING 4.6

Show that the equation $2x^4 - x + 10 = 0$ can have at most two distinct solutions.

Answer: See page A-18.

We employ the Intermediate Value Theorem to establish the following *Fixed-Point* result:

The notation $f: [0, 1] \to [0, 1]$ indicates that for every $x \in [0, 1] : f(x) \in [0, 1]$.

EXAMPLE 4.2 Let $f: [0, 1] \to [0, 1]$ be continuous (see margin). Show that there is at least one $c \in [0, 1]$ such that $f(c) = c$.

SOLUTION: **Case 1**. If $f(0) = 0$ or $f(1) = 1$, then we are done.

Case 2. If $f(0) \neq 0$ and $f(1) \neq 1$ then, since $f: [0, 1] \to [0, 1]$: $f(0) > 0$ and $f(1) < 1$. It follows that the continuous function $g(x) = f(x) - x$ is positive at 0 and negative at 1:

$$g(0) = f(0) - 0 > 0 \text{ and } g(1) = f(1) - 1 < 0$$

Applying the Intermediate Value Theorem to the function g we conclude that for some $c \in (0, 1)$ $g(c) = 0$; which is to say:

$$f(c) - c = 0$$
$$\boldsymbol{f(c) = c}$$

CHECK YOUR UNDERSTANDING 4.7

Let f and g be continuous on $[a, b]$ and such that:

$$f(a) < g(a) < g(b) < f(b).$$

Show that there exists some $c \in (a, b)$ for which $f(c) = g(c)$.

Answer: See page A-18.

PROOF OF THE MEAN VALUE THEOREM

The following important result tells us that a continuous function on a closed interval will attain a maximum and a minimum value:

THEOREM 4.7
MAXIMUM-MINIMUM THEOREM

If f is continuous on the closed interval $[a, b]$, then there exists some $c \in [a, b]$ such that $f(c) \geq f(x)$ for every $x \in [a, b]$, and some number $d \in [a, b]$ such that $f(d) \leq f(x)$ for every $x \in [a, b]$.

A proof of this result lies outside the scope of this text.

Alright, we're cheating a bit by not proving the above result, but its all for a good cause, for we can now establish Rolle's Theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b) = 0$, then $f'(c) = 0$ for some $c \in (a, b)$.

PROOF: If f is the constant zero function on $[a, b]$, then any $c \in (a, b)$ will do the trick.

If f is not the constant zero function, then it must take on some positive or negative values in $[a, b]$. Assume the former (you are asked to deal with the other case in CYU 4.8).

Theorem 4.7 assures us that f assumes its maximum value at some $c \in [a, b]$. Indeed, c must be contained in (a, b), for f is assumed to take on positive values in $[a, b]$, and $f(a) = f(b) = 0$.

Can $f'(c)$ be positive? No, for Theorem 4.4(a) would imply that $f(x) > f(c)$ for some $x \in (c, b)$.

Can $f'(c)$ be negative? No, for Theorem 4.4(b) would imply that $f(x) > f(c)$ for some $x \in (a, c)$.

Since $f'(c)$ can not be positive or negative: $f'(c) = 0$.

CHECK YOUR UNDERSTANDING 4.8

Modify the above proof to accommodate the assumption that $f(x) < 0$ for some $x \in [a, b]$.

Answer: See page A-18.

EXERCISES

Exercises 1-9. (Theory) Verify that the function satisfies the conditions of the mean-value theorem on the indicated interval $[a, b]$ and then find a c for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

1. $f(x) = x^2; [-1, 1]$
2. $f(x) = x^2; [-1, 0]$
3. $f(x) = x^2; [-3, 2]$
4. $f(x) = x^3; [-1, 1]$
5. $f(x) = x^3; [-1, 0]$
6. $f(x) = x^3; [-3, 2]$
7. $f(x) = \sqrt{x + 5}; [-1, 4]$
8. $f(x) = 3 + \sqrt{1 - x^2}; [0, 1]$
9. $f(x) = \frac{x}{x + 1}; [0, 1]$

Exercises 10-30. (Consequences of the Intermediate and Mean Value Theorems)

10. (a) Show that if *f* is differentiable on $[a, b]$ and if its derivative is never 0, then $f(a) \neq f(b)$.

    (b) Show that if $f'(x) < 0$ on $[a, b]$ then $f(a) - f(b) > 0$.

11. Show that the function $f(x) = \frac{1}{x^2} - \frac{1}{4}$ does not have a zero derivative in the interval $[-2, 2]$, even though $f(2) = f(-2) = 0$. Explain how this does not violate Rolle's theorem.

12. Show that the function $f(x) = \frac{1}{x}$ does not have a derivative equal to $\frac{f(1) - f(-1)}{-1 - 1}$ in the interval $[-1, 1]$. Explain how this does not violate the Mean-Value Theorem.

13. (a) Sketch the graph of $f(x) = \begin{cases} 2x + 2 & \text{if } x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$.

    (b) Does the function satisfy the conditions of the Mean-Value Theorem over the interval [0,3]?

    (c) Does there exist a c for which $f'(c) = \frac{f(b) - f(a)}{b - a}$ in the interval [0,3]?

14. (a) Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$.

    (b) Does the function satisfy the conditions of the Mean-Value Theorem over the interval [0,2]?

    (c) Does there exist a c for which $f'(c) = \frac{f(b) - f(a)}{b - a}$ in the interval [0,3]?

15. Let *f* be differentiable on (a, b). Prove that if $f(x) = 0$ has two distinct solutions in (a, b), then $f'(x) = 0$ has at least one solution in (a, b).

16. Show that the equation $6x^5 + 13x + 1 = 0$ has exactly one real root.

17. Show that the equation $x^3 + 6x^2 + 15x - 23 = 0$ has exactly one real root.

18. Show that the equation $6x^4 - 7x + 1 = 0$ cannot have more that two distinct real roots.

19. Show that the equation $2x = 1 + \sin x$ has exactly one real root.

20. (a) Show that the equation $x^2 = x\sin x + \cos x$ can have at most two real roots.

    (b) Show that the equation $x^2 = x\sin x + \cos x$ has exactly two real roots.

21. Show that the equation $x^4 + 50x^2 - 300 = 0$ cannot have more that two distinct real roots.

22. Let f be differentiable on (a, b). Prove that if $f(x) = 0$ has two distinct solutions in (a, b), then $f'(x) = 0$ has at least one solution in (a, b).

23. Show that $|\sin b - \sin a| \le |b - a|$ for any real numbers a and b.

24. Let f and g be differentiable on $[a, b]$ with $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Show that $f(b) < g(b)$.

25. Two runners start the 100-yard dash and finish in a tie. Prove that at some time during the race they are running at the same speed.

26. Suppose that $f(0) = 6$ and $f'(x) \le 1$ for $0 \le x \le 2$. What is the largest possible value of $f(2)$?

27. Suppose that $f(0) = 6$ and $f'(x) \ge 1$ for $0 \le x \le 2$. What is the smallest possible value of $f(2)$?

28. A **fixed point** for a function f is a number c for which $f(c) = c$.

    (a) Let f be differentiable on $[a, b]$. Show that if f has two distinct fixed-points in $[a, b]$, then $f'(x)$ must equal 1 for some $x \in (a, b)$.

    If $f'(x) = 1$ for some $x \in (a, b)$ must f have at least one fixed-point in $[a, b]$?

    (a) Let f: $[0, 1] \to [0, 1]$ be continuous. Show that f has a fixed point $0 \le c \le 1$.

29. Suppose that f and g are differentiable on $[a, b]$ and that the graphs of the two functions intersect at $x = a$ and at $x = b$. Show that there is some point between a and b where the tangents to the graphs of f and g are parallel.

There is no question that graphing calculators can graph most functions better and faster than any of us, but this does not diminish the importance of this section, for:

Learning how to graph a function by hand reinforces an understanding of important concepts.

§2. GRAPHING FUNCTIONS

As you will see, the sign of f, f', and f'' (labeled SIGN f, SIGN f' and SIGN f'')[1] will come into play when sketching the graph of a function f. You already know that the graph lives above the x-axis where $f(x)$ is positive, and below the x-axis where it is negative. You also know that the graph is increasing where $f'(x)$ is positive and decreasing where it is negative. To see what additional information can be inferred from the sign of $f''(x)$ we call your attention to Figure 4.4

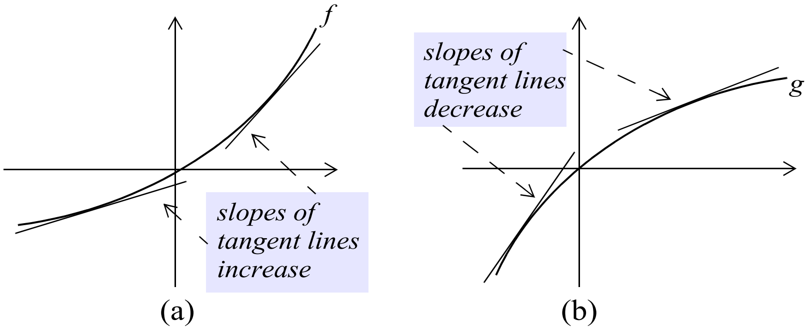


Figure 4.4

While SIGN f and SIGN g coincide (both functions are positive to the right of 0, and negative to its left), and while SIGN f' and SIGN g' also coincide (both are increasing functions), the graph of f is concave up (bending up, as you move from left to right) while that of g is concave down (bending down).

In this discussion we are assuming the existence of the second derivative.

Note that the slope of the tangent line increases as you move along the concave up curve of Figure 4.4(a), while it decreases along the concave down curve of Figure 4.4(b). But to say that the slope of the tangent line is increasing (or decreasing) is to say that $f'(x)$ is increasing (or decreasing); which is to say that $f''(x)$ is positive (or negative).

In the way of a definition:

If the second derivative of a function f exists at each point of an interval I, then

f is **concave up on I** if $f''(x) > 0$ for every $x \in I$.

f is **concave down on I** if $f''(x) < 0$ for every $x \in I$.

A point on the graph of a differentiable function about which concavity changes is called a **point of inflection.**

Summarizing:

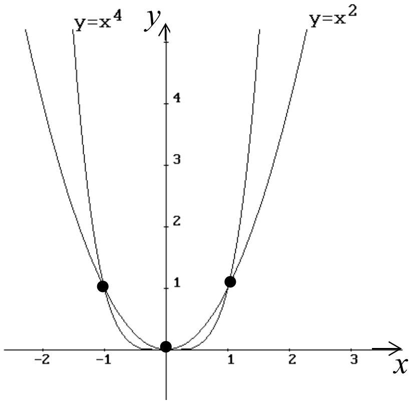
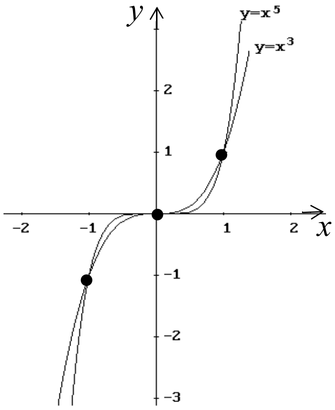
---

[1] We suggest you review the discussion on inequalities appearing in Section 1.3; specifically: pages 20-23 and pages 25-26.

SIGN f **+: graph lies above the x-axis** **-: graph lies below the x-axis**
(and consequently where the graph crosses the x-axis)
SIGN f' **+: graph is increasing** **-: graph is decreasing**
(and consequently where maximums and minimums occur)
SIGN f'' **+: graph is concave up** **-: graph is concave down**
(and consequently where inflection points occur)

GRAPHING POLYNOMIAL FUNCTIONS

We begin by noting that the graphs of the functions $f(x) = x^n$ (for n a positive integer) fall within two categories:

***n* even**	***n* odd**
The graph of every $y = x^{\text{even}}$ is similar to those of the functions $y = x^2$, and $y = x^4$ (below). Each such graph passes through the origin, and the points $(-1,1)$ and $(1,1)$. The larger the exponent, the flatter is the graph over $(-1, 1)$ and the steeper outside of $(-1, 1)$.	The graph of every $y = x^{\text{odd}}$ is similar to those of the functions $y = x^3$, and $y = x^5$ (below). Each such graph passes through the origin, and the points $(-1,-1)$ and $(1,1)$. The larger the exponent, the flatter is the graph over $(-1, 1)$ and the steeper outside of $(-1, 1)$
	

It is important to note that, in general:

Far away from the origin the graph of the polynomial function:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

resembles, in shape, that of its leading term $g(x) = a_n x^n$

The graph of a polynomial function and that of its leading term need not get arbitrarily close to each other as x tends to $\pm\infty$, but they will have similar shapes.

For example, as $x \to \pm\infty$, the graph of $p(x) = 6x^4 - 3x^3 - 6x + 1$ resembles that of $g(x) = 6x^4$. This makes sense since as x gets larger and larger in magnitude, the term $6x^4$ becomes more and more dominant.

Here is another way to arrive at the same observation:

$$p(x) = 6x^4 - 3x^3 - 6x + 1 = 6x^4\left(1 - \frac{3x^3}{6x^4} - \frac{6x}{6x^4} + \frac{1}{6x^4}\right)$$

$$= 6x^4\left(1 - \underbrace{\frac{1}{2x} - \frac{1}{x^3} + \frac{1}{6x^4}}_{\text{tend to 0 as } x \to \pm\infty}\right)$$

CHECK YOUR UNDERSTANDING 4.9

Determine a function $g(x) = ax^n$, whose graph resembles that of the given polynomial function f as $x \to \pm\infty$.

(a) $f(x) = x^3 - x$ (b) $f(x) = (2x^3 + x)(x^2 - 5x + 1)(x - 1)$

Answers: (a) $g(x) = x^3$
(b) $g(x) = 2x^6$

We now describe a procedure that can be used to graph polynomial functions when expressed in factored form. For our part:

WHEN GRAPHING A FUNCTION, WE WILL LET THE FUNCTION ITSELF DIRECT US, AS BEST IT CAN, TO ITS GRAPH; AND WILL THEN CALL ON THE CALCULUS TO CHALLENGE AND REFINE OUR INITIAL EFFORT.

EXAMPLE 4.3

Sketch the graph of :

$$f(x) = x^3 - 2x^2 - 15x$$

SOLUTION:

Step 1. **Factor:** $f(x) = x(x-5)(x+3)$

Step 2. **y-intercept:** $f(0) = 0$ [Y of Figure 4.5(a)].

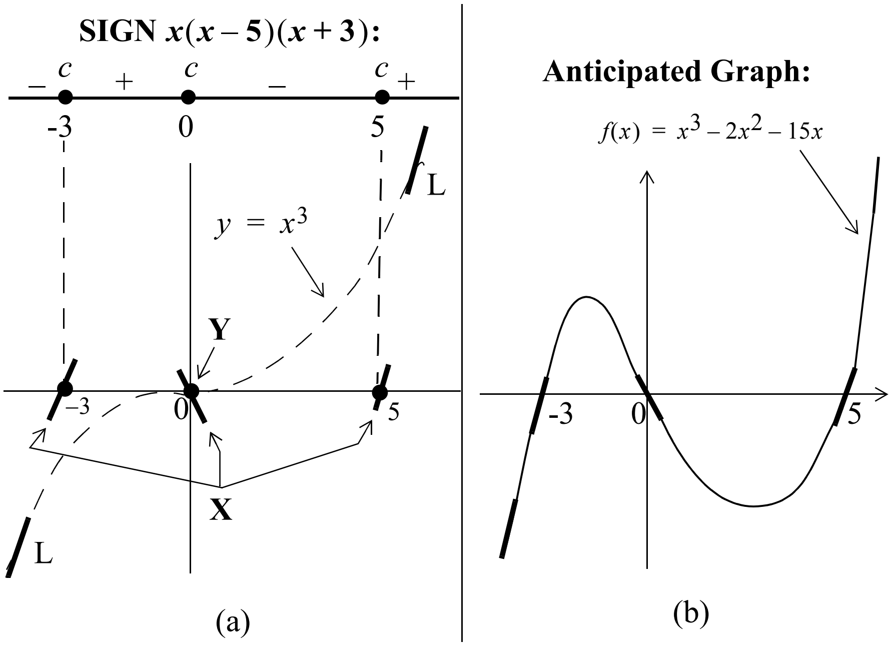
x-intercepts: $f(x) = x(x-5)(x+3) = 0$: at $x = 0$, 5, and -3 [X's of Figure 4.5(a)].

SIGN *f*: From the SIGN information at the top of Figure 4.5(a), we see that as you move from left to right, the graph crosses from below the x-axis to above the x-axis at -3; from above to below at 0; and from below to above at 5 [X's of Figure 4.5(a)].

The first three steps of the inequality procedure of Example 1.12, page 21, is used to determine **SIGN *f***.

Step 3. **As $x \to \pm\infty$:** The graph of f resembles that of its leading term $y = x^3$ [L's of Figure 4.5(a)].

Step 4. **Anticipated Graph:** With the above portion of the graph at hand, we can anticipate the general shape of the graph throughout its domain [Figure 4.5(b)].



(a) (b)

Figure 4.5

We now turn to the calculus to challenge the anticipated graph of Figure 4.5(b), and to determine precisely where maxima, minima and inflection points occur.

Step 5 (a) SIGN f' (Increasing/Decreasing; Maxima/Minima).

Figure 4.5(b) suggests that the function increases to a maximum value somewhere between −3 and 0, and then decreases to a minimum value somewhere between 0 and 5, and then increases forever more. And so it does:

$$f(x) = x^3 - 2x^2 - 15x$$

$$f'(x) = 3x^2 - 4x - 15 = (3x+5)(x-3)$$

	increasing	c	decreasing	c	increasing
SIGN f':	+	●	−	●	+
		$-\frac{5}{3}$		3	
		maximum		minimum	

Values: $f\left(-\frac{5}{3}\right) = \frac{400}{27}$ $\qquad f(3) = -36$

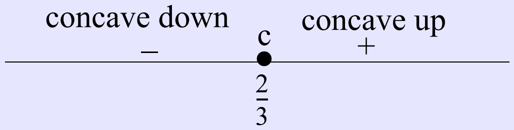
(as you can verify)

Step 5 (b) SIGN f'' (Concavity and Inflection Points). Figure 4.5(b) suggests that the graph is concave down from minus infinity to some point between its maximum and minimum points, and that it is concave up from that point to infinity. And so it is:

$$f'(x) = 3x^2 - 4x - 15$$

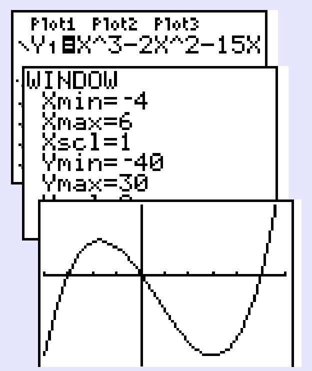
$$f''(x) = 6x - 4 \quad (f''(x) = 0 \text{ at } x = \tfrac{2}{3})$$

SIGN f'':



Inflection point: $(\frac{2}{3}, f(\frac{2}{3})) = (\frac{2}{3}, -\frac{286}{27})$

as you can verify



```
Plot1 Plot2 Plot3
\Y1■X^3-2X^2-15X
WINDOW
 Xmin=-4
 Xmax=6
 Xscl=1
 Ymin=-40
 Ymax=30
```

Yes, but how much does one learn?

Figure 4.6 reveals the fruit of our labor .

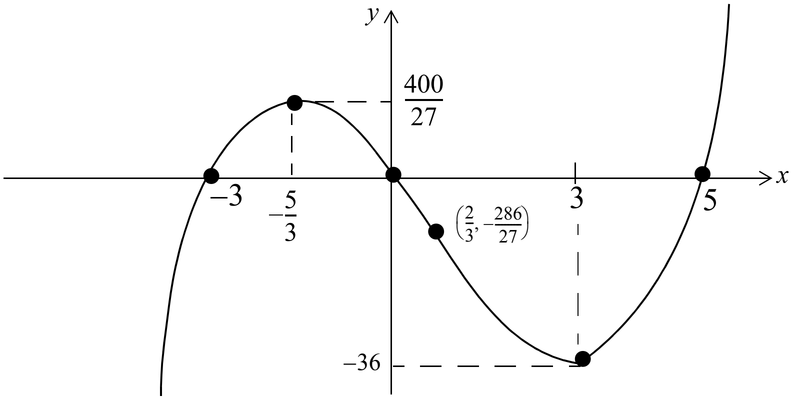


Figure 4.6

Note: Even though you may not be able to factor the cubic polynomial in $g(x) = x^3 - 2x^2 - 15x + 9$, you can still get a good sense of its graph by simply lifting the graph of the function $f(x) = x^3 - 2x^2 - 15x$ in Figure 4.6 9 units.

CHECK YOUR UNDERSTANDING 4.10

Sketch the graph of the given function:

(a) $f(x) = 3x^5 - 5x^3$ (b) $g(x) = 3x^5 - 5x^3 + 3$

Answer: See page A-19.

ENDPOINT EXTREMES

As you already know, a local maximum or local minimum can occur at an interior point c in the domain of a function f only if $f'(c) = 0$ or if f fails to be differentiable at c. You also know that **SIGN f'** reveals the specific nature of the function at those interior critical points. For example, **SIGN f'** in Figure 4.7(a) reveals the fact that the function f

has a (local) maximum at the interior point *a* and a (local) minimum at the interior point *c*.

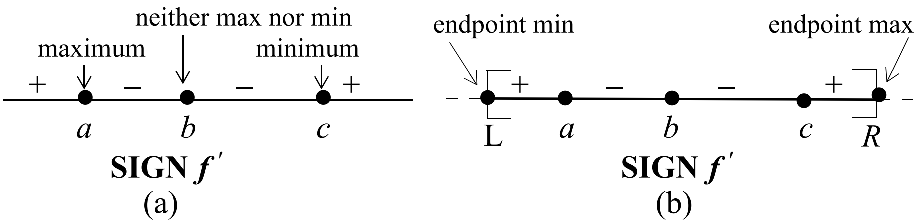
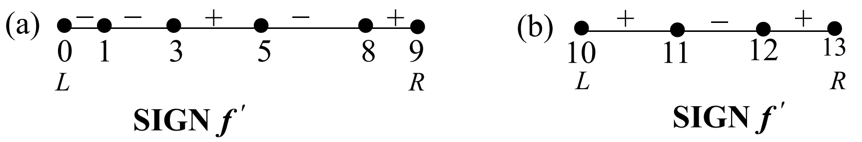


Figure 4.7

Now consider Figure 4.7(b) where the domain of *f* has been restricted to the closed interval with left endpoint *L* and right endpoint *R*. Note that since *f* is increasing to the right of *L* (why?), *L* is an **endpoint minimum**. Similarly, since *f* is increasing to the left of *R*, *R* is an **endpoint maximum**. Note that the maximum and minimum values of *f* over the closed interval $[L, R]$ of Figure 4.7(b) must occur at critical points or at endpoints (see Theorem, 4.7, page 128).

CHECK YOUR UNDERSTANDING 4.11

Indicate, from the given **SIGN** f', where *f* assumes a local or endpoint maximum and where it assumes a local or endpoint minimum value.



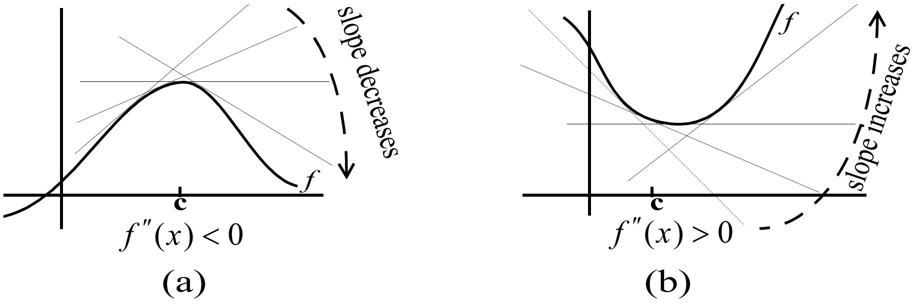
Answers: (a) Endpoint max at 0 and 9; local min at 3 and 8; local max at 5.

(b) Endpoint min at 10 and endpoint max at 13; local max at 11 and local min at 12.

SECOND DERIVATIVE TEST

So far, SIGN f' was used to find where the function *f* assumes maximum or minimum values — a method called the **first derivative test**. Here is another approach for your consideration:

> If a differentiable function *f* achieves a maximum value at *c*, then the slopes of the tangent lines: $f'(x)$, **decrease** as you move from the left to the right of *c* [see Figure 4.8(a)]. Consequently the derivative of $f'(x)$ at *c*, which is to say $f''(c)$, must be **negative** (providing it exists). If the function *f* achieves a minimum value at *c*, then the slopes of the tangent lines increase as you move from the left to the right of *c* [see Figure 4.8(b)]. Consequently, $f'(x)$ is **increasing** about the point *c*, and $f''(x)$ must be **positive** at *c* (providing it exists).



(a) (b)

Figure 4.8

Turning the above observations around, we have:

THEOREM 4.8
SECOND DERIVATIVE TEST

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(The second derivative test is inconclusive if $f''(c) = 0$, or if it does not exist.)

Exercise 54 addresses the $f''(c) = 0$-inconclusive issue.

EXAMPLE 4.4

Use the second derivative test to locate where the function, $f(x) = 3x^5 - 5x^3$, assumes maximum or minimum values.

SOLUTION: First:

$$f'(x) = (3x^5 - 5x^3)' = 0$$
$$15x^4 - 15x^2 = 0$$
$$15x^2(x^2 - 1) = 0$$
$$15x^2(x + 1)(x - 1) = 0$$
$$x = 0, x = -1, x = 1$$

Then:

$f''(x) = (15x^4 - 15x^2)' = 60x^3 - 30x$
Since $f''(0) = 0$, the second derivative test is **inconclusive**. [While the first derivative test is not — see CYU 4.10(a)]
Since $f''(-1) = -30$ is **negative**, a local **maximum** occurs at -1.
Since $f''(1) = 30$ is **positive**, a local **minimum** occurs at 1.

CHECK YOUR UNDERSTANDING 4.12

Use both the first and second derivative tests to determine where the function $f(x) = \dfrac{x^2}{x+1}$ assumes (local) maximum or minimum values. Any preference?

Answer: max: 0; min: –2

GRAPHING RATIONAL FUNCTIONS

In many ways, the procedure for sketching the graph of a rational function is quite similar to that for polynomial functions. To begin with:

Since the leading term of a polynomial function dominates its behavior far away from the origin:

As $x \to \pm\infty$, the graph of the rational function:

$$f(x) = \frac{\boldsymbol{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{\boldsymbol{b_m x^m} + b_{m-1}x^{m-1} + \cdots + b_0}$$

will resemble, **in shape**, that of: $g(x) = \dfrac{\boldsymbol{a_n x^n}}{\boldsymbol{b_m x^m}}$

For example, as $x \to \pm\infty$ the graphs of:

$$f(x) = \frac{\mathbf{3x^5} - 2x^3 - 3x}{\mathbf{2x^2} + x - 5} \text{ and } g(x) = \frac{\mathbf{3x^5}}{\mathbf{2x^2}} = \frac{3}{2}x^3$$

have similar shapes.

A SPECIAL CASE: When the degree of the numerator of a rational function f is less than or equal to the degree of the denominator, the graph will approach a horizontal line, called a **horizontal asymptote** for the graph of f.

For example, as $x \to \pm\infty$ the graph of $f(x) = \dfrac{\mathbf{2x^3} + 3x^2 - 5}{\mathbf{5x^3} + x}$ approaches the horizontal line $y = \dfrac{\mathbf{2x^3}}{\mathbf{5x^3}} = \dfrac{2}{5}$, and we say that $y = \dfrac{2}{5}$ is a horizontal asymptote for the graph of f.

The graph of $h(x) = \dfrac{\mathbf{2x^3} + 3x^2 - 5}{\mathbf{5x^4} + x}$ approaches that of the function $y = \dfrac{\mathbf{2x^3}}{\mathbf{5x^4}} = \dfrac{2}{5x}$, which approaches 0 as $x \to \pm\infty$. Consequently, the x-axis ($y = 0$) is a horizontal asymptote for the graph of h.

ANOTHER SPECIAL CASE: When the degree of the numerator of a rational function f is one more than that of the denominator, the graph will approach an oblique line, called an **oblique asymptote** for the graph of f.

For example, as $x \to \pm\infty$, the shape of the graph of $f(x) = \dfrac{\mathbf{2x^2}}{\mathbf{x+2}}$ will **resemble** that of a line of slope $\dfrac{\mathbf{2x^2}}{\mathbf{x}} = 2$. To be more precise, since:

$$f(x) = \frac{2x^2}{x+2} = 2x - 4 + \frac{8}{x+2} \quad \text{(see margin)}$$

the graph of f will get arbitrarily close to the line $y = 2x - 4$, and that line is the oblique asymptote for the graph of f.

$$\begin{array}{r|l} & \quad 2x - 4 \\ \hline x + 2 & 2x^2 \\ & 2x^2 + 4x \\ \hline & \quad -4x \\ & \quad -4x - 8 \\ \hline & \qquad 8 \end{array}$$

CHECK YOUR UNDERSTANDING 4.13

If the graph of the given function has a horizontal or oblique asymptote, determine the equation of the asymptote. If not, then specify a function whose graph resembles, in shape, that of the given function as $x \to \pm\infty$.

(a) $f(x) = \dfrac{3x^4 + 2x}{6x^4 - 5}$ (b) $f(x) = \dfrac{3x^4 + 2x}{6x^5 - 5}$

(c) $f(x) = \dfrac{4x^3 - 1}{2x^2 + x}$ (d) $f(x) = \dfrac{6x^6 - 5}{3x^4 + 2x}$

Answers: (a) Horizontal asymptote: $y = \frac{1}{2}$.
(b) Horizontal asymptote: $y = 0$.
(c) Oblique asymptote: $y = 2x - 1$.
(d) Resembles $y = 2x^2$.

VERTICAL ASYMPTOTES

The main difference between graphing polynomial functions and rational functions is that **vertical asymptotes** might come into play when graphing a rational function; where:

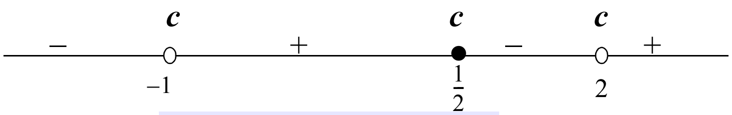
A **vertical asymptote** for the graph of a function *f* is a vertical line about which the graph tends to either plus or minus infinity.

The graph of a rational function $f(x) = \frac{p(x)}{q(x)}$ will approach a vertical asymptote at c where $q(c) = 0$ and $p(c) \neq 0$. A vertical asymptote need not occur at a point at which both the numerator and denominator of the rational expression are zero. (see Exercise 53).

Consider, for example, the function:

$$f(x) = \frac{2x - 1}{(x-2)(x+1)}$$

As x gets closer and closer to 2, the numerator $2x - 1$ approaches 3 while its denominator $(x-2)(x+1)$ shrinks to zero. It follows that, as x approaches 2, the quotient $\dfrac{2x-1}{(x-2)(x+1)}$ must tend to plus or minus infinity. From the sign information:

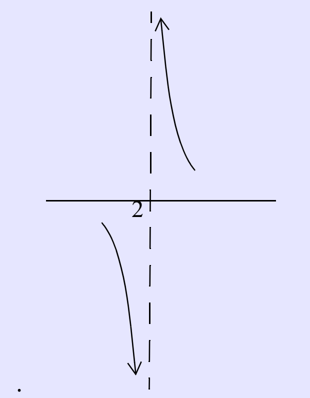


SIGN $f(x) = \dfrac{2x-1}{(x-2)(x+1)}$

we conclude that:

As x approaches 2 from the left, the values of $f(x)$, being negative, tend to $-\infty$ (see margin).

As x approaches 2 from the right, the values of $f(x)$, being positive, tend to ∞ (see margin).



Your turn:

CHECK YOUR UNDERSTANDING 4.14

Indicate the nature of the graph of $f(x) = \dfrac{2x-1}{(x-2)(x+1)}$ about the vertical asymptote at $x = -1$.

Answer: The graph tends to $-\infty$ as x approaches -1 from the left, and it tends to $+\infty$ as x approaches -1 from the right.

Returning to the five step procedure for graphing polynomial functions, we now modify Step 2 to accommodate vertical asymptotes.

EXAMPLE 4.5 Sketch the graph of the function:

$$f(x) = \frac{4x-7}{2x+5}$$

SOLUTION:

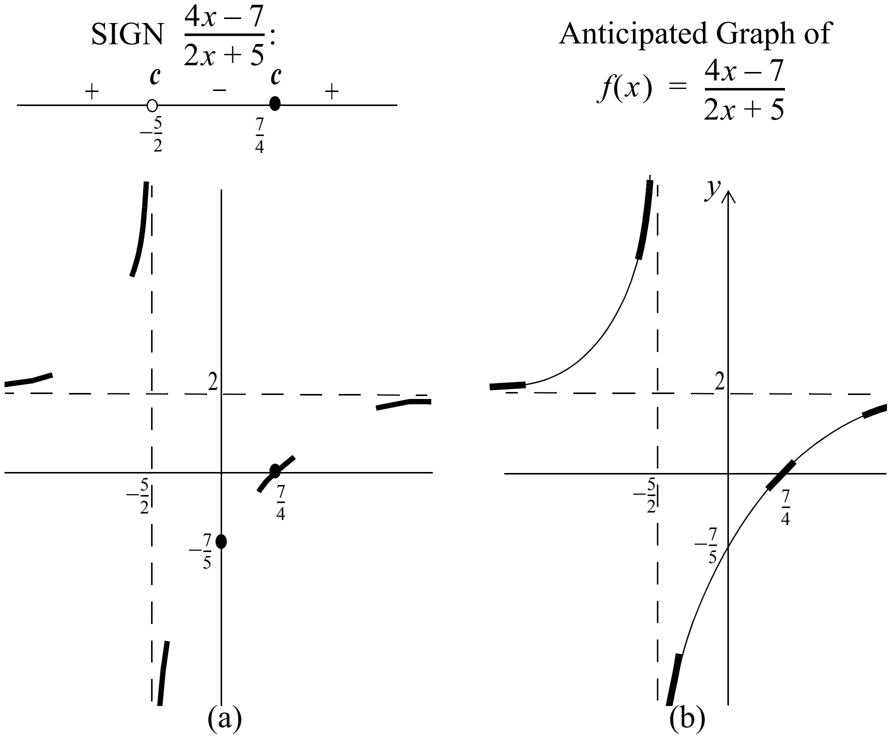
Step 1. Factor: Already in factored form.

Step 2. y-intercept: $y = f(0) = -\frac{7}{5}$ [Figure 4.9(a)].

x-intercepts: $f(x) = 0$ at $x = \frac{7}{4}$ [Figure 4.9(a)].

Vertical Asymptotes: The line $x = -\frac{5}{2}$ [Figure 4.9(a)].

SIGN $f(x)$: From the sign information at the top of Figure 4.9(a), we conclude that the graph goes from below the x-axis to above the x-axis as you move from left to right across the x-intercept at $x = \frac{7}{4}$. Since the function is positive to the left of the vertical asymptote at $x = -\frac{5}{2}$, the graph must tend to $+\infty$ as x approaches $-\frac{5}{2}$ from the left. Since the function is negative just to the right of $x = -\frac{5}{2}$, the graph tends to $-\infty$ as x approaches $-\frac{5}{2}$ from the right.



SIGN $\frac{4x-7}{2x+5}$:

Anticipated Graph of $f(x) = \frac{4x-7}{2x+5}$

(a) (b)

Figure 4.9

Step 3. As $x \to \pm\infty$: The graph of $f(x) = \dfrac{\mathbf{4x} - 7}{\mathbf{2x} + 5}$ approaches the horizontal asymptote $y = \dfrac{\mathbf{4x}}{\mathbf{2x}} = 2$ [Figure 4.9(a)].

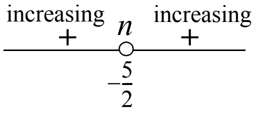
We tried to get a sense of the graph of a function prior to invoking the calculus, for:
Differentiation tends to be a "vulnerable" activity

Step 4. Anticipated Graph: With the above portion of the graph at hand, we can anticipate the general shape of the graph throughout its domain [Figure 4.9(b)].

Step 5: We now turn to the calculus to challenge the anticipated graph of Figure 4.9(b). If that graph is "on target," then the first derivative has to be positive everywhere, and the second derivative has to be positive up to $-\dfrac{5}{2}$ and negative to the right of that point, right? And that is indeed the case:

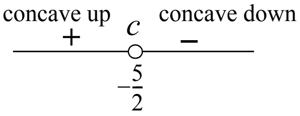
$$f'(x) = \left(\frac{4x-7}{2x+5}\right)' = \frac{(2x+5)\cdot 4-(4x-7)\cdot 2}{(2x+5)^2} = \frac{34}{(2x+5)^2}$$

SIGN f':



$$f''(x) = \left(\frac{34}{(2x+5)^2}\right)' = 34[(2x+5)^{-2}]'$$
$$= -68(2x+5)^{-3}\cdot 2 = -\frac{136}{(2x+5)^3}$$

SIGN f'':



EXAMPLE 4.6

Sketch the graph of the function:

$$f(x) = \frac{2x^2}{x+1}$$

SOLUTION:

Step 1. Factor: The function is already in factored form.

Step 2. y-intercept: $y = f(0) = \dfrac{0}{1} = 0$ [Figure 4.10(a)].

x-intercepts: Where $f(x) = 0 : x = 0$ [Figure 4.10(a)].

Vertical Asymptotes: The line $x = -1$ [Figure 4.10(a)].

SIGN $f(x)$: From the sign information at the top of Figure 4.10(a), we conclude that the graph lies above the x-axis on both sides of its x-intercept at $x = 0$ [Figure 4.13(a)]. Since the function is negative to the left of the vertical asymptote at $x = -1$, the graph must tend to $-\infty$ as x approaches -1 from the left [Figure 4.10(a)]. By the same token, since the function is positive to the right of -1, the graph tends to $+\infty$ as x approaches -1 from the right.

Step 3. As $x \to \pm\infty$: The graph of $f(x) = \frac{2x^2}{x+1}$ will resemble, in shape, that of a line of slope 2 ($\frac{2x^2}{x} = 2x$) [Figure 4.10(a)].

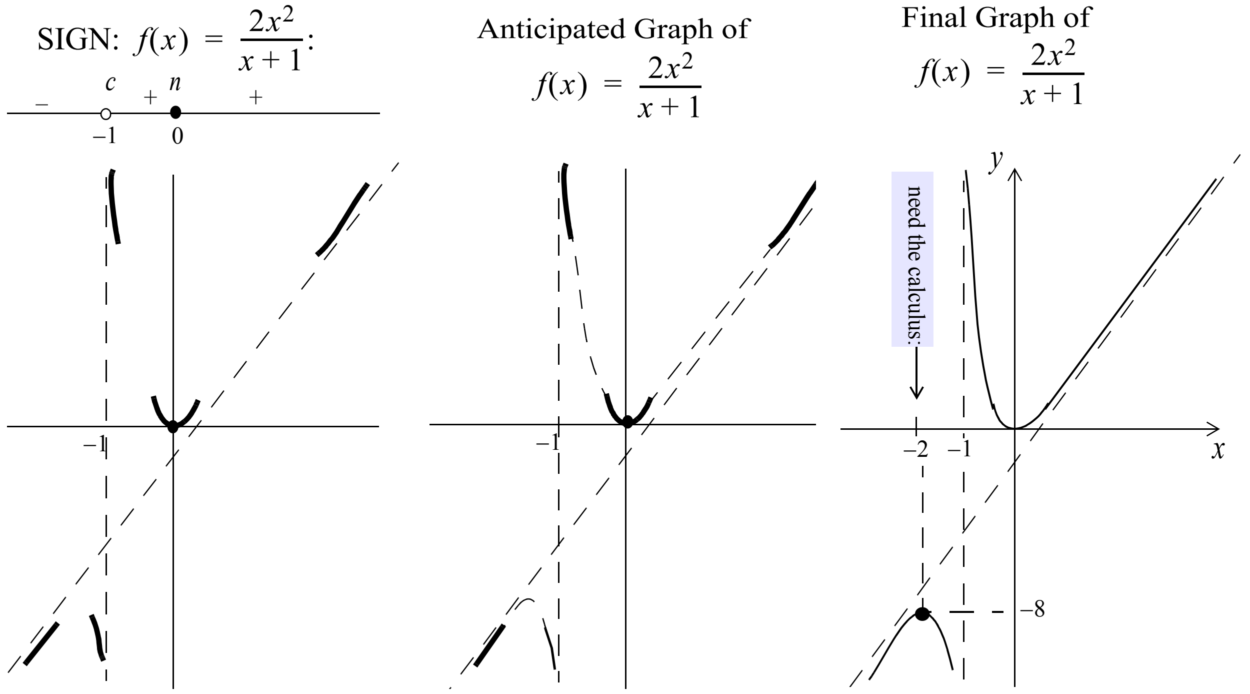
$$\begin{array}{r|l} & 2x-2 \\ \hline x+1 & 2x^2 \\ & \underline{2x^2+2x} \\ & \quad -2x \\ & \quad \underline{-2x-2} \\ & \qquad\quad 2 \end{array}$$

Additional information can be derived by observing that:

$$f(x) = 2x - 2 + \frac{2}{x+1} \quad \text{(see margin)}$$

From the above form, we can conclude that the graph will approach the oblique asymptote $y = 2x - 2$ from above as $x \to \infty$ (for $\frac{2}{x+1}$ will be positive), and from below as $x \to -\infty$ (for $\frac{2}{x+1}$ will be negative).

Step 4. Sketch the anticipated graph: Figure 4.10(b).



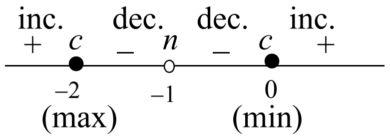
(a) (b) (c)

Figure 4.10

Step 5: Turning to the calculus:

$$f'(x) = \left(\frac{2x^2}{x+1}\right)' = \frac{(x+1)4x - 2x^2}{(x+1)^2} = \frac{2x^2+4x}{(x+1)^2} = \frac{2x(x+2)}{(x+1)^2}$$

SIGN f':



Conforms with Figure 4.10(b)

maximum point: $(-2, -8)$, minimum point: $(0, 0)$ [Figure 4.10(c)]

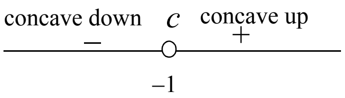
$f(-2) = \frac{2(-2)^2}{-2+1}$

$$f''(x) = \left[\frac{2x^2+4x}{(x+1)^2}\right]' = \frac{(\mathbf{x+1})^2(4x+4)-(2x^2+4x)2(\mathbf{x+1})}{(x+1)^4}$$

pull out the common factor $(\mathbf{x+1})$:
$$= \frac{\cancel{(\mathbf{x+1})}[(x+1)(4x+4)-2(2x^2+4x)]}{\cancel{(x+1)^4}}$$

$$= \frac{4x^2+8x+4-4x^2-8x}{(x+1)^3} = \frac{4}{(x+1)^3}$$

SIGN f'':



Conforms with Figure 4.10(b)

CHECK YOUR UNDERSTANDING 4.15

Sketch the graph of the function:

$$f(x) = \frac{x^2}{x^2-4}$$

Answer: See page A-21.

GRAPHING RADICAL FUNCTIONS

When graphing a radical function, the first order of business is to determine its domain. Consider the following example.

EXAMPLE 4.7 Sketch the graph of the function:

$$f(x) = 2\sqrt{x}-x$$

SOLUTION: Here is the domain of f: $D_f = [0, \infty)$.

Step 1. Factor: $f(x) = 2x^{\frac{1}{2}} - x = x^{\frac{1}{2}}\left(2 - x^{\frac{1}{2}}\right)$.

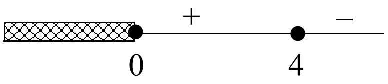
Step 2. y-intercept: $y = f(0) = 0$ [Figure 4.11(a)].

x-intercepts: $f(x) = 0 : x = 0$ or $x = 4$ [Figure 4.11(a)].

Vertical Asymptotes: None.

SIGN

$f(x) = x^{\frac{1}{2}}\left(2 - x^{\frac{1}{2}}\right)$:



Step 3. As $x \to \infty$: Since the term with largest exponent in $f(x) = 2\sqrt{x}-x$, namely $-x$, dominates the nature of the graph, the shape of the graph will resemble that of a line of slope -1 as $x \to \infty$ [Figure 4.11(a)].

Step 4. Sketch the anticipated graph [Figure 4.11(a)]:

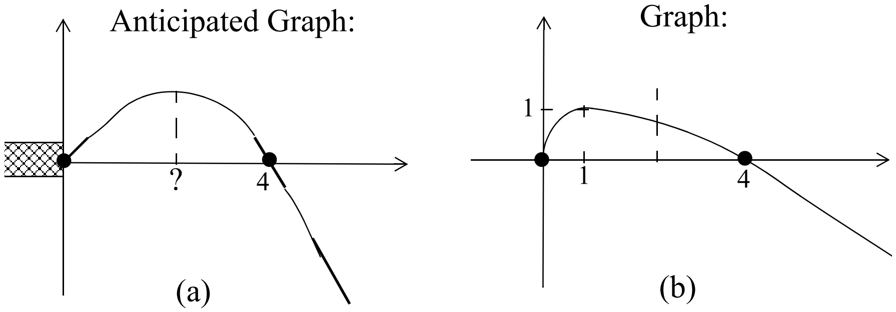
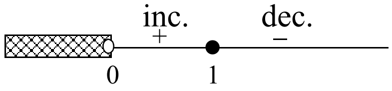


Figure 4.11

Step 5: Turning to the calculus:

$$f'(x) = \left(2x^{\frac{1}{2}} - x\right)' = 2 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 1 = \frac{1 - x^{1/2}}{x^{1/2}} = \frac{1 - \sqrt{x}}{\sqrt{x}}$$

SIGN f' :



At this point we know that the maximum point on the anticipated graph occurs at $x = 1$. Moreover, since the derivative is not defined at the origin, a vertical tangent line must occur at that point. This added information is reflected in Figure 4.11(b).

Our anticipated graph features a curve that is concave down throughout its domain. This is indeed the case:

$$f''(x) = \left(x^{-\frac{1}{2}} - 1\right)' = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{3/2}} \quad \text{always negative}$$

EXAMPLE 4.8 Sketch the graph of the function:

$$f(x) = x^{5/3} - 5x^{2/3}$$

SOLUTION: Domain: $(-\infty, \infty)$.

Step 1. Factor: $f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x - 5)$

Step 2. y-intercept: $y = f(0) = 0$ [Figure 4.12(a)].

x-intercepts: $f(x) = 0 : x = 0, x = 5$ [Figure 4.12(a)].

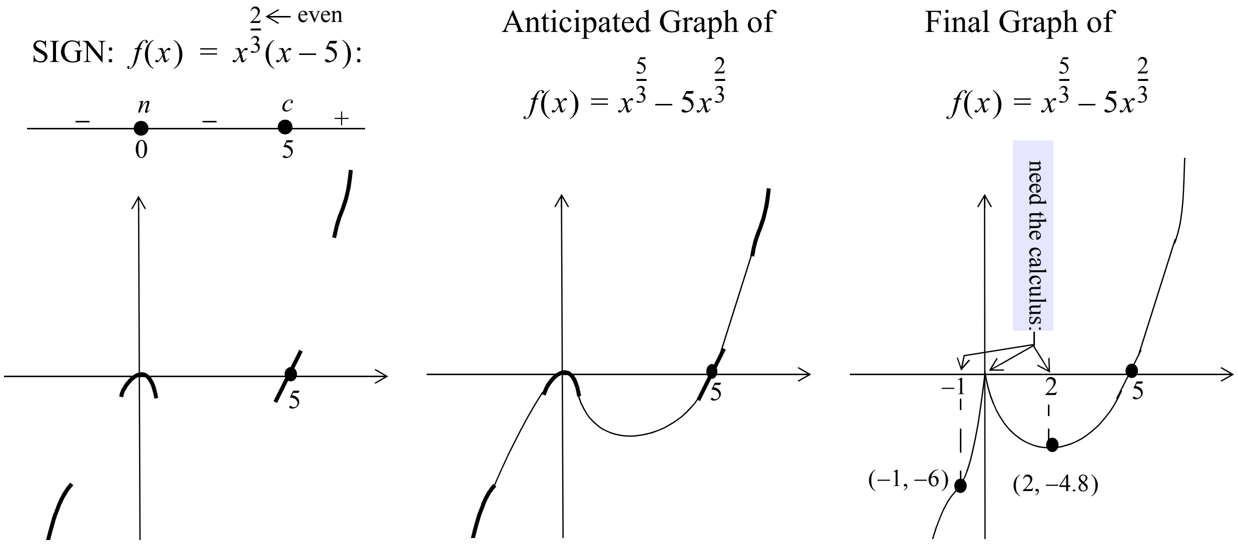
Vertical Asymptotes: None.

SIGN $f(x)$: From the sign information at the top of Figure 4.12(a), we conclude that the graph goes from below the x-axis to above the x-axis as you move from left to right across the x-intercept at $x = 5$, and that it is negative on both sides of the x-intercept at 0 (see margin). [Figure 4.12(a)].

The factor $x^{\frac{2}{3}} = \left(x^{\frac{1}{3}}\right)^2$ is always positive.

Step 3. As $x \to \pm\infty$: Far from the origin the graph of $f(x) = x^{5/3} - 5x^{2/3}$ resembles, in shape, that of its leading (dominant) term $x^{5/3}$, which tends to $+\infty$ as $x \to \infty$ and to $-\infty$ as $x \to -\infty$ [Figure 4.12(a)].

Step 4. Sketch the anticipated graph: Figure 4.12(b).

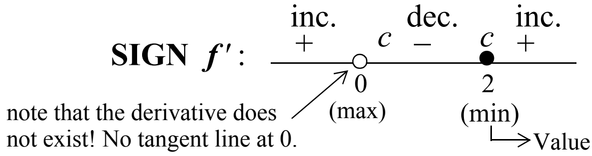


(a) (b) (c)

Figure 4.12

Step 5: Turning to the calculus:

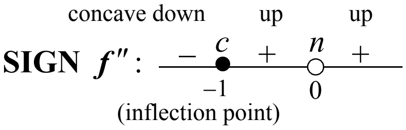
$$f'(x) = \left(x^{\frac{5}{3}} - 5x^{\frac{2}{3}}\right)' = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = \frac{5}{3}x^{-\frac{1}{3}}(x-2) = \frac{5(x-2)}{3x^{\frac{1}{3}}}$$



Value: $f(2) = 2^{\frac{5}{3}} - 5 \cdot 2^{\frac{2}{3}} \approx -4.8$

$$f''(x) = \left(\frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}\right)' = \frac{10}{9}x^{-\frac{1}{3}} + \frac{10}{9}x^{-\frac{4}{3}}$$

$$= \frac{10}{9}x^{-\frac{4}{3}}(x+1) = \frac{10(x+1)}{9x^{\frac{4}{3}}}$$



Value: $f(-1) = (-1)^{\frac{5}{3}} - 5 \cdot (-1)^{\frac{2}{3}} = -6$

We anticipated that a minimum occurs somewhere between 0 and 5, and the calculus showed us that it takes place at 2. Our anticipated graph in Figure 4.12(b), however, did not reveal the fact that a tangent line does not exist at the origin, nor that an inflection point occurs at -1. And so we refined our anticipated graph accordingly [Figure 4.12(c)]. Did we waste our time in constructing the anticipated graph in (b)? No, for it gave us a sense of what to expect and then used the calculus to refine our anticipated graph.

To accommodate the above calculus information, we adjusted the anticipated graph of Figure 4.11(b) and arrived at the "final graph" in Figure 4.11(c) (see margin).

CHECK YOUR UNDERSTANDING 4.16

Sketch the graph of the function:

$$f(x) = (x-2)^{1/3}$$

Answer: See page A-22.

EXERCISES

Exercises 1-29. Sketch the graph of the given function. Label the y-intercept (if it exists), x-intercepts, vertical asymptotes, local maximum and minimum points, and inflection points. Identify any horizontal or oblique asymptote.

1. $f(x) = 2x^2 + 7x + 4$
2. $f(x) = 4x^2 - 7x - 2$
3. $f(x) = x^3 + 2x^2$
4. $f(x) = 2x^4 + x$
5. $f(x) = -\frac{1}{3}x^3 + 3x^2 - 8x$
6. $f(x) = x^4 - 4x^3$
7. $f(x) = \frac{1}{4}x^4 + x^3$
8. $f(x) = 4x^4 + 4x^3 + x^2$
9. $f(x) = \frac{x}{2x+1}$
10. $f(x) = \frac{x^2}{2x+1}$
11. $f(x) = \frac{2x+1}{x}$
12. $f(x) = \frac{x^2+x}{x^2-1}$
13. $f(x) = x + \frac{1}{x^2}$
14. $f(x) = x^2 - \frac{1}{x^2}$
15. $f(x) = \frac{x}{x^2-1}$
16. $f(x) = x + \frac{1}{x} - \frac{1}{x^2}$
17. $f(x) = \frac{x^2}{x^2-9}$
18. $f(x) = \frac{x^3}{x^2-4}$
19. $f(x) = \frac{x^3-x}{x^3-x^2}$ (Function has a removable discontinuity)
20. $f(x) = \frac{x^3}{2x^2+x}$ (Function has a removable discontinuity)
21. $f(x) = \sqrt{x^2-1}$
22. $f(x) = \sqrt{x^2+1}$
23. $f(x) = x\sqrt{x-1}$
24. $f(x) = x\sqrt{x+1}$
25. $f(x) = x^{\frac{1}{3}}(x+4)$
26. $f(x) = \frac{x}{\sqrt{x^2-1}}$
27. $f(x) = x - 6x^{\frac{1}{3}}$
28. $f(x) = 3\left(x^{\frac{5}{3}} - x^{\frac{4}{3}}\right)$
29. $f(x) = x^{\frac{2}{3}}\left(x - \frac{5}{2}\right)$

Exercise 30-32. (“Unmanageable” Second Derivative) Sketch the anticipated graph of the given function. Label the y-intercept (if it exists) and x-intercepts, vertical asymptotes, and local maximum and minimum points.

30. $f(x) = \frac{x^2-x}{x^2-4}$
31. $f(x) = \frac{x^2-1}{2x^2+x}$
32. $f(x) = \frac{x+2}{x^2-1}$

Exercises 33-38 (Absolute Maximum and Minimum) A function f defined on an interval I is said to have an **absolute maximum** at $c \in I$ if $f(c) \geq f(x)$ for every $x \in I$, and an **absolute minimum** at $c \in I$ if $f(c) \leq f(x)$ for every $x \in I$.

Find where the absolute maximum and absolute minimum values of the given function occur in the specified interval. (Don’t forget to consider the endpoint extremes.)

33. $f(x) = x^3 - 3x^2 + 3$, $[-1, 1]$
34. $f(x) = x^3 - 3x^2 + 3$, $[1, 4]$
35. $f(x) = 3x^5 - 20x^3 + 2$, $[-3, 3]$
36. $f(x) = 3x^5 - 20x^3 + 2$, $[-2, 2]$

37. $f(x) = 3x^5 - 20x^3 + 2$, $[-1, 3]$

38. $f(x) = x^4 - x^3$, $[-1, 1]$

Exercises 39-50. (Construction) Sketch the graph of a function satisfying the given conditions.

39. Has a local maximum at 0, a local minimum at 5, and does not have an absolute maximum nor an absolute minimum anywhere. (See Exercises 33-38.)
40. Has domain $[-10, 10]$. Has a local maximum at 0, a local minimum at 5. The absolute minimum of the function occurs at -10, and the absolute maximum occurs at 10. (See Exercises 25-30.)
41. The first and second derivatives of the function are positive everywhere.
42. The first and second derivatives of the function are negative everywhere.
43. The first derivative is positive everywhere, and the second derivative is negative everywhere.
44. The first derivative is negative everywhere, and the second derivative is positive everywhere.
45. The function has a maximum value at $x = 1$ and an inflection point at $x = 3$; the first derivative is negative immediately to the left of 3, and positive immediately to the right of 3.
46. The function has a minimum value at $x = 1$ and an inflection point at $x = 3$; the first derivative is negative immediately to the left of 3, and positive immediately to the right of 3.
47. Has a vertical asymptote at $x = 1$, an x-intercept at $x = 2$, and a horizontal asymptote $y = 3$.
48. Has a vertical asymptote at $x = 1$, x-intercepts at $x = 2$ and $x = 3$, and a horizontal asymptote $y = 4$.
49. Has vertical asymptotes at $x = 1$ and $x = 2$, x-intercepts at $x = 3$ and $x = 4$, and a horizontal asymptote $y = 5$.
50. Has a vertical asymptote at $x = 1$, an x-intercept at $x = 2$, and an oblique asymptote $y = x + 2$.

51. **(Learning Process)** Experimentation has shown that the learning performance of rats for a particular task can be approximated by the function $L(t) = 15t^2 - t^3$, where t denotes the number of weeks the rat has been exposed to the learning process, for $t \le 7$. At what point in time does the rat's rate of change of learning begin to decline?

52. **(Fruit Flies)** In the early 1900, the biologist Raymond Pearl discovered that the growth rate of the population P of fruit flies with respect to time t, in days, can be approximated by the function, $\frac{dP}{dt} = 0.2P - \frac{0.2}{1035}P^2$, for $P \le 1035$.

    (a) Find the values of the population for which the growth rate of the population is increasing, and the values for which the growth rate is decreasing.

    (b) Show that the population growth rate reaches a maximum at the point of inflection of the graph of the population function.

Exercise 53-58. (Theory)

53. Show that the function $\frac{x^2-1}{x^2+x-2}$ has a vertical asymptote at $x = -2$ but not at $x = 1$.

54. Show that $f''(0) = 0$ for each of the following functions, and then go on to show that one of the functions has a maximum at 0, another has a minimum at 0, and the remaining one has neither a maximum nor a minimum at 0.

    (i) $f(x) = x^3$ (ii) $f(x) = x^4$ (iii) $f(x) = -x^4$

55. (a) Prove that the graph of a cubic polynomial cannot have more that two distinct horizontal tangent lines.

    (b) Give an example of a cubic polynomial whose graph has:

    (i) No horizontal tangent line.

    (ii) Exactly one horizontal tangent line.

    (iii) Two distinct horizontal tangent lines.

    (c) Prove that the graph of a cubic polynomial can have at most one maximum point and at most one minimum point.

56. (a) Prove that the vertex of the parabolic graph of a quadratic function $f(x) = ax^2+bx+c$ occurs at $x = -\frac{b}{2a}$.

    (b) Show that the parabola opens upward if $a > 0$ and opens downward if $a < 0$.

57. (a) Prove that the graph of the cubic function $f(x) = ax^3+bx^2+cx+d$ has but one inflection point, and that it occurs at $x = -\frac{b}{3a}$.

    (b) Give an example of a cubic polynomial whose graph has:

    (i) A point of inflection at $(1, 2)$.

    (ii) A point of inflection at $(1, 2)$ and a maximum at $x = -1$

    (iii) A point of inflection at $(1, 2)$, a maximum at $x = -1$ and a minimum at $x = 3$.

58. (a) Prove that the graph of a polynomial of degree $n > 1$ can have at most $n-1$ maximum or minimum points.

    (b) Prove that the graph of a polynomial of degree n can have at most $n-2$ inflection points.

§3 OPTIMIZATION

An optimization problem is one in which the maximum or minimum value of a quantity is to be determined. The main step in the solution process is to express the quantity to be optimized as a function of one variable. To achieve that end, we suggest the following 4-step procedure:

Step 1. See the problem.

Step 2. Express the quantity to be maximized or minimized in terms of any convenient number of variables.

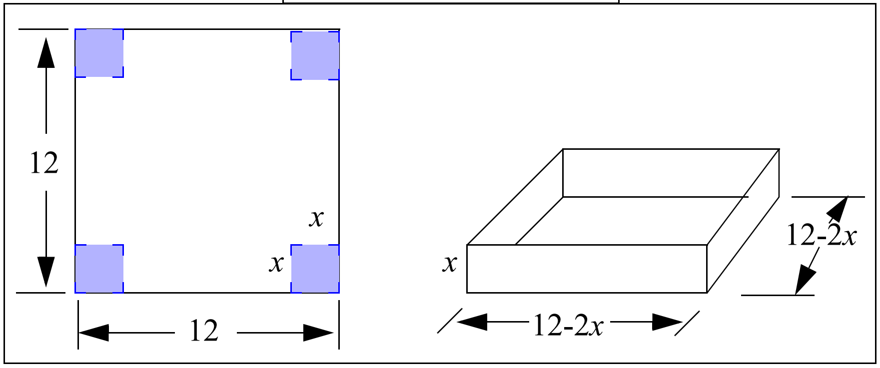
Step 3. In the event that the expression in Step 2 involves more than one variable, use the given information to arrive at an expression involving but one variable.

Step 4. Differentiate, set equal to zero, and solve (to find where horizontal tangent lines occur). Analyze the nature of the critical points.

EXAMPLE 4.9 Best Box Company is to manufacture open-top boxes from 12 in. by 12 in. pieces of cardboard. The construction process consists of two steps: (1) cutting the same size squares from each corner of the cardboard, and (2) folding the resulting cross-like configuration into a box. What size square should be cut out, if the resulting box is to have the largest possible volume?

SOLUTION:

Step 1: SEE THE PROBLEM



Step 2: Volume is to be maximized, and from the above, we see that:

$$V(x) = (12-2x)(12-2x)x = 4x^3 - 48x^2 + 144x$$

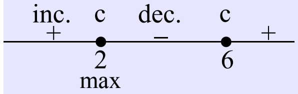
Step 3: Not applicable, since volume is already expressed as a function of **one** variable.

Step 4: Differentiate, set equal to 0, and solve:

$$\begin{aligned} V'(x) &= 12x^2 - 96x + 144 = 0 \\ 12(x-6)(x-2) &= 0 \\ \text{Critical points:} \quad x = 6 &\quad \text{or} \quad x = 2 \end{aligned}$$

At this point, we know that the graph of the volume function has a horizontal tangent line at $x = 6$ and at $x = 2$. You can forget about the 6, for if you cut a square of length 6 from the piece of cardboard, nothing will remain. Since a box of maximum-volume certainly exists, and since we are down to one horizontal tangent line, $x = 2$ inches must be the answer.

Lest you doubt that a maximum really does occur at 2:



SIGN $12(x-6)(x-2)$

CHECK YOUR UNDERSTANDING 4.17

A rectangular field is to be enclosed on all four sides with a fence. One side of the field borders a road, and the fencing material to be used for that side costs \$8 per foot. The fencing material for the remaining sides costs \$6 per foot. Find the maximum area that can be enclosed for \$2800.

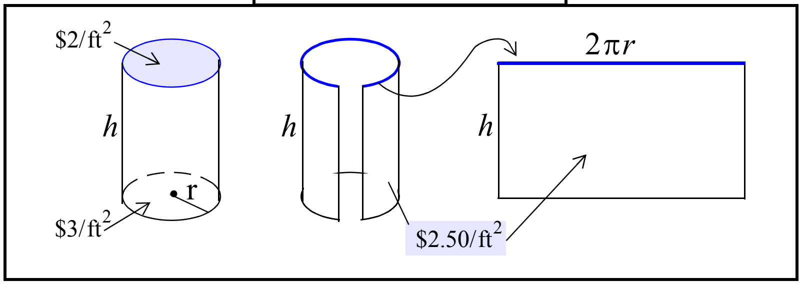
Answer: $\frac{35000}{3}$ ft^2

EXAMPLE 4.10

A cylindrical drum is to hold 65 cubic feet of chemical waste. Metal for the top of the drum costs \$2 per square foot, and \$3 per square foot for the bottom. Metal for the side of the drum costs \$2.50 per square foot. Find the dimensions for minimal material cost.

SOLUTION:

Step 1: SEE THE PROBLEM



Step 2: Total cost C is to be minimized, and from the above we see that:

$$C = \text{cost of the top} + \text{cost of the bottom} + \text{cost of the side}$$

$$\$C = \$2.00(\pi r^2) + \$3.00(\pi r^2) + \$2.50(2\pi rh)$$

$$C = 5\pi r^2 + 5\pi rh \qquad (*)$$

Recall that the area of a circle of radius r is πr^2, and that the volume of the cylinder is the area of its base, times its height: $V = \pi r^2 h$.

Step 3: We can express C as a function of only one variable by eliminating h. Since the drum is to have a volume of 65 ft^3:

$$\pi r^2 h = 65$$

$$h = \frac{\mathbf{65}}{\pi \boldsymbol{r^2}} \qquad (**)$$

Substituting in (*):

$$C = 5\pi r^2 + 5\pi r\left(\frac{\mathbf{65}}{\pi \boldsymbol{r^2}}\right) = 5\pi r^2 + \frac{325}{r}$$

Step 4: Differentiate, set equal to 0, and solve:

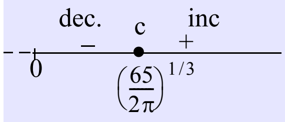
$$\frac{dC}{dr} = \frac{d}{dr}(5\pi r^2 + 325r^{-1}) = 10\pi r - 325r^{-2} = 0$$

$$10\pi r - \frac{325}{r^2} = 0$$

$$\text{multiply both sides by } r^2\text{:} \quad 10\pi r^3 - 325 = 0$$

$$\text{Critical Point:} \quad r = \left(\frac{325}{10\pi}\right)^{1/3} = \left(\frac{65}{2\pi}\right)^{1/3}$$

SIGN $\frac{dC}{dr}$:



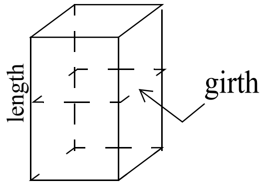
Returning to (**), we find that:

$$h = \frac{65}{\pi r^2} = \frac{65}{\pi\left(\frac{65}{2\pi}\right)^{2/3}} = \frac{65(2\pi)^{2/3}}{\pi(65)^{2/3}} = \frac{65^{1/3}\cdot 4^{1/3}}{\pi^{1/3}} = \left(\frac{260}{\pi}\right)^{1/3}$$

Conclusion: The dimensions for minimal material cost are a radius of $\left(\frac{65}{2\pi}\right)^{1/3}$ feet and a height of $\left(\frac{260}{\pi}\right)^{1/3}$ feet.

CHECK YOUR UNDERSTANDING 4.18

A box with a square base is to be constructed. For mailing purposes, the perimeter of the base (the girth of the box), plus the length of the box, cannot exceed 108 in. Find the dimensions of the box of greatest volume.



Answer:
18 in. × 18 in. × 36 in. .

EXAMPLE 4.11

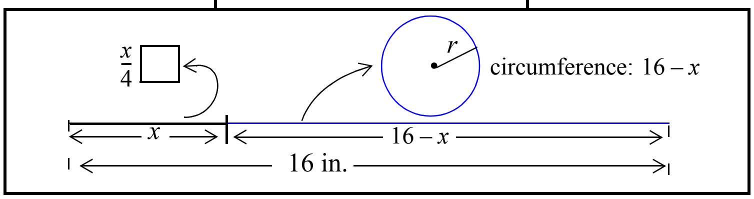
A 16 inch wire is to be cut into two pieces. One piece is to be bent into a square and the other into a circle. How should the wire be cut, if at all, in order for the resulting combined areas to be:

(a) Maximum? (b) Minimum?

SOLUTION:

Step 1:

SEE THE PROBLEM



Step 2: Combined area:

$$A = \overset{\text{area of the square}}{\left(\frac{x}{4}\right)^2} + \overset{\text{area of the circle}}{\pi r^2} \qquad (*)$$

Step 3: Using the fact that $16 - x$ is the circumference of the circle:

$$16 - x = 2\pi r$$

$$r = \frac{16 - x}{2\pi}$$

Substituting in (*):

$$A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{16 - x}{2\pi}\right)^2$$

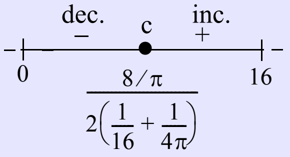
$$= \frac{x^2}{16} + \frac{1}{4\pi}(256 - 32x + x^2) = \left(\frac{1}{16} + \frac{1}{4\pi}\right)x^2 - \frac{8}{\pi}x + \frac{64}{\pi}$$

Step 4: Exercise 56, page 148, tells us that the graph of the above quadratic function is a parabola that opens upward with vertex at:

$$x = -\frac{b}{2a} = -\frac{-\frac{8}{\pi}}{2(\frac{1}{16} + \frac{1}{4\pi})} \approx 9.0$$

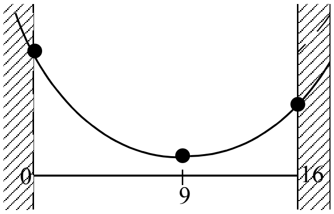
$$\left[\left(\frac{1}{16} + \frac{1}{4\pi}\right)x^2 - \frac{8}{\pi}x + \frac{64}{\pi}\right]'$$

$$= 2\left(\frac{1}{16} + \frac{1}{4\pi}\right)x - \frac{8}{\pi}$$



That a minimum occurs at $\frac{8/\pi}{2\left(\frac{1}{16} + \frac{1}{4\pi}\right)}$ is also established in the margin.

The adjacent graph of the parabola reveals the fact that the minimum combined area occurs when 9 inches is used for the square and 7 inches for the circle, and that the maximum occurs at $x = 0$ (don't cut the wire at all, but bend it all into a circle).



CHECK YOUR UNDERSTANDING 4.19

Determine the maximum area of an isosceles triangle that can be constructed from a 12 inch wire.

Answer: $4\sqrt{3}$ in.^{2}

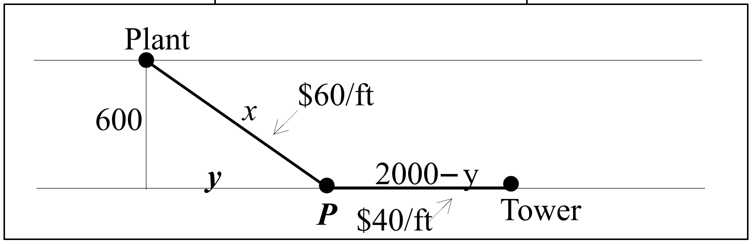
EXAMPLE 4.12

A cable is to be run from a power plant, on one side of a river that is 600 feet wide, to a tower on the other side, which is 2000 feet downstream. The cable is to be laid from the plant to a point P on the other side of the river and then from P to the tower. Determine the location of P for minimum cost, if the cost of laying the cable in the water is \$60 per foot, and the cost on land is \$40 per foot.

SOLUTION:

Step 1:

SEE THE PROBLEM



Carrying units:

$6\frac{\$}{\cancel{\text{ft}}}\cdot x\cancel{\text{ft}}+4\frac{\$}{\cancel{\text{ft}}}\cdot(2000-y)\cancel{\text{ft}}$

Step 2: Cost: $C = 60x+40(2000-y)$.

Step 3: $(x=\sqrt{y^2+(600)^2})$: $C(y) = 60\sqrt{y^2+(600)^2}+40(2000-y)$

↑

Pythagorean Theorem

Step 4:

$$\begin{aligned}
C'(y) &= 0\\
(60\sqrt{y^2+(600)^2}+40(2000-y))' &= 0\\
60\left[(y^2+(600)^2)^{\frac{1}{2}}\right]'+(80000-40y)' &= 0\\
60\cdot\frac{y}{\sqrt{y^2+(600)^2}}-40 &= 0\\
36y^2 &= 16y^2+16(600)^2\\
20y^2 &= 16(600)^2\\
y^2 &= \frac{16(600)^2}{20}
\end{aligned}$$

ignoring the negative root: $y = 240\sqrt{5}$

Theorem 4.7 page 127 assures us that the cost function assumes a minimum value for some value of $y\in[0,2000]$. Can it occur at the endpoints? A direct calculation reveals that $C(240\sqrt{5})\approx 106,833$ is less than both $C(0)\approx 116{,}000$ and $C(2000)\approx 125,283$. We can therefore conclude that the point P should be $2000-240\sqrt{5}\approx 1463$ feet from the tower.

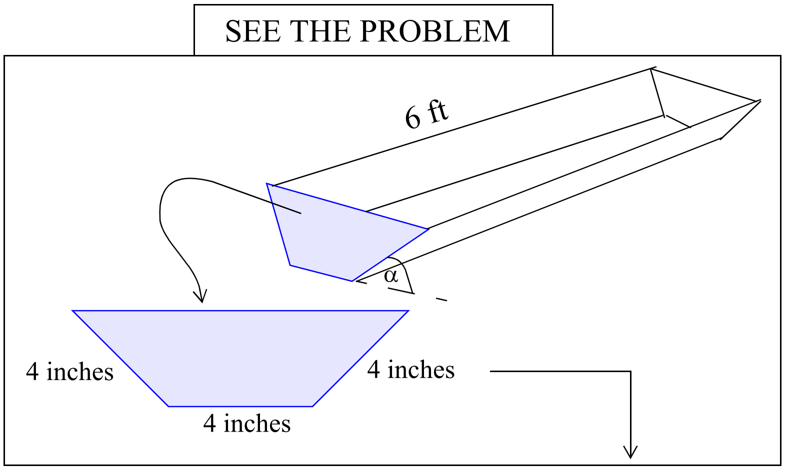
CHECK YOUR UNDERSTANDING 4.20

A man in a boat 3 miles from a straight coastline wants to get to a dock that is 10 miles down the coast in the shortest time, by rowing to some point P on the coast and running the rest of the way. The journey is to consist of two linear paths, one from the boat to P, and the other from P to the dock. Assuming that the man rowed at a rate of 4 miles per hour, and ran at a rate of 5 miles per hour, determine the distance between the point P and the dock.

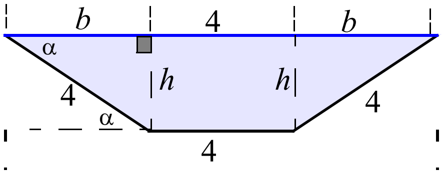
Answer: 6 miles

EXAMPLE 4.13 A water trough is to be constructed from a 12 inch by 6 foot metal sheet by bending up one-third of the shorter side through an angle α. Determine α so that the resulting trough has maximum volume.

SOLUTION:



Since the volume of the trough is the area A of its cross-section times its length, to maximize the volume is to maximize the area of the adjacent region — which is the sum of the areas of the two indicated triangles and the rectangular region between those triangles:



$$A = \underbrace{2\left[\frac{1}{2}bh\right]}_{\text{area of the two triangles}} + \overbrace{4\cdot h}^{\text{area of middle rectangular part}} = \underbrace{b}_{\cos\alpha\,=\,\frac{b}{4}}h + 4h = (4\cos\alpha)(4\sin\alpha) + 4(4\sin\alpha) \qquad \sin\alpha = \frac{h}{4}$$

Now that we have the area expressed as a function of one variable:

$$A(\alpha) = 16(\cos\alpha\sin\alpha + \sin\alpha)$$

we turn to the routine task of locating where its graph has a horizontal tangent line; which is to say, where $A' = 0$:

$$
\begin{aligned}
[16(\cos\alpha\sin\alpha+\sin\alpha)]' &= 0\\
(\mathbf{\cos}\,\alpha\,\mathbf{\sin}\,\alpha+\sin\alpha)' &= 0\\
[\mathbf{\cos}\,\alpha(\mathbf{\cos}\,\alpha)+\mathbf{\sin}\,\alpha(-\mathbf{\sin}\,\alpha)]+\cos\alpha &= 0\\
\cos^2\alpha-\mathbf{\sin}^{\mathbf{2}}\alpha+\cos\alpha &= 0\\
\cos^2\alpha-(\mathbf{1}-\mathbf{\cos}^{\mathbf{2}}\alpha)+\cos\alpha &= 0\\
2\cos^2\alpha+\cos\alpha-1 &= 0\\
(2\cos\alpha-1)(\cos\alpha+1) &= 0
\end{aligned}
$$

why can we “drop” the 16?

$\sin^2\alpha+\cos^2\alpha = 1$:

$$\cos\alpha = \frac{1}{2}$$

only acute angle: $\alpha = 60°$

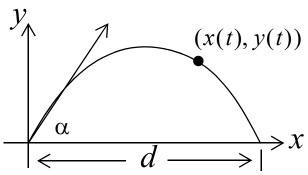
$\cos\alpha = -1$
no acute angle

Conclusion: To maximize the volume of the trough, bend the sheet through an angle of $60°$ (see margin).

Clearly $0° \le \alpha \le 90°$. A direct calculation reveals that $A(60°) = 12\sqrt{3}$ is greater than $A(0°) = 0$ and $A(90°) = 16$. Applying Theorem 4.7, page 127, we conclude that the absolute maximum of the area cross-section (and therefore of the volume of the trough) occurs when $\alpha = 60°$.

CHECK YOUR UNDERSTANDING 4.21

If a projectile is fired with initial velocity $V_0\dfrac{\text{ft}}{\text{sec}}$ at an angle of elevation α then, assuming negligible air resistance, its position, $(x(t), y(t))$, t seconds later is given by:



$$x(t) = (V_0\cos\alpha)t \qquad y(t) = (V_0\sin\alpha)t - 16t^2$$

(a) Determine α so as to maximize d.
(b) Find the maximum height of the projectile when fired at that angle of elevation.

Answer: (a) $\alpha = 45°$
(b) $\dfrac{V_0^2}{128}$ ft.

EXAMPLE 4.14

When 20 peach trees are planted per acre, each tree will yield 200 peaches. For every additional tree planted per acre, the yield of each tree diminishes by 5 peaches. How many trees per acre should be planted to maximize yield?

SOLUTION:

Step 1:

SEE THE PROBLEM

20 trees/acre ⟶ 200 per tree
$20+x$ trees/acre ⟶ $200-5x$ per tree

Step 2: Letting x denote the number of trees above 20 to be planted per acre, we express the yield per acre as a function of x:

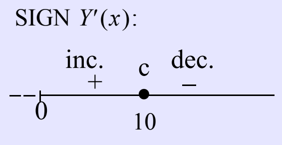
$$Y(x) = \overbrace{(20+x)}^{\text{number of trees per acre}}\overbrace{(200-5x)}^{\text{number of peaches per tree}} = -5x^2 + 100x + 4000$$

Step 3: Not applicable (The function to be maximized, $Y(x)$, is already expressed in terms of one variable).

Step 4:

$$Y'(x) = -10x + 100 = 0$$

$$x = \frac{100}{10} = 10$$



We conclude that maximum yield will be attained with the planting of 30 trees per acre.

CHECK YOUR UNDERSTANDING 4.22

The weekly demand function for a certain company is given by:

$$p = 50 - \frac{x^2}{6000}, \quad \text{for } x < 500$$

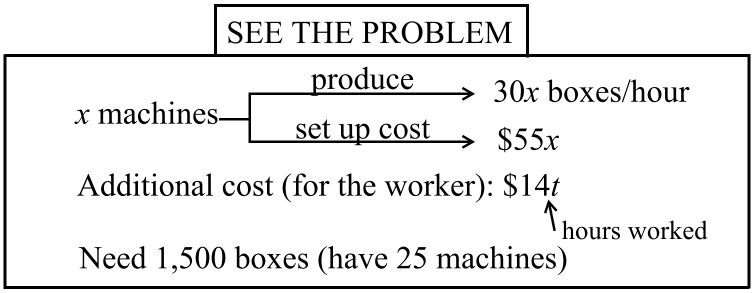
where x is the number of units sold each week, and p is the price per unit (in dollars). Determine the number of units that should be produced to maximize the (weekly) profit for the company, if it costs the company \$30 to produce one unit.

Answer: 200 units

EXAMPLE 4.15

The Best-Box manufacturing firm has received an order for 1,500 shipping boxes. The firm has 25 machines that can be used to manufacture the boxes, each of which can produce 30 boxes per hour. It will cost the firm \$55 to set up each machine. Once set up, the machines are fully automated, and can all be supervised by a single worker, earning \$14 per hour. How many machines should be used to minimize cost of production?

SOLUTION:



Using x machines, we have:

$$\text{Cost} = C = 55x + 14t \text{ (dollars)}$$

We express the variable t in terms of x, and find it helpful to consider units along the way:

$$\text{time } t\text{, to produce 1,500 boxes, using x machines} = \frac{1500 \text{ boxes}}{30x \dfrac{\text{boxes}}{\text{hour}}}$$

$$= \frac{50 \cancel{\text{boxes}}}{x} \cdot \frac{\text{hour}}{\cancel{\text{boxes}}} = \frac{50}{x} \text{ hours}$$

We can now express cost as a function of one variable:

$$C(x) = 55x + 14 \cdot \frac{50}{x} = 55x + 700x^{-1}$$

Bringing us to the routine part of the solution process:

$$C'(x) = 55 - 700x^{-2} = 0$$

$$55 = \frac{700}{x^2}$$

$$x^2 = \frac{700}{55} = \frac{140}{11}$$

$$x = \sqrt{\frac{140}{11}} \approx 3.6$$

Since x has to be an integer one must calculate both $C(3)$ and $C(4)$ directly prior to making a final decision. If you do, you will find that $C(3) \approx 398$ and that $C(4) = 395$.

Conclusion (see margin): To minimize cost of production, the company should utilize four machines.

CHECK YOUR UNDERSTANDING 4.23

A sailboat company can manufacture up to 200 boats per year. The number of boats, n, that the company can sell per year can be approximated by the function $n = 200 - \frac{p}{1000}$, where p is the price (in dollars) for a single boat. What yearly production level will maximize profit, if it costs the company $100{,}000 + 75{,}000n$ dollars to produce n boats?

Answer: 62 boats

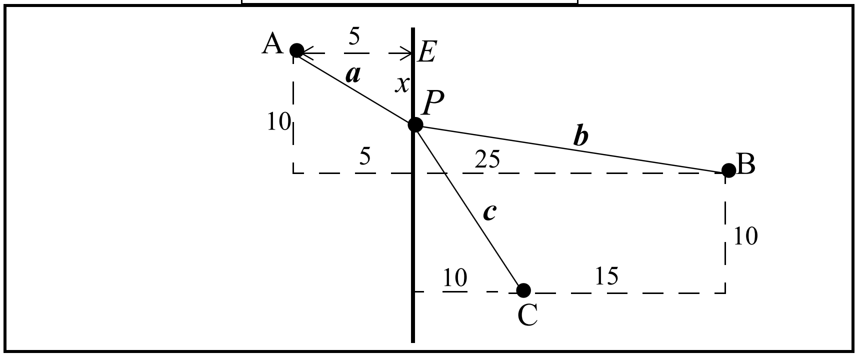
WITH THE HELP OF A GRAPHING UTILITY

By now you know that the real challenge of solving an optimization problem is that of expressing the quantity to be optimized as a function of one variable. But what if you are not able to calculate where the derivative of the function is zero? You invoke some battery power, that's what. Consider the following example.

EXAMPLE 4.16 A straight road runs from North to South. Point A is 5 miles due West of point E on the road. If you walk 10 miles south of A and then go 30 miles due East, you will reach point B. If you walk 10 miles due South of B and then go 15 miles due west, then you will reach point C. Use a graphing utility to determine, to 2 decimal places, the point P on the road whose combined distance from the three points A, B, and C is minimal.

SOLUTION:

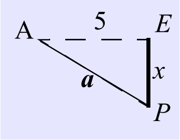
SEE THE PROBLEM

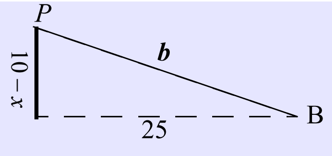


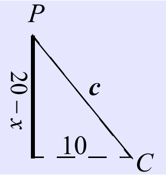
We want to minimize the combined distance:

$$s = a + b + c$$

and choose to express it in terms of the one represented variable x:
Focusing on the three right triangles:



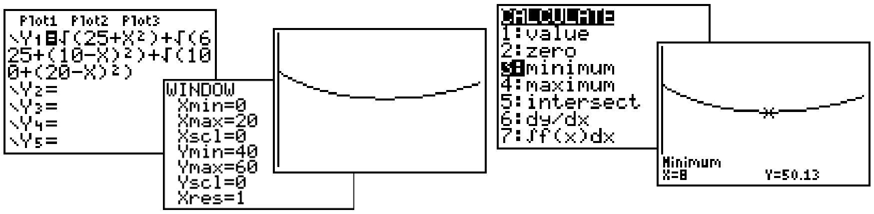




we have:

$$s = a + b + c = \sqrt{5^2 + x^2} + \sqrt{25^2 + (10 - x)^2} + \sqrt{10^2 + (20 - x)^2}$$

Then:



```
Plot1 Plot2 Plot3
\Y1■√(25+X²)+√(6
25+(10-X)²)+√(10
0+(20-X)²)
\Y2=
\Y3=
\Y4=
\Y5=
```

```
WINDOW
 Xmin=0
 Xmax=20
 Xscl=0
 Ymin=40
 Ymax=60
 Yscl=0
 Xres=1
```

```
Minimum
X=8          Y=50.13
```

Conclusion: A minimum combined distance of approximately 50.13 miles will be achieved when P is positioned 8.00 miles South of point E.

Answer: 8.8 miles from plant A.

CHECK YOUR UNDERSTANDING 4.24

Two chemical plants are located 12 miles apart. The pollution count from plant A, in parts per million, at a distance of x miles from plant A, is given by $\frac{K}{x^2 + 10}$ for some constant K. The pollution count from the cleaner plant B, at a distance of x miles from plant B, is one quarter that of A. A third plant C is located on a road perpendicular to the road joining A and B and is 5 miles from A and 10 miles from B. Assuming that the pollution count of plant C is twice that of B, determine, to one decimal place, the point on the road joining A and B where the pollution count from the three plants is minimal.

EXERCISES

1. **(Maximize Profit)** A company can produce up to 500 units per month. Its profit, in terms of number of units produced is given by $P(x) = -\frac{x^3}{30} + 9x^2 + 400x - 75000$. How many units should the company produce to maximize profit?
2. **(Minimize Cost)** The total operating cost, per hour, to operate a freight train is given by $C(s) = 250 + \frac{s^2}{4}$, where s is the speed of the train in miles per hour. Find minimum cost for a 400 mile trip.
3. **(Maximum Drug Concentration)** The concentration (in milligrams per cubic centimeter) of a particular drug in a patient's bloodstream, t hours after the drug has been administered has been modeled by $C(t) = \frac{0.2t}{0.9t^2 + 5t + 3}$. How many hours after the drug is administered will the concentration be at its maximum? What is the maximum concentration?
4. **(Air Velocity in the Trachea)** When a person coughs, the radius r of the trachea decreases. The velocity of air in the trachea during a cough can be approximated by the function $v(r) = ar^2(r_0 - r)$, where a is a constant, and r_0 is the radius of the trachea in a relaxed state. Determine the radius at which the velocity is greatest.
5. **(Bacterial Growth)** A pond is treated to control bacterial growth. After t days, the concentration of bacteria per cubic centimeter can be approximated by the function $K(t) = 25t^2 - 150t + 700$, $0 \le t \le 7$. Determine (a) the minimal bacterial concentration and (b) the maximal bacterial concentration, in the seven day period.
6. **(Minimum Force)** An object of weight W is being pulled along a horizontal plane by a force F acting along a rope attached to the object which makes an angle α with the plane. Find the angle for which the force is smallest, given that $F = \frac{kW}{\cos\alpha + k\sin\alpha}$, where the constant k denotes the coefficient of friction.
7. **(Sensitivity)** The reaction to a dosage x of a drug administered to a patient is given by $R(x) = x^2\left(\frac{a}{2} - \frac{x}{3}\right)$, where x is the amount of the drug administered, and a is the maximum dosage of the drug that can be administered. The rate of change of R with respect to the dose x is called the sensitivity of the patient to the dosage x. Find the dosage at which the sensitivity is greatest.
8. **(Maximize Revenue)** A car-rental agency can rent 150 cars per day at a rate of \$15 per day. Assume that for each price increase of \$1 per day, 3 less cars will be rented, while for each \$1 decrease 2 additional cars will be rented. What rate should be charged to maximize the revenue of the agency?
9. **(Maximize Revenue)** A chemical company charges \$90 per pound for a product. The decision is made to discount each pound in any order that exceeds 10 pounds by \$3 per additional pound; up to and including $10 + x$ pounds. Find the value of x beyond which revenue will start to decrease.

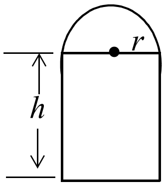
10. **(Maximize Profit)** It costs the college bookstore \$7 for a student supplement to one of its mathematics texts. The bookstore is currently selling 300 copies at \$12 per book, and it estimates that it will be able to sell 10 additional copies for each 25-cent reduction in price, and will sell 10 copies less for each 25-cent increase in price. At what price should the bookstore sell the books in order to maximize profit?
11. **(Maximize Revenue)** A computer manufacturer will, on the average, sell 25,000 units per month at \$950 per unit. It is estimated that 250 additional units will be sold per month for each \$5 decrease in price. Find the price that will maximize revenue.
12. **(Minimum Distance)** Find the point on the line $y = 2x + 1$ that is closest to the point $(1, 0)$.
13. **(Smallest Sum)** Determine the positive number which, when added to its reciprocal, yields the smallest sum.
14. **(Greatest Difference)** Determine the positive number which exceeds its cube by the greatest amount.
15. **(Maximum Area)** Find the largest possible area of a rectangle with base on the x-axis and upper vertices on the curve $y = 4 - x^2$.
16. **(Minimum Area)** Determine the right triangle of largest area that can be inscribed in a circle of radius r.
17. **(Maximum Area)** Determine the maximum area of a right triangle with hypotenuse of length 4 inches.
18. **(Maximum Area)** Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.
19. **(Minimum Area)** A poster is to surround 1200 in.^2 of printing material with a top and bottom margin of 4 in. and side margins of 3 in. Find the outside dimensions of the poster that will require the minimum amount of paper.
20. **(Maximum Volume)** Determine the maximum volume of a right circular cylinder that can be inscribed in a sphere of radius r.
21. **(Maximum Volume)** A shipping crate with base twice as long as it is wide is to be shipped by freighter. The shipping company requires that the sum of the three dimensions of the crate cannot exceed 288 inches. What are the dimensions of the crate of maximum volume?
22. **(Minimum Surface Area)** Find the dimensions of a 4ft^3 open-top rectangular box with square base requiring the least amount of material.
23. **(Minimum Cost)** A fenced-in rectangular garden is divided into 2 areas by a fence running parallel to one side of the rectangle. Find the dimensions of the garden that minimizes the amount of fencing needed, if the garden is to have an area of 15,000 square feet.
24. **(Minimum Cost)** A fenced-in rectangular garden is divided into 3 areas by two fences running parallel to one side of the rectangle. The two fences cost \$6 per running foot, and the outside fencing costs \$4 per running foot. Find the dimensions of the garden that minimizes the total cost of fencing, if the garden is to have an area of 8,000 square feet.
25. **(Minimize Cable Length)** A power line runs north-south. Town A is 3 miles due east from a point a on the power line, and town B is 5 miles due west from a point b on the power line that is 9 miles north of a. A transformer, on the power line, is to accommodate both towns. Where should it be located so as to minimize the combined cable lengths to A and B?

26. (**Shortest Ladder)** A ladder is to reach over a 8 ft fence to a wall 2 ft behind the fence. What is the length of the shortest ladder that can be used?

27. **(Minimum Commuting Time)** A lighthouse lies 2 miles offshore directly across from point A of a straight coastline. The lighthouse keeper lives 5 miles down the coast from point A. What is the minimum time it will take the lighthouse keeper to commute to work, rowing his boat at 3 miles per hour, and walking at 5 miles per hour?

28. **(Minimal Distance Between Two Cars)** At noon, car A is 10 miles due west of car B, and traveling east at a constant speed of 55 miles per hour. Meanwhile, car B is traveling north at 40 miles per hour. At what time will the two cars be closest to each other?

29. **(Maximum Light Emission)** A Norman window is a window in the shape of a rectangle surmounted by a semicircle. Find the dimensions of the base of the window that admits the most light if the perimeter of the window (total outside length) is 15 feet. (Assume that the same type of glass is used for both parts of the window.)



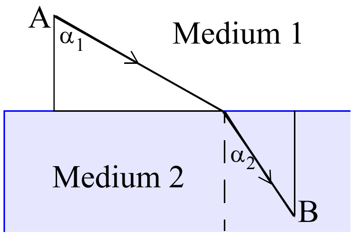
30. **(Optimizing Area**) A 16 inch wire is to be cut into two pieces. One piece is to be bent into an equilateral triangle and the other into a square. How should the wire be cut in order for the resulting combined areas to be: (a) Maximum? (b) Minimum?

31. **(Minimum Production Cost)** A union agreement stipulates that the worker of Example 4.15 will now be paid \$14 per hour plus \$4 per hour for each machine in operation. How many machines should be used to minimize cost of production?

32. **(Minimum Production Cost)** A manufacturer receives an order for N units. He can use any number of machines for the project, each capable of producing n units per hour, and each costing c dollars to be set up for the job. Once set up, the machines are fully automated, and can be supervised by a single worker, earning q dollars per hour. Derive a formula for the number of machines that should be used to minimize production cost. Show that production costs are minimum when the cost of setting up the machines equals the cost of running the machines.

33. **(Beam Strength)** A rectangular beam is to be cut from a log with circular cross section. If the strength of the beam is proportional to its width and the square of its depth, find the dimensions of the strongest beam.

34. **(Fermat's Principle and Snell's Law)** The speed of light depends on the medium through which it travels. Fermat's Principle in optics asserts that light will travel along the quickest route. Assume that the speed of light in medium 1 and medium 2 in the adjacent figure is v_1 and v_2 respectively. Show that angle of incidence α_1 and the angle of refraction α_2 will be such that $\dfrac{\sin\alpha_1}{v_1} = \dfrac{\sin\alpha_2}{v_2}$ (called **Snell's law** or the **law of refraction**).



35. **(Minimum Perimeter)** Prove that among all rectangles of a given area, the square has the smallest perimeter.

36. **(Maximum Area)** Prove that among all rectangles of a given perimeter, the square has the largest area.

37. **(Maximum Area)** Prove that among all rectangles that can be inscribed in a given circle, the square has the largest area.

38. **(Minimum Area)** Prove that the length of the square of minimal area that can be inscribed in a square of length L is of length $\frac{L}{\sqrt{2}}$.

Exercises 39-43. Use a **GRAPHING UTILITY** to find an approximate answer for the given optimization problem.

39. **(Shortest Distance)** Determine, to two decimal places, the shortest distance between a point on the curve $y = 2x^3 + 3x - 1$ and the point $\left(0, \frac{1}{2}\right)$.

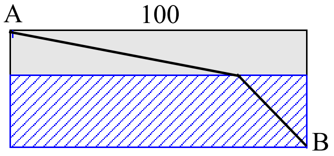
40. **(Shortest Distance)** Determine, to two decimal places, the value of b such that the distance between the points where the line $y = -x + b$ intersects the graphs of the functions $y = \sqrt{x}$ and $y = x^3 + 2$ is smallest.

41. **(Shortest Distance)** Determine, to two decimal places, the value of b such that the distance between the points where the line $y = -x + b$ intersects the graphs of the functions $y = \sqrt{x}$ and $y = x + 3$ is smallest.

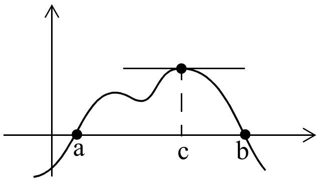
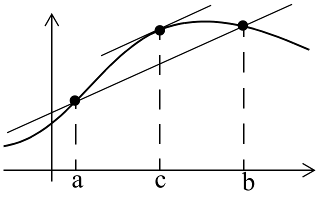
42. **(Shortest Distance)** In Example 4.16, insert an additional point D midway between plants B and C. Determine, to 2 decimal places, the point P on the road whose combined distances from the four points A, B, C, and D is minimal.

43. **(Minimum Pollution Count)** In CYU 4.24, introduce a fourth plant D that is on the same road as C and midway between C and the line joining A and B. Assuming that the pollution emission of D equals that of B, determine, to one decimal place, the point on the road joining A and B where the pollution count from the four plants is minimal.

44. **(Minimum Cost)** Point A is at ground level, and point B that is 35 feet below ground level, and 100 feet away from A (at ground level). The first 15 feet below ground level is soil, after which there is shale. A pipe is to join the two points. It costs \$76 per foot to lay piping in the soil layer, and \$245 per foot to lay piping in the shale layer. Find the minimum labor cost of the project.



CHAPTER SUMMARY

ROLLE'S THEOREM	Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) for which $f'(c) = 0$. 
MEAN VALUE THEOREM	If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. 
INTERMEDIATE VALUE THEOREM	If f is continuous on the closed interval $[a, b]$ and if r is a number lying between $f(a)$ and $f(b)$, then there exists at least one c between a and b such that $f(c) = r$.
THEOREM	Let f be differentiable on the open interval $I = (a, b)$ (or (a, ∞) or $(-\infty, b)$. (a) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I. (b) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I. (c) If $f'(x) = 0$ for all $x \in I$, then f is constant on I.
LOCAL MAXIMUM AND LOCAL MINIMUM	A function f has a **local** (or **relative**) maximum at an interior point c in its domain if $f(c) \geq f(x)$ for all x sufficiently close to c. A function f has a **local** (or **relative**) minimum at c if $f(c) \leq f(x)$ for all x sufficiently close to c.
THEOREM	Let f be differentiable in some open interval containing c. If f has a local maximum or a local minimum at c, then $f'(c) = 0$.
CRITICAL POINT	If c is an interior point in the domain of a function f at which a local maximum or minimum occurs, then either $f'(c) = 0$ or $f'(c)$ does not exist. The points at which $f'(x) = 0$ or f' does not exist are called **critical points**.
MAX/MIN THEOREM	A continuous function on a closed interval $[a, b]$ achieves its maximum value and its minimum value on $[a, b]$.

GRAPHING FUNCTIONS	**SIGN $\boldsymbol{f}$, SIGN $\boldsymbol{f'}$**, and **SIGN $\boldsymbol{f''}$** **PLAY ROLES WHEN GRAPHING A FUNCTION $\boldsymbol{f}$:** **SIGN $\boldsymbol{f}$** +: graph lies above -: lies below the x-axis **SIGN $\boldsymbol{f'}$** +: graph is increasing -: is decreasing **SIGN $\boldsymbol{f''}$** +: graph is concave up -: is concave down
FAR FROM THE ORIGIN	As $x \to \pm\infty$, the graph of the polynomial function: $p(x) = \boldsymbol{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ resembles, **in shape**, that of its leading term $g(x) = \boldsymbol{a_n x^n}$. As $x \to \pm\infty$, the graph of the rational function: $f(x) = \dfrac{\boldsymbol{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{\boldsymbol{b_m x^m} + b_{m-1}x^{m-1} + \cdots + b_0}$ will resemble, **in shape**, that of: $g(x) = \dfrac{\boldsymbol{a_n x^n}}{\boldsymbol{b_m x^m}}$
ASYMPTOTES	When the degree of the numerator of a rational function f is less than or equal to the degree of the denominator, the graph will approach a horizontal line, called a **horizontal asymptote** for the graph of f. When the degree of the numerator of a rational function f is one more than that of the denominator, the graph will approach an oblique line, called an **oblique asymptote** for the graph of f.

CHAPTER 5
INTEGRATION

§1. THE INDEFINITE INTEGRAL

A question for you:

$$(\mathbf{?})' = 3x^2$$

One answer: x^3 [since $(x^3)' = 3x^2$].

A similar question:
$(?)^2 = 49$
Answer: 7 and -7.

We say that x^3 is **an antiderivative** of $3x^2$. We do not call it "the antiderivative," since there are infinitely many functions whose derivatives are $3x^2$; here are a couple more: $x^3 + 9$, and $x^3 - 173$.

In general:

DEFINITION 5.1
ANTIDERIVATIVE

An **antiderivative** of a function f is a function whose derivative is f.

CHECK YOUR UNDERSTANDING 5.1

Find two different antiderivatives for the function $f(x) = 8x^7$.

One possible answer:
x^8 and $x^8 + 1$

Are there antiderivatives of $f(x) = 3x^2$ that are not of the type $x^3 + c$ for some constant c? No:

THEOREM 5.1 If $f'(x) = g'(x)$ then $f(x) = g(x) + C$ for some constant C.

PROOF: CYU 4.3, page 124.

The fact that all antiderivatives of a function can be generated by adding an arbitrary constant to any one of its antiderivatives enables us to formulate the following definition:

DEFINITION 5.2
INDEFINITE INTEGRAL

The collection of all antiderivatives of f is called the **indefinite integral** of f and is denoted by $\int f(x)dx$. In other words:

$$\int f(x)dx = g(x) + C$$

where $g(x)$ is any antiderivative of $f(x)$. The number C in the above notation represents an arbitrary (real) number and is called the **constant of integration.**

The reason for the form $\int f(x)dx$ will surface in the next section.

For example:

Since x^3 is an antiderivative of $3x^2$: $\int 3x^2dx = x^3 + C$:

Since $(x^8)' = 8x^7$: $\int 8x^7dx = x^8 + C$

How can you justify the claim that $\sqrt{49} = 7$? Easy: $7^2 = 49$. By the same token:

Here is a special case:
$\int 1\,dx = x + C$
(Recall that $1 = x^0$)

THEOREM 5.2 For any number $r \neq -1$:

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C$$

PROOF:

$$\left(\frac{x^{r+1}}{\boldsymbol{r+1}}\right)' = \frac{1}{\boldsymbol{r+1}}(x^{r+1})' = \frac{1}{r+1}\cdot(r+1)x^r = x^r$$

$[c\cdot f(x)]' = c\cdot f'(x)$

$(x^n)' = nx^{n-1}$

For example:

up one

$$\int x^9 dx = \frac{x^{10}}{10} + C$$

up one

$$\int x^{-5} dx = \frac{x^{-4}}{-4} + C$$

up one

$$\int x^{\frac{2}{3}} dx = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C$$

divided by the “upped one”

Turning the differentiation theorems around:

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$
$$[cf(x)]' = cf'(x)$$

brings us to the following result:.

THEOREM 5.3

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$
$$\int cf(x)dx = c\int f(x)dx$$

For example:

$$\int (5x^3 + x^2 - 2x + 3)dx = 5\int x^3 dx + 1\int x^2 dx - 2\int x dx + 3\int 1\,dx$$

The four constants associated with the above integrals are combined into one constant C:

$$= 5\cdot\frac{x^4}{4} + \frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 3x + C$$
$$= \frac{5}{4}x^4 + \frac{x^3}{3} - x^2 + 3x + C$$

Check: $\left(\frac{5}{4}x^4 + \frac{x^3}{3} - x^2 + 3x + C\right)' = 5x^3 + x^2 - 2x + 3$

CHECK YOUR UNDERSTANDING 5.2

Determine:

(a) $\int 5x^4 dx$ (b) $\int -4x^{-5} dx$ (c) $\int (2x^5 + 4x^3 - \frac{1}{3}x^2 + 2)dx$

Answers:
(a) $x^5 + C$ (b) $x^{-4} + C$
(c) $\frac{x^6}{3} + x^4 - \frac{x^3}{9} + 2x + C$

Please note that **only constant factors** can be "extracted" from an integral. In particular, as you can easily verify:

$$\int(4+x^7)dx \neq 4+\int x^7 dx \text{ and } \int x(x^2+x)dx \neq x\int(x^2+x)dx$$

Moreover, as it is with derivatives, it is important for you to remember that:

The integral of a product, (or quotient) is NOT the product (or quotient) of the integrals.

In particular, as you can easily verify:

$$\int(2x-5)(x+4)dx \neq \int(2x-5)dx\cdot\int(x+4)dx$$

and: $$\int\frac{2x^5-3x+1}{x^3}dx \neq \frac{\int(2x^5-3x+1)dx}{\int(x^3)dx}$$

But not all is lost:

EXAMPLE 5.1

Determine:

(a) $\int(2x-5)(x+4)dx$

(b) $\int\frac{2x^5-3x+1}{x^3}dx$

SOLUTION: The "trick" is to rewrite the given expression as powers of x, and then apply Theorems 5.2 and 5.3:

(a) $$\int[(2x-5)(x+4)]dx = \int(2x^2+3x-20)dx = \frac{2x^3}{3}+\frac{3x^2}{2}-20x+C$$

(b) $$\int\left(\frac{2x^5-3x+1}{x^3}\right)dx = \int\left(\frac{2x^5}{x^3}-\frac{3x}{x^3}+\frac{1}{x^3}\right)dx = \int(2x^2-3x^{-2}+x^{-3})dx$$

"up one divided by that up one:"

$$= \frac{2x^3}{3}-\frac{3x^{-1}}{-1}+\frac{x^{-2}}{-2}+C = \frac{2x^3}{3}+\frac{3}{x}-\frac{1}{2x^2}+C$$

CHECK YOUR UNDERSTANDING 5.3

Determine:

(a) $\int(3x^2-2x+1)(x-5)dx$ (b) $\int\frac{x^4-2x-6}{x^4}dx$

Answers:

(a) $\frac{3}{4}x^4-\frac{17}{3}x^3+\frac{11}{2}x^2-5x+C$

(b) $x+\frac{1}{x^2}+\frac{2}{x^3}+C$

Turning around the following derivative formulas:

$$\text{(a) } \frac{d}{dx}(\sin x) = \cos x \qquad \text{(b) } \frac{d}{dx}(\cos x) = -\sin x$$

$$\text{(c) } \frac{d}{dx}(\tan x) = \sec^2 x \qquad \text{(d) } \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\text{(e) } \frac{d}{dx}(\sec x) = \sec x \tan x \qquad \text{(f) } \frac{d}{dx}(\csc x) = -\csc x \cot x$$

we have:

THEOREM 5.4

$$\text{(a) } \int \sin x\, dx = -\cos x + C \qquad [\text{since } (-\cos x)' = \sin x]$$

$$\text{(b) } \int \cos x\, dx = \sin x + C \qquad [\text{since } (\sin x)' = \cos x]$$

$$\text{(c) } \int \sec^2 x\, dx = \tan x + C$$

$$\text{(d) } \int \csc^2 x\, dx = -\cot x + C$$

$$\text{(e) } \int \sec x \tan x\, dx = \sec x + C$$

$$\text{(f) } \int \csc x \cot x\, dx = -\csc x + C$$

CHECK YOUR UNDERSTANDING 5.4

Determine:

(a) $\int (\sin x + 2\cos x)dx$ (b) $\int (x^2 - \sec x \tan x)dx$

Answers:
(a) $-\cos x + 2\sin x + C$
(b) $\frac{x^3}{3} - \sec x + C$

DIFFERENTIAL EQUATIONS

A **differential equation** is an equation that involves derivatives of an unknown function (or functions). Consider the following example:

EXAMPLE 5.2 Solve the differential equation:

$$f'(x) = 2x^2 + 3x - 1, \text{ if } f(1) = 2$$

SOLUTION: $f(x) = \int (2x^2 + 3x - 1)dx = \frac{2x^3}{3} + \frac{3x^2}{2} - x + C$

To find C, we use the given information that $f(1) = 2$:

If $x = \mathbf{1}, f(x) = \mathbf{2}$:

$$\mathbf{2} = \frac{2 \cdot \mathbf{1}^3}{3} + \frac{3 \cdot \mathbf{1}^2}{2} - \mathbf{1} + C$$

$$C = 2 - \frac{2}{3} - \frac{3}{2} + 1 = \mathbf{\frac{5}{6}}$$

Solution: $f(x) = \frac{2x^3}{3} + \frac{3x^2}{2} - x + \mathbf{\frac{5}{6}}$

EXAMPLE 5.3 Solve the second-order differential equation:

$$f''(x) = 2x + 2\cos x$$

if $f'(0) = \frac{\pi}{2}$ and $f(0) = 1$

SOLUTION: Since $[f'(x)]' = 2x + 2\cos x$:

$$f'(x) = \int(2x + 2\cos x)dx = x^2 + 2\sin x + C$$

Since $f'(0) = \frac{\pi}{2}$: $\frac{\pi}{2} = 0^2 + 2\sin 0 + C \Rightarrow C = \frac{\pi}{2}$.

We now have: $f'(x) = x^2 + 2\sin x + \frac{\pi}{2}$.

Integrating: $f(x) = \frac{x^3}{3} - 2\cos x + \frac{\pi}{2}x + C$

Since $f(0) = 1$: $1 = \frac{0^3}{3} - 2\cos 0 + \frac{\pi}{2} \cdot 0 + C \Rightarrow C = 3$

recall that $\cos 0 = 1$

Thus: $f(x) = \frac{x^3}{3} - 2\cos x + \frac{\pi}{2}x + 3$.

CHECK YOUR UNDERSTANDING 5.5

Solve the differential equation:

$$f'(x) = 5x^4 - 2, \text{ if } f(0) = 1$$

Answer:
$f(x) = x^5 - 2x + 1$

FREE FALLING OBJECTS

Due to the force of gravity, an object released near the surface of the earth will accelerate at a rate of (approximately) 32 feet per second per second: $a(t) = -32 \text{ ft/sec}^2$ (or $-9.8 \text{ } m/\text{sec}^2$). The negative sign indicates that the object is accelerating in a downward direction.

Based solely on the above measured force of gravity and the force of mathematics, we are able to express velocity and position of the object (while in flight) as a function of time:

By convention, a positive velocity indicates an upward movement, while a negative velocity indicates a downward movement. Also, a positive position indicates "up" from the reference point, and a negative position indicates "down."

THEOREM 5.5 If an object is thrown, in a vertical direction, with initial velocity v_0 (in feet per second), from a point that is s_0 feet from a fixed reference point, then t seconds later the velocity (in feet per second) of the object is given by:

$$v(t) = -32t + v_0$$

and the position (in feet) of the object from the fixed reference point is given by:

$$s(t) = -16t^2 + v_0t + s_0$$

PROOF: Since acceleration is the derivative of velocity with respect to time, velocity is the integral of acceleration:

acceleration due to gravity

$$v(t) = \int a(t)dt = \int(-32)dt = -32t + C$$

When $t = 0$, $v = v_0$. So, $v_0 = -32 \cdot 0 + C$, or $C = v_0$, and this brings us to the velocity equation: $v(t) = -32t + v_0$.

Since velocity is the derivative of position with respect to time, position is the integral of velocity:

$$s(t) = \int v(t)dt = \int(-32t + v_0)dt = -16t^2 + v_0t + C$$

When $t = 0$, $s = s_0$. So, $s_0 = -16 \cdot 0^2 + v_0 \cdot 0 + C$; or: $C = s_0$, and this brings us to the position equation: $s(t) = -16t^2 + v_0t + s_0$.

EXAMPLE 5.4 A stone is dropped from a height of 1600 feet. What is its speed on impact with the ground?

SOLUTION: Since the stone is dropped, $v_0 = 0$, and the velocity and position functions of Theorem 5.5 take the form:

$$v(t) = -32t$$

and:

$$s(t) = -16t^2 + 1600$$

We don't have to tell you that the ground is our reference point, as this is implied by the above equation (how?).

Since both the velocity and position functions are functions of time, the critical step in most gravity problems, is to **find the particular t of interest**.

Setting position to zero, we determine the time it takes for the stone to hit the ground:

$$0 = -16t^2 + 1600$$

$$t^2 = 100, \text{ or } t = 10 \text{ (seconds)}$$

Evaluating the velocity function at $t = 10$, we find the impact velocity:

$$v(10) = -32 \cdot 10 = -320 \text{ (feet per second)}$$

By definition, speed is the magnitude of velocity. Thus, the impact speed is 320 feet per second.

CHECK YOUR UNDERSTANDING 5.6

A stone is thrown upward from the roof of a 80 foot building at a speed of 64 feet per second.

(a) Find the maximum height of the stone (with respect to the ground).

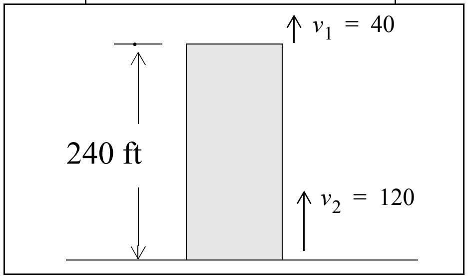
(b) At what speed will the stone hit the ground?

Answers: (a) 144 feet
(b) 96 ft/sec.

EXAMPLE 5.5 Object-one is thrown upward from the top of a 240-foot building at a speed of 40 feet per second. At the same time, object-two is catapulted up from the ground at 120 feet per second along the same vertical line. Will the objects collide? If so, determine the directions of the objects at collision.

SOLUTION:

SEE THE PROBLEM



Here are the velocity and position equations governing the fate of the two objects:

Object-one	Object-two
$v_1(t) = -32t + 40$	$v_2(t) = -32t + 120$
$s_1(t) = -16t^2 + 40t + 240$	$s_2(t) = -16t^2 + 120t$

As is evident from the position functions, the ground is our chosen reference point.

Solving $s_1(t) = s_2(t)$ will yield the time of impact (objects occupy the same point in space):

$$\begin{aligned} s_1(t) &= s_2(t) \\ -16t^2 + 40t + 240 &= -16t^2 + 120t \\ 80t &= 240 \\ t &= 3 \end{aligned}$$

At this point we know that collision, if it occurs, must take place three seconds into flight. Will they collide? Yes:

$$s_2(3) = -16 \cdot 3^2 + 120 \cdot 3 = 216$$

Collision occurs 216 feet above the ground.

(We used $s_2(3)$ to find the point of collision. Could we have gone with $s_1(3)$?)

Had $s_2(3)$ turned out to be negative, then collision **would not occur** (why not?).

At collision:

$$v_1(3) = -32 \cdot 3 + 40 = -56 \quad \text{and} \quad v_2(3) = -32 \cdot 3 + 120 = 24$$

From the above, we see that at collision the first object is falling at a speed of 56 feet per second, while the second object is rising at a speed of 24 feet per second.

CHECK YOUR UNDERSTANDING 5.7

An object is propelled upward from a 128-foot building at a speed of 32 feet per second. At the same time, a second object is catapulted upward from ground level along the same vertical line. Determine the speed of the second object if collision is to occur precisely when the first object reaches its maximum height.

Answer: 160 feet per second.

EXERCISES

Exercises 1-26. Determine:

1. $\int 3\,dx$
2. $\int (3+3x)dx$
3. $\int (6x^5+5x^4)dx$
4. $\int (4x^3-3x^2+5x-2)dx$
5. $\int \left(\frac{x^4}{5}-\frac{3}{x^5}\right)dx$
6. $\int (x^9-x^{-9})dx$
7. $\int \left(3x^4-4x^{-4}+\frac{2}{x^5}\right)dx$
8. $\int x(3x-2)dx$
9. $\int x^2(2x-5)dx$
10. $\int (3x^2-2)(x^3+x)dx$
11. $\int x(x-1)(x+1)dx$
12. $\int \frac{3x^5+2x-1}{x^4}dx$
13. $\int \frac{x^6+x^2-x^{-2}}{2x^4}dx$
14. $\int \frac{(2x^3+1)(x^4+x^2)}{2x}dx$
15. $\int \frac{(x^4+x)(x+1)}{x^4}dx$
16. $\int \sqrt{x}\,dx$
17. $\int x^{-3/5}dx$
18. $\int (2x^{1/3}+x^3)dx$
19. $\int \sqrt{x}(x^2+x-3)dx$
20. $\int \frac{x^2+x-5}{\sqrt{x}}dx$
21. $\int \frac{x(2x^{1/3}+x^3)}{x^{2/3}}dx$
22. $\int \left(3\sin x-\frac{1}{2}\cos x+1\right)dx$
23. $\int (\sec x\tan x-\sec^2 x)dx$
24. $\int \frac{\sin x+\sqrt{x}}{5}dx$
25. $\int \frac{\sec x-\tan x}{\cot x}dx$
26. $\int \frac{\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}{1+\cos 2x}dx$

Exercises 27-38. **(Differential Equations)** Solve:

27. $f'(x) = 3x+5,\ f(5) = 1$
28. $f'(x) = 3x+5,\ f(1) = 5$
29. $f'(x) = 3x^2+5x,\ f(1) = 1$
30. $f'(x) = 3x^2+5x,\ f(1) = 5$
31. $f'(x) = x^3+5x-2,\ f(0) = 1$
32. $f'(x) = x^3+5x-2,\ f(1) = 0$
33. $f'(x) = \frac{3x^2+5x}{x^4},\ f(1) = 2$
34. $f'(x) = \frac{3x^2+5x}{x^4},\ f(2) = 1$
35. $f'(x) = (2x+3)(x-1),\ f(1) = 0$
36. $f'(x) = (2x+3)(x-1),\ f(2) = 1$
37. $f''(x) = 3x+5,\ f'(0) = 1, f(1) = 1$
38. $f''(x) = 3x^2+5x,\ f'(1) = 1, f(2) = 1$

Exercises 39-42. Verify the given claim:

39. $\int x(x^2-1)^4dx = \frac{(x^2-1)^5}{10}+C$

40. $\int x^2\sqrt{x^3+4}\,dx = \frac{2}{9}(x^3+4)^{\frac{3}{2}}+C$

41. The function $y = \cos x$ is a solution of the differential equation $(y')^2+y^2-1 = 0$.

42. The function $y = -\frac{1}{2}x + \frac{1}{4}$ is a solution of the second order differential equation $y'' - y' - 2y = x$.

43. **(From Slope to Function)** The slope of the tangent line to the graph of a function f at $(x, f(x))$ is x^2. Find the function, if its graph passes through the point (1,5).

44. **(From Slope to Function)** The slope of the tangent line to the graph of a function f at $(x, f(x))$ is $2x^3 + x - 1$. Find the function, if its graph passes through the point (0,1).

45. **(Impact Speed)** A stone is dropped from a height of 3200 feet. What is its speed on impact with the ground?

46. **(Initial Speed)** At what speed should an object be tossed upwards, in order for it to reach a maximum height of 160 feet from the point of its release?

47. **(Bouncing Height)** An object is thrown downward from a 96 foot building at a speed of 16 feet per second. Upon hitting the ground, it bounces back up at three-quarters of its impact speed. How high will it bounce?

48. **(Collision Velocity)** An object is thrown downward from a 264 foot building at a speed of 24 feet per second, at the same time that an object is thrown up from the ground at 64 feet per second. Assuming that the two objects are in line with each other, determine the velocity of both objects when they collide.

49. **(Particle Position)** Let $s(t) = t^3 - t$ represent the position function of a particle moving along the x-axis, where $t > 0$ is measured in minutes and s in meters.
    - (a) Draw a diagram to represent the motion of the particle.
    - (b) When is the particle moving to the right? Moving to the left?
    - (c) When is the particle speeding up? When is it slowing down?
    - (d) Determine the total distance traveled by the particle during the first five minutes.

50. **(Particle Position)** Repeat Exercise 49 for the position function $s(t) = t^4 - 2t^3 - 3t^2$.

51. **(Stopping Distance)** After its brakes are applied, a car decelerates at a constant rate of 30 feet per second per second. Compute the stopping distance, if the car was going 60 miles per hour (88 ft/sec) when the brakes were applied.

52. **(Stopping Distance)** After its brakes are applied, a car decelerates at a constant rate of 30 feet per second per second. Compute the speed of the car at the point at which the brakes were applied, if the stopping distance turned out to be 120 feet.

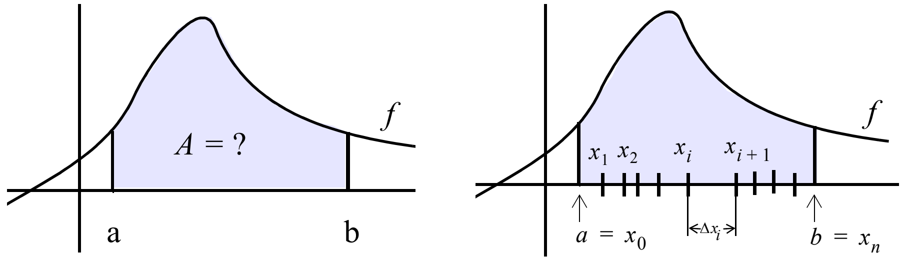
53. **(Theory)** An object is tossed upward from the ground with an initial velocity of v_0 feet per second.
    - (a) Determine the maximum height M reached by the object.
    - (b) Prove that at any height h, with $0 < h < M$, the object's speed when going up is equal to its speed when going down.

§2. THE DEFINITE INTEGRAL

Our geometrical quest for slopes of tangent lines led us to the definition of the derivative. We now go on another quest, that of finding the area A in Figure 5.1(a), which is bounded above by the graph of the function $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. As it was with the tangent line situation, we know what we are looking for, but still have to find it (to **define** it). Here goes:

This "area quest" will lead us to the definition of another immensely useful object — the definite integral. At first blush, the definite integral does not appear to have any connection whatsoever with the indefinite integral of the previous section. But, as you will see, there is a beautiful connection, and it is called the Fundamental Theorem of Calculus.

Loosely speaking, partition the interval $[a, b]$ into subintervals $[x_i, x_{i+1}]$ of length $\Delta x_i = x_{i+1} - x_i$ [see Figure 5.1(b)].



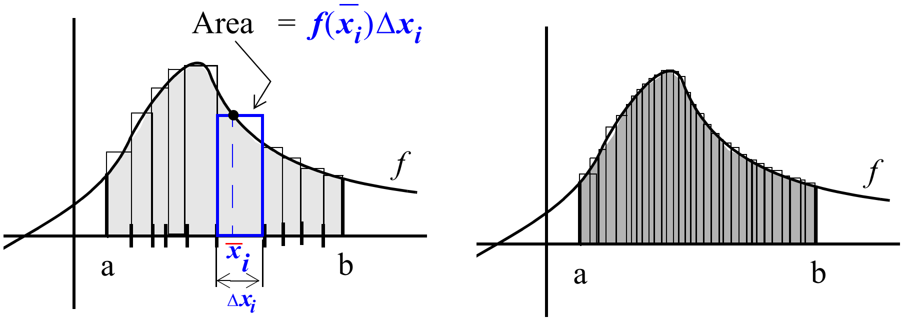
(a) (b)

Figure 5.1

Pick an arbitrary point $\bar{x}_i$ in each subinterval $[x_i, x_{i+1}]$, and construct the rectangles of base Δx_i and height $f(\bar{x}_i)$ [see Figure 5.2(a)]. Let's denote the sum of the areas of all those rectangles by the symbol $\sum_a^b f(x)\Delta x$ — a sum that gives us an approximation for the area in question.

The Greek letter sigma, denoted by $\sum$, indicates a sum.

We use $\sum_a^b f(x)\Delta x$ to represent the more intimidating form $\sum_{1=1}^n f(\bar{x}_i)\Delta x_i$, where Δx_i and x_i are depicted in Figure 5.2(a).



(a) (b)

Figure 5.2

The sums $\sum_a^b f(x)\Delta x$ are called **Riemann sums**, after the German mathematician Georg Riemann (1826-1866).

Clearly, the smaller we make those Δx_i's, the closer $\sum_a^b f(x)\Delta x$ will get to the area we are looking for [see Figure 5.2(b)]. And so, we (naturally) define the area A to be:

$$A = \lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x$$

By "$\lim_{\Delta x \to 0}$" we mean the limit as the length of the largest Δx_i tends to 0.

Limits of Riemann sums play many important roles throughout mathematics, bringing us to the following definition:

The symbol $\int_a^b f(x)dx$ is one "word." In particular, "dx" is just a "letter" in that word, that's all. The notation does, however, recall its origin: the sum symbol $\sum_a^b$ "evolving" into $\int_a^b$, and the Δx into "dx."

DEFINITION 5.3
DEFINITE INTEGRAL

A function f is said to be **integrable** over the interval $[a, b]$ if $\lim\limits_{\Delta x \to 0} \sum_a^b f(x)\Delta x$ exists. In this case, we write:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x$$

and call the number $\int_a^b f(x)dx$ the **integral** of f over $[a, b]$.

In addition: $\int_a^a f(x)dx = 0$ for any function with a in its domain.

We would not want it any other way, since "there is no area between a and a."

As it turns out, it is "easier" for a function to be integrable than it is for it to be differentiable. In particular, though a continuous function need not be differentiable [see Figure 3.4(b), page 71], it can be shown that:

THEOREM 5.6 If f is continuous on $[a, b]$, then it is integrable over that interval.

Both the definition of the derivative and that of the definite integral involve limits. The limit situation for the integral, however, is much more complicated than that of the derivative: we have to worry about partitioning the given interval, and then we have to compute the Riemann sum for that partition, and then we have to see if all the Riemann sums approach something as the largest Δx of the partition tends to zero. This gets way out of hand, even for relatively simple functions like $f(x) = x^2 + x$. Help is on the way.

The Principal and Fundamental Theorems of Calculus

The derivative and the definite integral are really quite different objects. The derivative gives slopes of tangent lines to a curve, while the integral yields the area under a curve (at least for positive functions). At first glance, one would not assume that these two concepts are related to each other; but they are:

THEOREM 5.7
PRINCIPAL THEOREM OF CALCULUS

For f continuous on $[a, b]$, let:

$$T(x) = \int_a^x f(t)dt$$

Then T is differentiable on (a, b) and:

$$T'(x) = f(x)$$

We content ourselves by offering a geometrical argument suggesting the validity of the above amazing result which links the concepts of the derivative with that of the integral. Our first order of business is to explain the nature of that strange looking function $T(x) = \int_a^x f(t)dt$. To keep our discussion on a geometric level, we assume that the graph of the function f lies above the t-axis over some interval $[a, b]$ [see Figure 5.3]. Note that the function $T(x)$ simply gives the indicated "**this Area**" over the interval $[a, x]$,.

Note that the horizontal axis is labeled t. We can't call it x, since we chose the variable x for our "main" function T.

We like to call T a *Trombone* function — as you slide the variable x back and forth, you get less or more area from the "integral instrument:"

$$T(x) = \int_a^x f(t)dt$$

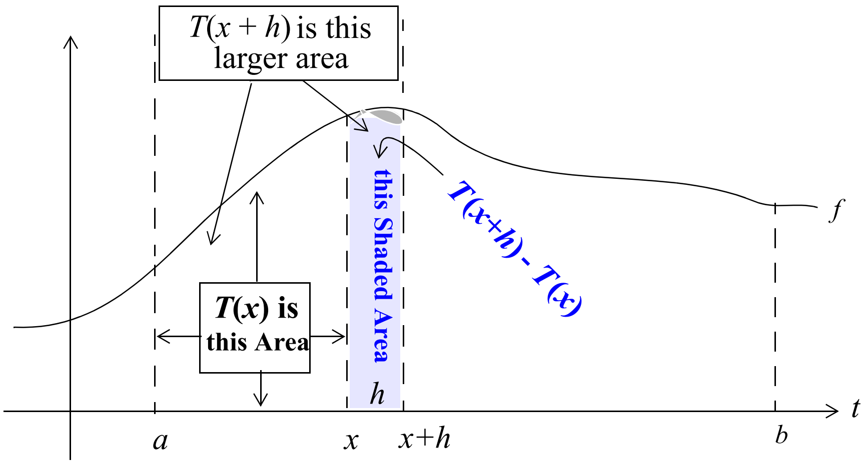


Figure 5.3

From the above figure, we see that:

$$\frac{\overbrace{T(x+h)-T(x)}^{\text{area of shaded region in Figure 5.3}}}{\underset{\text{base of shaded region}}{h}} \Longrightarrow \text{average height of shaded region}$$

As h approaches 0, the average height of the shaded region must approach the height at x: $f(x)$. Hence, as advertised in Theorem 5.7:

$$T'(x) = \lim_{h \to 0} \frac{T(x+h)-T(x)}{h} = f(x)$$

EXAMPLE 5.6 Find the derivative of the function:

$$T(x) = \int_1^x \frac{(t+5)^5}{t^2+2} dt$$

SOLUTION: Applying Theorem 5.7 with $f(t) = \frac{(t+5)^5}{t^2+2}$, we have:

$$T'(x) = f(x) = \frac{(x+5)^5}{x^2+2}$$

Answer: $(3x^2+2)^7$

CHECK YOUR UNDERSTANDING 5.8

Find the derivative of the function:

$$T(x) = \int_3^x (3t^2+2)^7 dt$$

What is so great about Theorem 5.7? For one thing it will enable us to establish the next theorem which says that:

> IF you can find an antiderivative g of a function f, then you can determine the complicated limit $\lim\limits_{\Delta x \to 0} \sum\limits_a^b f(x)\Delta x$ by simply subtracting the number $g(a)$ from the number $g(b)$:

You can now see why similar notation and terminology is used for both the definite and indefinite integral. The connection is this theorem which links the definite integral $\int_a^b f(x)dx$ with a $g(x)$ of the indefinite integral $\int f(x)dx$.

THEOREM 5.8
FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on $[a, b]$ and if $g'(x) = f(x)$, then:

$$\int_a^b f(x)dx = g(b) - g(a)$$

PROOF: We are given that g is an antiderivative of f, and Theorem 5.7 gives us another. Theorem 5.1, page 167, tells us that these two antiderivatives can differ only by a constant C, bringing us to:

$$g(x) = \int_a^x f(t)dt + C$$

Evaluating both sides of the above equation at $x = a$, we have:

$$g(a) = \underbrace{\int_a^a f(t)dt}_{\text{Definition 5.3}} + C = 0 + C = C\text{, i.e: } C = g(a)$$

At this point we know that $g(x) = \int_a^x f(t)dt + \overset{\mathbf{C}}{\boldsymbol{g(a)}}$. Evaluating both sides of this equation at $x = b$ brings us to:

$$g(b) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(t)dt} + g(a)$$

$$\textbf{\textit{or:}}\quad \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(t)dt} = g(b) - g(a) \quad (*)$$

Since the variable x is no longer in use, we can choose to substitute x for t in (*) to arrive at our desired result:

$$\int_a^b f(x)dx = g(b) - g(a)$$

NOTATION: The difference $g(b)-g(a)$ is denoted by the symbol $g(x)\big|_a^b$, leading us to the form:

$$\int_a^b f(x)dx = g(x)\big|_a^b = g(b)-g(a)$$

EXAMPLE 5.7

Evaluate:

$$\int_1^2 (3x^2+2)dx$$

SOLUTION: Since $g(x) = x^3+2x$ is an antiderivative of $f(x) = 3x^2+2$, we have:

$$\int_1^2 (3x^2+2)dx = (x^3+2x)\big|_{\mathbf{1}}^{\mathbf{2}} = \overbrace{(\mathbf{2}^3+2\cdot\mathbf{2})}^{g(\mathbf{2})} - \overbrace{(\mathbf{1}^3+2\cdot\mathbf{1})}^{g(\mathbf{1})} = 9$$

CHECK YOUR UNDERSTANDING 5.9

Referring to Example 5.7, see what happens if you use $x^3+2x+100$, instead of x^3+2x, as the chosen antiderivative of $3x^2+2$.

Answer: Same result.

EXAMPLE 5.8

Determine the area of the region over the interval $[1, 2]$ that is bounded above by the graph of the function:

$$f(x) = \frac{x^3+x^2+1}{x^2}$$

SOLUTION: Since the function f is positive (see margin) over the indicated interval, the area in question is given by the integral:

$$\int_1^2 \frac{x^3+x^2+1}{x^2}\,dx$$

Which we now evaluate:

$$\int_1^2 \frac{x^3+x^2+1}{x^2}\,dx = \int_1^2 \left(\frac{x^3}{x^2}+\frac{x^2}{x^2}+\frac{1}{x^2}\right)dx = \int_1^2 (x+1+x^{-2})\,dx$$

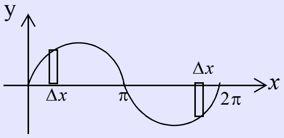
$$= \left(\frac{x^2}{2}+x+\frac{x^{-1}}{-1}\right)\Bigg|_1^2 = \left(\frac{x^2}{2}+x-\frac{1}{x}\right)\Bigg|_1^2$$

$$= \left(\frac{2^2}{2}+2-\frac{1}{2}\right)-\left(\frac{1}{2}+1-\frac{1}{1}\right) = 3$$

Note that:

$$\int_0^{2\pi} \sin x = -\cos x\big|_0^{2\pi}$$
$$= -\cos 2\pi - (-\cos 0)$$
$$= -1+1 = 0$$

does **not** represent the area bounded by the sine graph and the x-axis:



As you can see, each positive $f(x)\Delta x$ over the interval $[0, \pi]$ is counterbalanced by a negative $f(x)\Delta x$ over the interval $[\pi, 2\pi]$ — accounting for zero Riemann sums.

CHECK YOUR UNDERSTANDING 5.10

(a) Evaluate:

(i) $\int_0^1 (x^3 + x - 1)dx$ (ii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\ dx$

(b) Determine the area of the region over the interval $[-1, 1]$ that is bounded above by the graph of the function:

$$f(x) = (x^2 + 1)(x^2 + 3)$$

Answers:

(a) (i) $-\frac{1}{4}$ (ii) $\frac{1}{\sqrt{2}}$

(b) $\frac{136}{15}$ square units.

In the definition of $\int_a^b f(x)dx$, the lower limit of integration, a, was less than the upper limit of integration, b. Is there a reasonable way of defining an integral such as $\int_4^2 (2x + 1)dx$? Yes, for if we formally apply the Fundamental Theorem of Calculus to that expression, we obtain:

$$\int_4^2 (2x + 1)dx = (x^2 + x)\Big|_4^2 = (2^2 + 2) - (4^2 + 4) = 6 - 20 = -14$$

On the other hand:

$$\int_2^4 (2x + 1)dx = (x^2 + x)\Big|_2^4 = (4^2 + 4) - (2^2 + 2) = 20 - 6 = 14$$

The above observations leads us to:

DEFINITION 5.4 For f integrable on $[a, b]$:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

In words: Switching the limits of integration introduces a minus sign.

As it was with indefinite integrals:

THEOREM 5.9 If f and g are continuous on $[a, b]$ then:

(a) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(b) $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(c) $\int_a^b cf(x)\,dx = c\int_a^b f(x)$ for any constant c.

Theorem 5.7 assures us that f and g have antiderivatives.

PROOF: For F and G antiderivatives of f and g, respectively, $F + G$ is an antiderivative of $f + g$ [Theorem 3.2(d), page 78]. So:

$$\int_a^b [f(x) + g(x)]dx = [F(x) + G(x)]\Big|_a^b = [F(b) + G(b)] - [F(a) + G(a)]$$

$$= [F(b) - F(a)] + [G(b) - G(a)]$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

CHECK YOUR UNDERSTANDING 5.11

Prove Theorem 5.9(b) and (c).

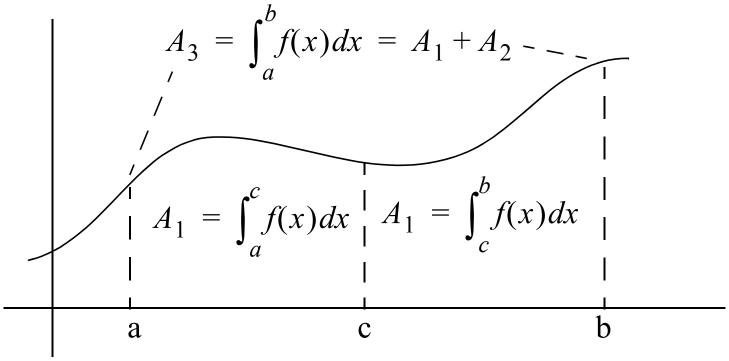
Answer: See page A-28.

The following theorem tells us that the "integral journey" from a to c can be broken down into pieces.

THEOREM 5.10 If f is continuous on $[a, b]$, and $a < c < b$, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

"PROOF:" We offer a geometrical argument without words. Think "Area:"



CHECK YOUR UNDERSTANDING 5.12

Let $\int_a^c f(x)dx = 5$, $\int_c^b f(x)dx = -3$ and $\int_a^b g(x)dx = 7$.

Evaluate:

(a) $\int_a^c -2f(x)dx + \int_b^a g(x)dx$ (b) $\int_a^b f(x)dx + \int_a^b 2g(x)dx$

Answers: (a) –17 (b) 16

NET-CHANGE DERIVED FROM RATE OF CHANGE

Question: Suppose that f is differentiable on $[a, b]$ what is the value of $\int_a^b f'(x)dx$?

Answer: $$\int_a^b f'(x)dx = f(b) - f(a).$$

Why? Because $f(x)$ is an antiderivative of $f'(x)$, that's why.

Calling the difference $f(b) - f(a)$ the **net-change** of the function f over the interval $[a, b]$, we have observed that:

THEOREM 5.11 The net-change of a differentiable function f from $x = a$ to $x = b$ is given by:

$$\text{Net-change} = \int_a^b f'(x)dx$$

EXAMPLE 5.9 Oil is leaking out of a ruptured tanker at a rate of $125 - \frac{t}{50}$ gallons per minute, where t is measured in minutes. How many gallons leak out during:

(a) the first hour? (b) the second hour?

Units can help point the way. We are given a rate in gallons per minute; and want to end up with total gallons over a specified period of time:

$$\text{gal} = \frac{\textbf{gal}}{\textbf{min}} \cdot \text{min}$$

$$\int_0^{60} \left(\mathbf{125} - \frac{\boldsymbol{t}}{\mathbf{50}}\right) dt$$

↖"add up those gallons"

SOLUTION: (a) Total quantity of oil leaked in the first hour:

$$\int_0^{60} \left(125 - \frac{t}{50}\right)dt = 125t - \frac{t^2}{100}\bigg|_0^{60} = 125(60) - \frac{60^2}{100} = 7464 \text{ gallons}$$

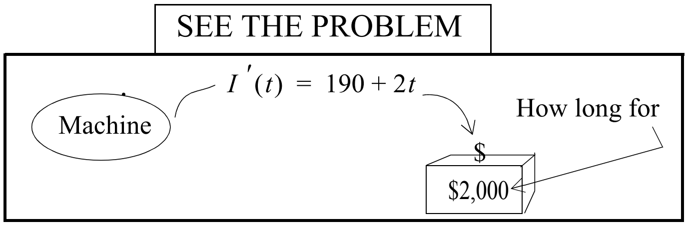
(b) Total quantity of oil leaked in the second hour

$$\int_{60}^{120} \left(125 - \frac{t}{50}\right)dt = 125t - \frac{t^2}{100}\bigg|_{60}^{120}$$

$$= \left(125(120) - \frac{120^2}{100}\right) - \left(125(60) - \frac{60^2}{100}\right) = 7392 \text{ gallons}$$

EXAMPLE 5.10 A printing company is considering purchasing a new hole-punching machine for $2,000. It estimates that with the purchase of the machine, monthly income will increase at a rate of $190 + 2t$ dollars per month (t in months). How many months will it take for the machine to pay for itself?

SOLUTION:



First find the total increase of income after T months:

$$\text{Total income increase} = \int_0^T (190 + 2t)dt = 190t + t^2\Big|_0^T = 190T + T^2$$

Then set that income to 2000, and solve for T:

$$190T + T^2 = 2000$$
$$T^2 + 190T - 2000 = 0$$
$$(T - 10)(T + 200) = 0$$
$$T = 10 \text{ or } T = -200$$

Ignoring the negative time period we conclude that the machine will pay for itself in 10 months.

CHECK YOUR UNDERSTANDING 5.13

The rate of production, in barrels per day, of oil from an oil well is anticipated to be $75 - \frac{t}{2500}$ (t in days). Find the total income produced by the well in its first 30 days of operation, if crude sells at $85 per barrel.

Answer: $191,250

EXERCISES

Exercises 1-21. Evaluate:

1. $\int_0^1 3\,dx$

2. $\int_1^2 3x\,dx$

3. $\int_{-1}^1 (3+3x)dx$

4. $\int_0^1 (x^2+3x-1)dx$

5. $\int_1^2 (x^2+3x-1)dx$

6. $\int_{-1}^1 x^3 dx$

7. $\int_1^2 x(3x-2)dx$

8. $\int_{-1}^0 (3x-1)(x-1)dx$

9. $\int_0^{-1} (3x-1)(x-1)dx$

10. $\int_1^2 \frac{x^3-2}{x^2}\,dx$

11. $\int_1^2 \frac{(x^4+x)(x+1)}{x^4}dx$

12. $\int_{-2}^{-1} \frac{(3x-1)(x-1)}{x^4}dx$

13. $\int_1^2 \sqrt{x}\,dx$

14. $\int_1^2 x^{-3/5}dx$

15. $\int_0^1 \sqrt{x}(x^2+x-3)dx$

16. $\int_1^2 \frac{x+1}{\sqrt{x}}\,dx$

17. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\ dx$

18. $\int_0^{\frac{\pi}{2}} (2\sin x - 5\cos x)dx$

19. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \sec x\,dx$

20. $\int_{-5}^5 x^2 \sin x\ dx$

    Consider the graph of $x^2 \sin x$

21. $\int_{-5}^5 x^5 \cos^2 x\,dx$

    Consider the graph of $x^5 \cos^2 x$

Exercises 22-27. (Area) Sketch the region bounded above by the graph of the given function over the specified interval, and below by the x-axis. Determine the area of that region.

22. $f(x) = x^2, -1 \le x \le 1$

23. $f(x) = x^3, 0 \le x \le 1$

24. $f(x) = \frac{1}{x^2}, 1 \le x \le 4$

25. $f(x) = \sqrt{x}, 1 \le x \le 2$

26. $f(x) = |x|, -1 \le x \le 2$

27. $f(x) = |x-1|, -1 \le x \le 2$

28. **(Cost Increase)** In July, the price of gas increased at the rate of $0.06t + 0.001t^2$ cents per gallon, where t denotes the number of days from June 1. How much did the cost of a gallon increase during the course of the month?

29. **(Depreciation)** The resale value of a car **decreases** at the rate of $1200 + 600t + 4t^3$ dollars per year, where $0 \le t \le 7$ denotes the number of years following the car's year of manufacture. How much did the car's value depreciate:

    (a) in the first three years?

    (d) during the third year?

30. **(Melting Ice)** A 360 cubic inch block of ice is melting at the rate of $\frac{t}{5}$ cubic inches per minute. How many minutes will it take for the block to totally melt?

31. **(Advertising)** A store is launching an aggressive advertising campaign, and anticipates that the number of daily customers, N, will grow from its current value of 200, at a rate of $N'(t) = \frac{t}{100}$, where t is the number of days from the beginning of the campaign. How many days from the beginning of the campaign will it take before the number of daily customers doubles?

32. **(Declining Sales)** Because of fierce competition, the weekly sales at an appliance store are expected to decline at the rate of $S'(t) = -\frac{t^2}{100}$ units per week, where t is number of weeks from the present date. The store plans to go out of business when weekly sales drop below 500. Currently, the shop sells 900 units weekly. How many more weeks will the company remain in business?

33. **(Income Stream)** A printing company can purchase a $2,000 hole-punching machine that will increase monthly earnings at a rate of $190 + 12t$ dollars per month, or a $3,000 machine that will increase monthly earnings at a rate of $250 + 20t$ dollars per month (t in months). Which should be purchased, given that the company anticipates using the machine for exactly five years?

34. **(Depreciation)** The resale value of a certain industrial machine decreases at a rate that depends on the age of the machine. When the machine is x years old, the rate at which its value is dropping during that year is $250(15-x)$ dollars per year. If the machine was originally worth $28,000, how much will it be worth when it is 3 years old?

Exercises 35-40. (Theory) Assume that: $\int_1^2 f(x)\,dx = 5$, $\int_2^4 f(x)\,dx = 7$, $\int_1^4 g(x)dx = 9$. Evaluate:

35. $\int_1^4 3g(x)dx$

36. $\int_1^4 3f(x)dx$

37. $\int_1^4 [2g(x)-f(x)]dx$

38. $\int_2^1 -f(x)dx$

39. $\left(\int_1^3 f(x)dx\right)\left(\int_3^3 g(x)dx\right)$

40. $\dfrac{\int_1^2 f(x)dx}{\int_4^2 f(x)\,dx} + 2\int_1^4 g(x)dx$

Exercises 41-43. (Principal Theorem of Calculus) Use Theorem 5.7 to find the derivative of the given function T.

41. $T(x) = \int_5^x \sqrt{3t^4+1}\,dt$

42. $T(x) = \int_x^5 \sqrt{3t^4+1}\,dt$

43. $T(x) = \int_1^x \frac{\sin t}{t^2+1}dt$

Exercises 44-46. (Theory)

(a) Use Theorem 5.7 to find the derivative of the given function $T(x)$.

(b) Use Theorem 5.8 to first express $T(x)$ in a form that does not involve an integral, and then differentiate that explicit function of x directly. Compare your answer with that of (a).

(c) Repeat parts (a) and (b), replacing the lower limit of integration “1” with 5.

44. $T(x) = \int_1^x t^2 dt$

45. $T(x) = \int_1^x (t^2+t)dt$

46. $T(x) = \int_1^x \frac{t^4+t}{t^3}dt$

47. **(Theory)** Let f be integrable, and g be differentiable. Use the Chain Rule (page 94) and Theorem 5.7, to show that for $H(x) = \int_a^{g(x)} f(t)dt$: $H'(x) = f[g(x)] \cdot g'(x)$.

48. **(Theory)** Let f be integrable, and g and k be differentiable. Use the Chain Rule (page 94) and Theorem 5.7, to show that for $H(x) = \int_{k(x)}^{g(x)} f(t)dt$: $H'(x) = f[g(x)] \cdot g'(x) - f[k(x)]k'(x)$.

Exercises 49-51. (Theory) Use the results of Exercises 47 and 48 to differentiate the function H.

49. $H(x) = \int_5^{2x} \sqrt{3t^4+1}\,dt$ 50. $H(x) = \int_5^{x^2} \frac{t}{t^4+1}dt$ 51. $H(x) = \int_{x^2}^{\sin x} \frac{dt}{x^2+1}$

Exercise 52-54. (Theory)

(a) Use the results of Exercise 47 and 48 to find the derivative of the given function $H(x)$.

(b) Use the Fundamental Theorem of Calculus to first express $H(x)$ in a form that does not involve an integral, and then differentiate that explicit function of x directly. Compare your answer with that of (a).

52. $H(x) = \int_5^{2x} (3t^2 + 2t)dt$ 53. $H(x) = \int_1^{x^2} t(t-5)dt$ 54. $H(x) = \int_{2x}^{x^2} (3t^2 - 1)dt$

Exercises 55-57. (Second Derivative) Determine $\frac{d^2y}{dx^2}$.

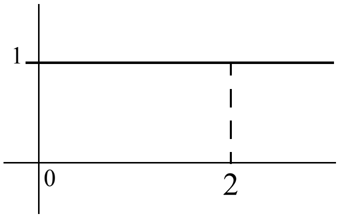
55. $y = \int_1^x t\sin t dt$ 56. $y = \int_1^x \frac{\sqrt{t+1}}{t^2}dt$ 57. $y = \int_1^{x^2} \tan t\,dt$

58. **(Theory)** Referring to Definition 5.3, offer an argument explaining why the function:

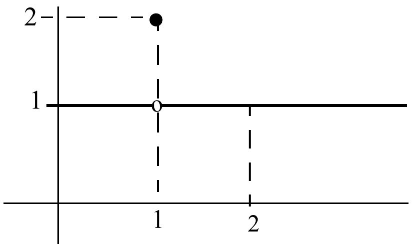
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ -1 & \text{if } x \text{ is not a rational number} \end{cases}$$

is **not** integrable over the interval $[0, 1]$ (or any other interval, for that matter).
(Use the fact that any interval, no matter how small, contains both rational and irrational numbers.)

59. **(Theory)** Referring to Definition 5.3, offer an argument explaining why $\int_0^2 f(x)dx = \int_0^2 g(x)dx$ for the two functions depicted below.



$f(x) = 1$



$$g(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

§3. THE SUBSTITUTION METHOD

Next semester you will encounter a half dozen or so integration techniques that will enable you to determine the integrals of a variety of functions. Here, we will content ourselves with just one technique, the so called ***u*-substitution method**. This method stems from the following theorem, which is really the Chain Rule "in reverse."

Assuming, of course, that the function g is differentiable, and that the integral exists.

THEOREM 5.12 If $F'(x) = f(x)$, then:

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + C$$

PROOF: We simply show that $F[g(x)]$ is an antiderivative of $f[g(x)]g'(x)$:

$$(F[g(x)])' \underset{\uparrow}{=} F'[g(x)]g'(x) \underset{\uparrow}{=} f[g(x)]g'(x)$$

The Chain Rule, page 94

since $F'(x) = f(x)$

Though easy to prove, Theorem 5.12 in its present form is not very useful because of its intimidating form. To soften its appearance, we make the substitution: $u = g(x)$, bringing us to a somewhat improved form:

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + C$$

$$\int f(u) \cdot u'\,dx = \boldsymbol{F(u)} + C$$

Still not great. But we now observe that the simpler looking integral $\int f(u)du$ is also equal to $\boldsymbol{F(u)} + C$:

$$\int f(u)\,du \underset{\uparrow}{=} \boldsymbol{F(u)} + C$$

since $F'(u) = F'[g(x)] = f[g(x)] = f(u)$

Please note that we attribute no meaning to either the expression $u'dx$ or the expression du. We simply replace the symbol $u'dx$ in the (meaningful) expression $\int f(u)u'\,dx$, with the symbol du, to arrive at another meaningful expression $\int f(u)du$.

It follows that if we let $u = g(x)$, and then **formally** make the substitution $du = u'dx$, we arrive at:

$$\boldsymbol{\int f[g(x)]g'(x)\,dx = \int f(u)du}$$

The following examples illustrate how the above ***substitution method*** can sometimes be used to transform a complicated integral into a simpler form.

EXAMPLE 5.11 Determine:

$$\int x(x^2-5)^7 dx$$

SOLUTION: The "trick" is to let u be some part of the integral, so that "$du = u'dx$" is "essentially the rest" (up to a multiplicative constant). Specifically:

Let $u = x^2-5$ — then (**formally**): $du = 2\boldsymbol{xdx}$, or: $\boldsymbol{xdx = \frac{1}{2}du}$.

So:

$\boldsymbol{xdx = \frac{1}{2}du}$ $\qquad\qquad$ $u = x^2-5$

$$\int \boldsymbol{x}(x^2-5)^7\boldsymbol{dx} = \boldsymbol{\frac{1}{2}}\int u^7\boldsymbol{du} = \frac{1}{2}\cdot\frac{u^8}{8}+C = \frac{(x^2-5)^8}{16}+C$$

And the end justifies the means. The substitution:

$$u = x^2-5$$
$$\text{“}du = 2xdx\text{”}$$

takes us from:

$$\int x(x^2-5)^7dx$$

to:

$$\frac{1}{2}\int u^7du$$

Check:

$$\left[\frac{1}{16}(x^2-5)^8+C\right]' = \frac{1}{16}\cdot 8(x^2-5)^7(x^2-5)' = \frac{1}{2}(x^2-5)^7\cdot 2x = x(x^2-5)^7$$

If you currently find yourself a bit uncomfortable with the u-substitution method, that's par for the course. A few more examples should remedy the situation.

EXAMPLE 5.12 Determine:

(a) $\displaystyle\int\frac{x^2}{\sqrt{x^3+5}}dx$ (b) $\displaystyle\int 3x(2x^2+7)^{\frac{2}{3}}dx$

(c) $\displaystyle\int x\cos x^2\,dx$

SOLUTION: (a) Let $u = x^3+5$ — then (**formally**): $du = 3x^2dx$, or: $\boldsymbol{x^2dx = \frac{1}{3}du}$. So:

$\boldsymbol{x^2dx = \frac{1}{3}du}$ $\qquad\qquad$ $u = x^3+5$

$$\int\frac{\boldsymbol{x^2}}{\sqrt{x^3+5}}\boldsymbol{dx} = \boldsymbol{\frac{1}{3}}\int\frac{\boldsymbol{du}}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\cdot\frac{u^{1/2}}{1/2}+C = \frac{2}{3}\sqrt{x^3+5}+C$$

Check: $\left[\frac{2}{3}(x^3+5)^{\frac{1}{2}}\right]' = \frac{2}{3}\left[\frac{1}{2}(x^3+5)^{-\frac{1}{2}}\cdot 3x^2\right] = x^2(x^3+5)^{-\frac{1}{2}} = \frac{x^2}{\sqrt{x^3+5}}$

(b) $$\int 3x(2x^2+7)^{\frac{2}{3}}dx = 3\int \boldsymbol{x}(2x^2+7)^{\frac{2}{3}}\boldsymbol{dx}$$

$$\boxed{\begin{aligned} u &= 2x^2+7 \\ du &= 4xdx \\ \boldsymbol{xdx} &= \boldsymbol{\frac{1}{4}du} \end{aligned}} \rightarrow = \frac{3}{\boldsymbol{4}}\int u^{\frac{2}{3}}\boldsymbol{du} = \frac{3}{4}\left(\frac{u^{\frac{5}{3}}}{\frac{5}{3}}\right) + C = \frac{9}{20}(2x^2+7)^{\frac{5}{3}} + C$$

Check: $\left[\frac{9}{20}(2x^2+7)^{\frac{5}{3}}\right]' = \frac{9}{20}\left[\frac{5}{3}(2x^2+7)^{\frac{2}{3}}\cdot 4x\right] = 3x(2x^2+7)^{\frac{2}{3}}$

(c) $$\int \boldsymbol{x}\cos x^2\,\boldsymbol{dx} \underset{\substack{\uparrow \\ \boxed{\begin{aligned} u &= x^2 \\ du &= 2xdx \\ \boldsymbol{xdx} &= \boldsymbol{\frac{1}{2}du} \end{aligned}}}}{=} \boldsymbol{\frac{1}{2}}\int \cos u\,\boldsymbol{du} = \frac{1}{2}\sin u + C = \frac{1}{2}\sin x^2 + C$$

Check: $\left(\frac{1}{2}\sin x^2\right)' = \frac{1}{2}(\cos x^2 \cdot 2x) = x\cos x^2$

CHECK YOUR UNDERSTANDING 5.14

Determine:

(a) $\displaystyle\int \frac{x}{5(x^2-10)^5}dx$ (b) $\displaystyle\int \frac{\cos x}{\sin^2 x}dx$

Answers:

(a) $-\dfrac{1}{40(x^2-10)^4} + C$

(b) $-\dfrac{1}{\sin x} + C$

Our next example is tricky in that it does not fit the typical u-substitution mode:

EXAMPLE 5.13

Determine:

$$\int \frac{x}{(x+1)^3}dx$$

Solution: If that denominator were $(x^2+1)^3$, then we could proceed as in the earlier examples, letting $u = x^2+1$, and so on. But it is not. And so:

$$u = x+1 \quad (*)$$
$$du = dx$$

But this leaves us with an “unresolved” x in the integral:

$$\int \frac{x}{(x+1)^3}dx \underset{\substack{\uparrow \\ \boxed{\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}}}}{=} \int \frac{x}{u^3}du$$

And so we return to (*) and solve for x in terms of u: $x = u-1$. Substituting, we then have:

$$\int \frac{x}{(x+1)^3}dx \underset{\uparrow}{=} \int \frac{x}{u^3}du \underset{\uparrow}{=} \int \frac{u-1}{u^3}du = \int\left(\frac{u}{u^3}-\frac{1}{u^3}\right)du$$

$u = x+1 \Rightarrow x = u-1$
$du = dx$

$$= \int (u^{-2}-u^{-3})du$$

$$= -\frac{1}{u}+\frac{1}{2u^2}+C$$

$$= -\frac{1}{x+1}+\frac{1}{2(x+1)^2}+C$$

You are invited to check the above result by showing that the derivative of $-\frac{1}{x+1}+\frac{1}{2(x+1)^2}$ is indeed $\frac{x}{(x+1)^3}$.

CHECK YOUR UNDERSTANDING 5.15

Determine $\int x\sqrt{x+1}\,dx$. Use differentiation to check your answer.

Answer:
$\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+C$

SUBSTITUTION AND DEFINITE INTEGRALS

We now illustrate how the u-substitution method can be used to evaluate certain definite integrals.

EXAMPLE 5.14 Evaluate:

$$\int_0^1 \frac{x^2}{(x^3+1)^3}dx$$

SOLUTION: **<u>One approach</u>** is to begin by finding an antiderivative of $\frac{x^2}{(x^3+1)^3}$:

$$\int \frac{x^2}{(x^3+1)^3}dx \underset{\uparrow}{=} \frac{1}{3}\int \frac{1}{u^3}\,du = \frac{1}{3}\int u^{-3}du = \frac{1}{3}\cdot\frac{u^{-2}}{-2}+C$$

$u = x^3+1$
$du = 3x^2dx$
$x^2dx = \frac{1}{3}du$

$$= \frac{-1}{6u^2}+C = \frac{-1}{6(x^3+1)^2}+C$$

Bringing us to:

$$\int_0^1 \frac{x^2}{(x^3+1)^3}dx = \left.\frac{-1}{6(x^3+1)^2}\right|_0^1 = \frac{-1}{6\cdot 2^2}-\frac{-1}{6\cdot 1^2} = \frac{1}{8}$$

A better approach is to use the u-substitution, $u = x^3 + 1$ to also change the limits of integration:

if $u = x^3 + 1$ and $x = 1$, then $u = 1^3 + 1 = 2$

$$\int_0^1 \frac{x^2}{(x^3+1)^3}dx = \frac{1}{3}\int_1^2 \frac{1}{u^3}\,du = \frac{1}{3}\int_1^2 u^{-3}du = \frac{1}{3}\cdot\frac{u^{-2}}{-2}\Bigg|_1^2$$

$u = 0^3 + 1 = 1$

$$= \frac{-1}{6u^2}\Bigg|_1^2 = -\frac{1}{24} - \frac{-1}{6} = \frac{1}{8}$$

EXAMPLE 5.15

Evaluate:

$$\int_0^2 \cos(3x+1)dx$$

SOLUTION:

$u = 3\cdot\mathbf{2} + 1 = 7$

$$\int_{\mathbf{0}}^{\mathbf{2}} \cos(3x+1)dx = \frac{1}{3}\int_1^7 \cos u\,du$$

$u = 3\cdot\mathbf{0} + 1 = 1$

$u = 3x + 1$
$du = 3dx$
$dx = \frac{1}{3}du$

$$= \frac{1}{3}(\sin u)\Bigg|_1^7 = \frac{1}{3}(\sin 7 - \sin 1) \approx -0.06$$

CHECK YOUR UNDERSTANDING 5.16

Evaluate:

(a) $\int_1^{\sqrt{2}} x\sqrt{x^2-1}\,dx$

(b) $\int_0^1 \frac{x}{(x^2+1)^2}dx$

Answers: (a) $\frac{1}{3}$ (b) $\frac{1}{4}$

EXERCISES

Exercises 1-15. (Indefinite Integrals) Determine:

1. $\int (x-5)^{15}dx$
2. $\int (2x-5)^{15}dx$
3. $\int \frac{dx}{(2x-5)^{15}}$
4. $\int 2x(x^2+5)^{15}dx$
5. $\int x(x^2+5)^{15}dx$
6. $\int \frac{x}{(x^2+5)^{15}}dx$
7. $\int \frac{x}{\sqrt{5x^2-4}}dx$
8. $\int \frac{6x^2+4x}{(x^3+x^2)^2}dx$
9. $\int \left(3x+\frac{x}{(x^2-3)^2}\right)dx$
10. $\int x^2 \sin x^3\, dx$
11. $\int \sec^2 x \tan x\, dx$
12. $\int \frac{x}{(x+2)^3}dx$
13. $\int \frac{x^2}{(x+1)^4}dx$
14. $\int (x+1)(x-1)^{17}dx$
15. $\int \frac{x^2+x-1}{\sqrt{x-3}}dx$

Exercises 16-27. (Definite Integrals) Evaluate:

16. $\int_1^2 \frac{x}{\sqrt{x^2+1}}\, dx$
17. $\int_{-1}^0 x^2(x^3+2)^5 dx$
18. $\int_1^2 \frac{6x+1}{(3x^2+x)^2}dx$
19. $\int_0^2 \frac{x}{\sqrt{5(x^2+4)}}dx$
20. $\int_1^2 \frac{6x^2+4x}{(x^3+x^2)^2}dx$
21. $\int_2^{-1} \frac{x}{(x^2+5)^2}dx$
22. $\int_{-1}^{\sqrt{5}} 2x(x^2-5)^{15}dx$
23. $\int_{-1}^1 x\cos x^2 dx$
24. $\int_1^4 \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
25. $\int_0^{\sqrt{\frac{\pi}{4}}} x\sec^2 x^2 dx$
26. $\int_0^1 x\sqrt{x+1}\,dx$
27. $\int_0^1 \sqrt{x^3+x^2}\,dx$

Exercises 28-29. (Area) Determine the area bounded above by the graph of the given function over the specified interval.

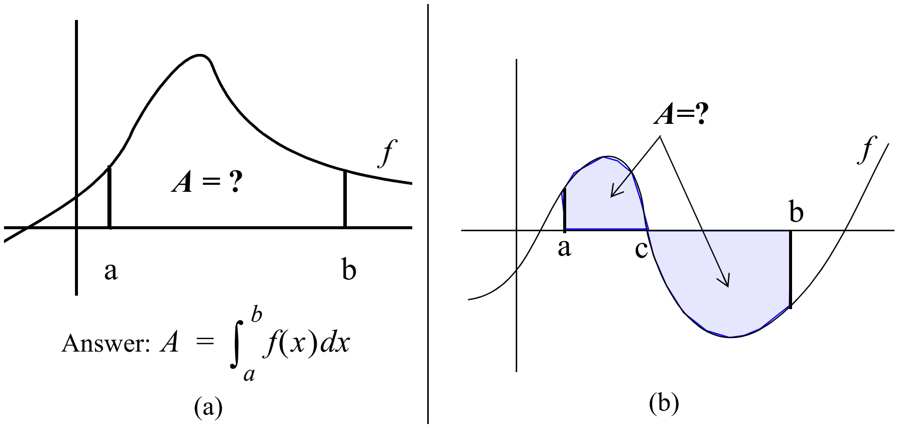
28. $f(x) = \frac{x}{x^2+1},\ 0 \le x \le 1$
29. $f(x) = x^2\sqrt{x^3+10},\ -1 \le x \le 1$

30. **(Theory)** Prove that if $F'(x) = f(x)$, then:

$$\int_a^b f[g(x)]g'(x)\,dx = F[g(b)] - F[g(a)]$$

§4. AREA AND VOLUME

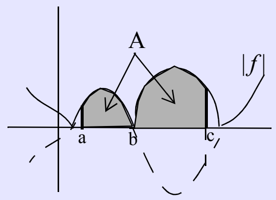
In section 2 we came up with a definition for the area in Figure 5.4(a), an area that is bounded above by the graph of the positive function $y = f(x)$; below by the x-axis; and on the sides by the vertical lines $x = a$ and $x = b$. How about the area of the shaded region in Figure 5.4(b)? (Please consider the question before moving on.)



Answer: $A = \int_a^b f(x)dx$

(a) (b)

Figure 5.4



graph of the absolute value of the function in Figure (b).

A correct answer is: $A = \int_a^b |f(x)|dx$ (see margin). Noting that

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

we have:

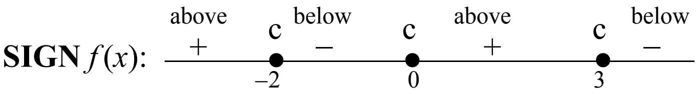
$$A = \int_a^b |f(x)|dx = \int_a^c f(x)dx + \int_c^b -f(x)dx = \int_a^c f(x)dx - \int_c^b f(x)dx$$

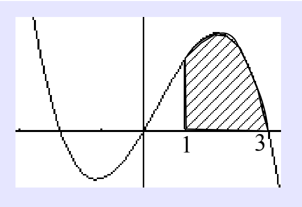
EXAMPLE 5.16 Find the area bounded by the x-axis, the graph of the function $f(x) = -x^3 + x^2 + 6x$, and the vertical lines:

(a) $x = 1$ and $x = 3$ (b) $x = -1$ and $x = 2$

SOLUTION: SIGN $f(x)$ reveals where the graph of the function lies above the x-axis, and where it lies below the x-axis:

$$f(x) = -x^3 + x^2 + 6x = -x(x^2 - x - 6) = -x(x - 3)(x + 2)$$

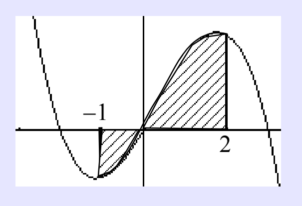




(a) The above information reveals the fact that the function is not negative anywhere in the interval $[1, 3]$. Hence:

$$A = \int_1^3 (-x^3 + x^2 + 6x)dx = \left(-\frac{x^4}{4} + \frac{x^3}{3} + 3x^2\right)\Bigg|_1^3$$

$$= \left[-\frac{3^4}{4} + \frac{3^3}{3} + 3 \cdot 3^2\right] - \left[-\frac{1}{4} + \frac{1}{3} + 3\right] = \frac{38}{3} \approx 12.7$$



(b) SIGN $f(x)$ tells us that the function is negative (or zero) over the interval $[-1, 0]$, and positive (or zero) over the interval $[0, 2]$. Hence:

since function is negative on (-1,0)

$$A = -\int_{-1}^{0} (-x^3 + x^2 + 6x)dx + \int_0^2 (-x^3 + x^2 + 6x)dx$$

$$= -\left(-\frac{x^4}{4} + \frac{x^3}{3} + 3x^2\right)\Bigg|_{-1}^{0} + \left(-\frac{x^4}{4} + \frac{x^3}{3} + 3x^2\right)\Bigg|_0^2$$

$$= -\left[0 - \left(-\frac{1}{4} - \frac{1}{3} + 3\right)\right] + \left[\left(-4 + \frac{8}{3} + 12\right) - 0\right] = \frac{157}{12} \approx 13.1$$

CHECK YOUR UNDERSTANDING 5.17

Find the area bounded by the x-axis, the graph of the function $f(x) = x^2 + 2x - 3$, and the vertical lines:

(a) $x = -3$ and $x = 1$ (b) $x = 0$ and $x = 2$

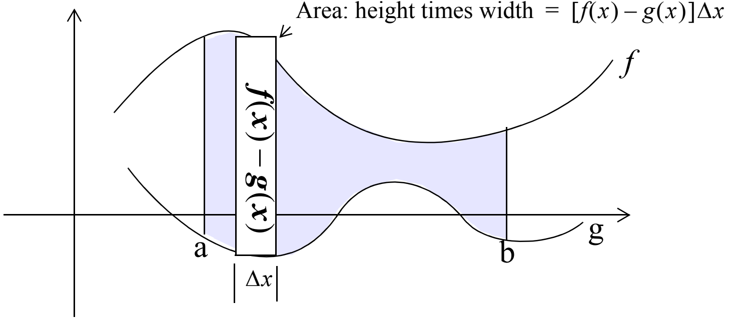
Answers: (a) $\frac{32}{3}$ (b) 4

AREA BETWEEN CURVES

Keeping in mind that the definite integral is the limit of Riemann sums, it is natural to define the area of the figure below to be:

$$A = \lim_{\Delta x \to 0} \sum_a^b [f(x) - g(x)]\Delta x = \int_a^b [f(x) - g(x)]dx$$

Area: height times width $= [f(x) - g(x)]\Delta x$



Note the height of the indicated rectangle:

dominant or higher function

$$\downarrow$$
$$f(x_i) - g(x_i)$$
$$\uparrow$$

minus **subordinate** or lower function

In general:

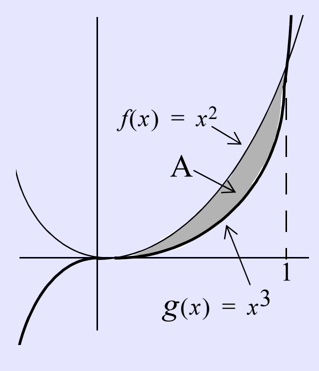
THEOREM 5.13
AREA BETWEEN CURVES

Let f and g be continuous over the interval $[a, b]$. The area between the graphs of those functions between $x = a$ and $x = b$ is given by:

$$A = \int_a^b |f(x) - g(x)|dx$$

EXAMPLE 5.17

Determine the area of the finite region bounded by the graphs of the functions $f(x) = x^2$ and $g(x) = x^3$.



SOLUTION: The first order of business is to determine the points of intersection of those two curves (see margin):

$$x^3 = x^2$$
$$x^3 - x^2 = 0$$
$$x^2(x-1) = 0$$
$$x = 0 \text{ and } x = 1$$

Since the graph of $f(x) = x^2$ lies above that of $g(x) = x^3$ over the interval $[0, 1]$ we have:

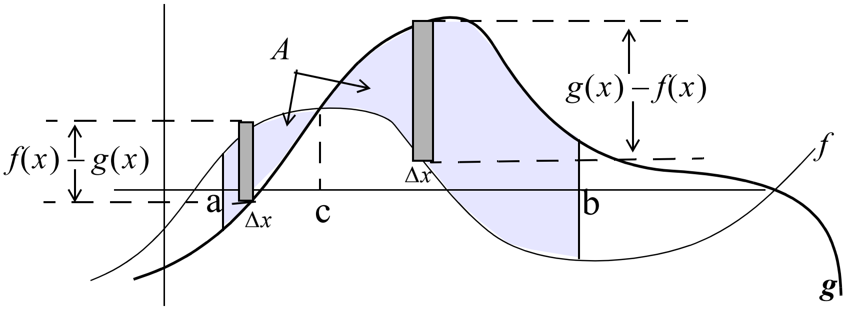
$$(*)\quad A = \int_0^1 (\overset{\substack{\text{dominant}\\ \downarrow}}{x^2} - \underset{\substack{\uparrow\\ \text{subordinate}}}{x^3})dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

CHECK YOUR UNDERSTANDING 5.18

Determine the area of the finite region bounded by the graphs of the functions $f(x) = x^2$ and $g(x) = x + 2$.

Answer: $\frac{9}{2}$

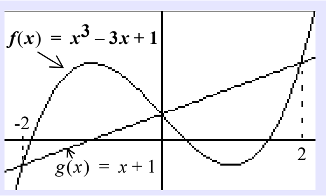
The graphs of f and g below switch dominance about the point $x = c$.



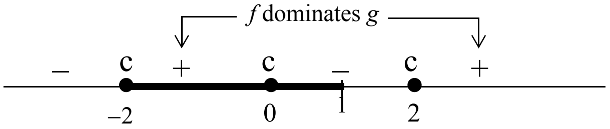
Consequently, two integrals are needed to calculate the area of the indicated shaded region:

$$A = \int_a^b |f(x) - g(x)|\,dx = \int_a^c [f(x) - g(x)]\,dx - \int_c^b [f(x) - g(x)]\,dx$$

EXAMPLE 5.18 Determine the area, A, bounded by the graphs of the functions $f(x) = x^3 - 3x + 1$ and $g(x) = x + 1$, between $x = -2$ and $x = 1$.



SOLUTION: While you are encouraged to consider the graphs of the two functions (margin), it is not necessary to do so to find the area in question. All you have to do is to find **SIGN** $f(x) - g(x)$:



SIGN $f(x) - g(x) = (x^3 - 3x + 1) - (x + 1) = x^3 - 4x = x(x + 2)(x - 2)$

We are interested in the area bounded by f and g between $x = -2$ and $x = 1$. From the above, we see that $f(x) - g(x)$ is positive on $(-2, 0)$, and negative on $(0, 1)$. Consequently:

$$\begin{aligned} A &= \int_{-2}^{0} [(x^3 - 3x + 1) - (x + 1)]\,dx - \int_0^1 [(x^3 - 3x + 1) - (x + 1)]\,dx \\ &= \int_{-2}^{0} (x^3 - 4x)\,dx - \int_0^1 (x^3 - 4x)\,dx \\ &= \left(\frac{x^4}{4} - 2x^2\right)\Bigg|_{-2}^{0} - \left(\frac{x^4}{4} - 2x^2\right)\Bigg|_{0}^{1} \\ &= [0 - (4 - 8)] - \left[\left(\frac{1}{4} - 2\right) - 0\right] = 4 + \frac{7}{4} = \frac{23}{4} \end{aligned}$$

$\sqrt{x} = -x + 6$

$x = x^2 - 12x + 36$

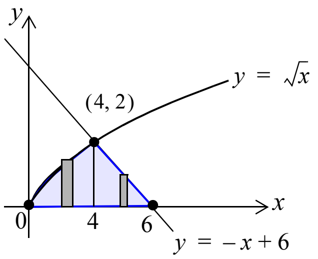
$x^2 - 13x + 36 = 0$

$(x - 9)(x - 4) = 0$

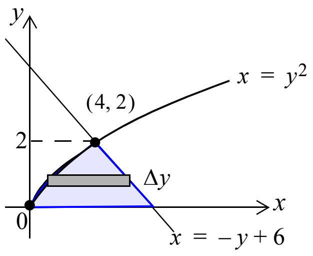
$\cancel{x = 9}$ $x = 4$

EXAMPLE 5.19 Express the area, A, bounded below by the x-axis, above by the graph of $f(x) = \sqrt{x}$, and on the side by the line $y = -x + 6$, in integral form.

SOLUTION: (Without words)



OR:



$$A = \int_0^4 \sqrt{x}\,dx + \int_4^6 (-x + 6)\,dx$$

$$A = \int_0^2 (-y + 6 - y^2)\,dy$$

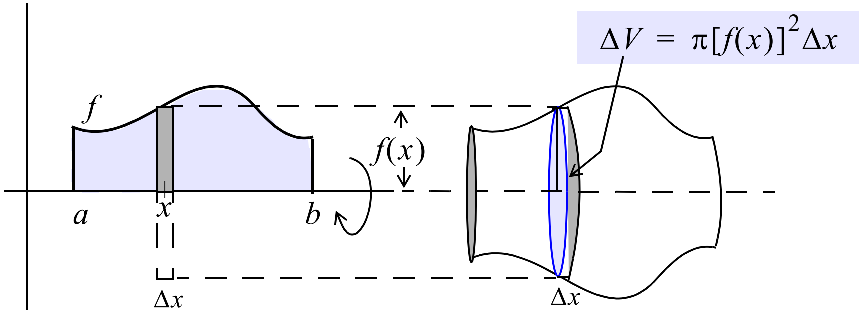
CHECK YOUR UNDERSTANDING 5.19

Determine the area, A, bounded by the graphs of the functions $f(x) = x^4 - x^2$ and $g(x) = x^3 - x^2$, between $x = -1$ and $x = 3$.

Answer: $\frac{289}{10}$

VOLUME OF SOLIDS OF REVOLUTION

If you take the shaded region of Figure 5.5(a) and revolve it about the x-axis, you will generate the solid represented in Figure5.5(b).



(a) (b)

Figure 5.5

The volume, ΔV, of the narrow disk in Figure (b) is the area of its base: $\pi r^2 = \pi[f(x)]^2$, times its thickness: Δx. We then define the volume of the solid to be:

$$V = \lim_{\Delta x \to 0} \sum_a^b \pi[f(x)]^2 \Delta x = \pi\int_a^b [f(x)]^2 dx$$

Generalizing:

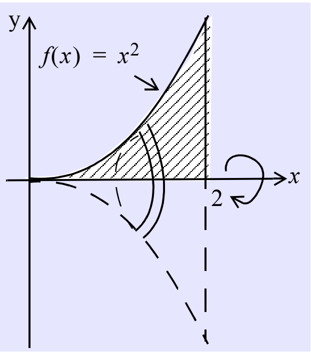
DEFINITION 5.5
VOLUME OF A SOLID OF REVOLUTION (DISK METHOD)

Let f be nonnegative and continuous over the interval $[a, b]$. The volume of the solid obtained by rotating, about the x-axis, the region bounded above by the graph of the function f, below by the x-axis, and on the sides by $x = a$ and $x = b$, is given by:

$$V = \pi\int_a^b [f(x)]^2 dx$$

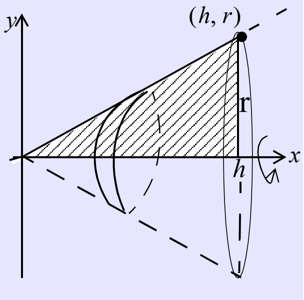
EXAMPLE 5.20

Determine the volume of the solid obtained by rotating, about the x-axis, the region bounded by the graph of the function $f(x) = x^2$, over the interval $[0, 2]$.



SOLUTION:

$$V = \pi\int_0^2 [x^2]^2 dx = \pi\int_0^2 x^4 dx = \pi \cdot \frac{x^5}{5}\Big|_0^2 = \pi \cdot \frac{2^5}{5} = \frac{32\pi}{5}$$



EXAMPLE 5.21 Determine a formula for the volume of a cone of height h and radius r.

SOLUTION: We can generate a cone of height h and radius r by rotating the region below the line passing through the origin and the point (h, r) about the x-axis (see margin). That line has slope $\frac{r}{h}$ and y-intercept 0, and is therefore given by:

$$y = f(x) = \frac{\boldsymbol{r}}{\boldsymbol{h}} \cdot \boldsymbol{x}$$

Bringing us to the formula:.

$$V = \pi\int_0^h \left(\frac{\boldsymbol{r}}{\boldsymbol{h}} \cdot \boldsymbol{x}\right)^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2 dx$$

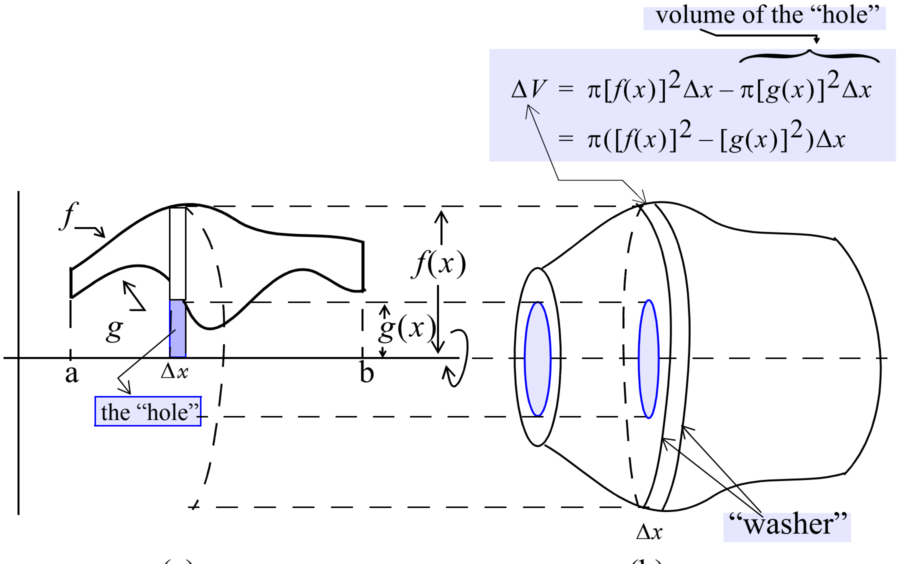
$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\Big|_0^h\right] = \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi r^2 h$$

CHECK YOUR UNDERSTANDING 5.20

Determine the volume of the solid obtained by rotating, about the x-axis, the region bounded by the graph of the function, $f(x) = x^3$ above the interval $[1, 2]$.

Answer: $\frac{127\pi}{7}$

If you take the shaded region of Figure 5.6(a) and revolve it about the x-axis, you will generate the solid represented in Figure 5.7(b).



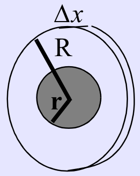
$$\Delta V = \pi[f(x)]^2\Delta x - \overbrace{\pi[g(x)]^2\Delta x}^{\text{volume of the “hole”}}$$

$$= \pi([f(x)]^2 - [g(x)]^2)\Delta x$$

(a) (b)

Figure 5.6

Volume of washer:



$$\pi R^2\Delta x - \pi r^2 \Delta x$$

$$= \pi(\underset{\substack{\uparrow \\ \textbf{Outside radius}}}{\boldsymbol{R}^2} - \underset{\substack{\uparrow \\ \textbf{Inside radius}}}{\boldsymbol{r}^2})\Delta x$$

As is depicted in Figure 5.6, the generated “washer” has volume:

$$\Delta V = \pi(\overset{\text{Outside radius squared}}{[\boldsymbol{f}(\boldsymbol{x})]^2} - \overset{\text{inside radius squared}}{[\boldsymbol{g}(\boldsymbol{x})]^2})\Delta x$$

Note that V can also be obtained by subtracting the volume generated by rotating g about the x-axis from that obtained by rotating f about the x-axis: the difference of the two volumes.

As is depicted in Figure 5.6, the generated "washer" has volume:

$$\Delta V = \pi([\overset{\text{Outside radius squared}}{\boldsymbol{f}(\boldsymbol{x})]^{\mathbf{2}}} - [\overset{\text{inside radius squared}}{\boldsymbol{g}(\boldsymbol{x})]^{\mathbf{2}}})\Delta x$$

Taking the limit of the sum of those ΔV's brings us to an integral representation for the volume in question:

$$V = \lim_{\Delta x \to 0} \sum_{a}^{b} \pi([f(x)]^2 \Delta x - [g(x)]^2)\Delta x = \pi\int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

Summarizing:

In the event that $g(x) = 0$ (the x-axis), then the "washer method" coincides with the "disk method" of Definition 5.5.

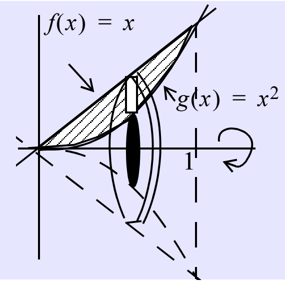
VOLUME OF A SOLID OF REVOLUTION (Washer Method)

Let f and g be nonnegative and continuous over the interval $[a, b]$ with $f(x) \geq g(x)$. The volume of the solid obtained by rotating, about the x-axis, the region bounded above by the graph of the function f, below by the graph of g, and on the sides by $x = a$ and $x = b$, is given by:

$$V = \pi\int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

EXAMPLE 5.22 Determine the volume of the solid obtained by rotating the finite region enclosed by the graphs of the functions $f(x) = x$ and $g(x) = x^2$ about:

(a) The x-axis (b) The line $y = -1$

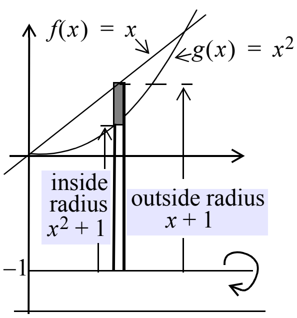


SOLUTION: Finding the points of intersection:

$$x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1 .$$

(a) $$V = \pi\int_0^1 [\overset{\text{outside radius}}{\overset{\downarrow}{(\boldsymbol{x})}}^2 - \overset{\text{inside radius}}{\overset{\downarrow}{(\boldsymbol{x^2})}}^2]dx = \pi\int_0^1 (x^2 - x^4)dx$$

$$= \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$



(b) $$V = \pi\int_0^1 [\overset{\text{outside radius}}{\overset{\downarrow}{(\boldsymbol{x}+1)}}^2 - \overset{\text{inside radius}}{\overset{\downarrow}{(\boldsymbol{x^2}+1)}}^2]dx = \pi\int_0^1 (-x^4 - x^2 + 2x)dx$$

$$= \pi\left(-\frac{x^5}{5} - \frac{x^3}{3} + x^2\right)\Bigg|_0^1 = \frac{7\pi}{15}$$

Answer: $\frac{768\pi}{7}$

CHECK YOUR UNDERSTANDING 5.21

Determine the volume of the solid obtained by rotating, about the x-axis, the finite region enclosed by the y-axis, the line $y = 8$, and the graph of the function $f(x) = x^3$.

THE SHELL METHOD

If you take the region at the top of Figure 5.7(a) and revolve it about the y-axis you will generate the solid ***S***, with the shaded rectangular region giving rise to the indicated canister below it. By snipping open that canister [Figure (b)] and then flattening it out [Figure (c)] we arrive at a formula for the indicated volume ΔV.]

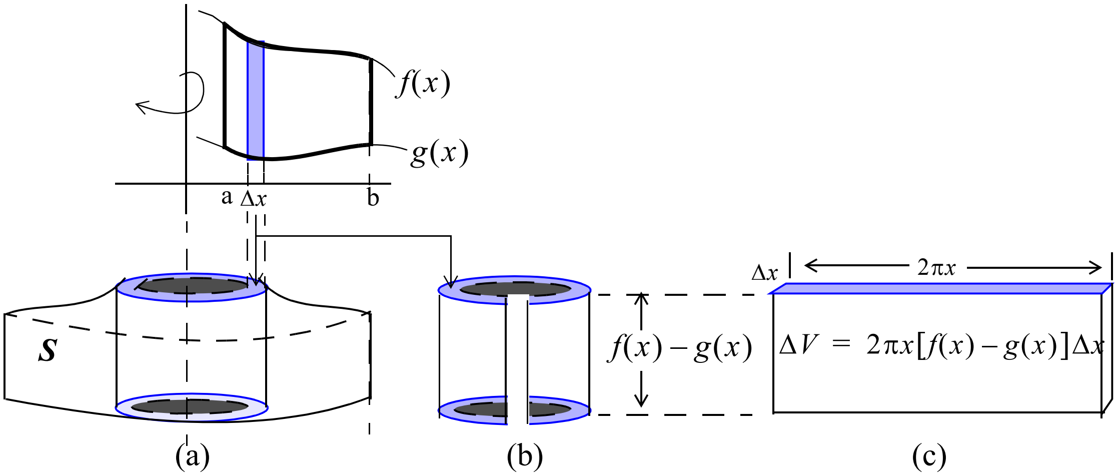


Figure 5.7

Stacking all of the canisters (one inside of another) you will arrive at a solid resembling ***S***. Clearly, by making the partition finer and finer, the sum of the volumes ΔV of the stacked canisters will get closer and closer to the volume of the solid ***S***. All of which brings us to:

$$\text{Volume}(\boldsymbol{S}) = \lim_{\Delta x \to 0} \sum_{a}^{b} 2\pi x[f(x) - g(x)]\Delta x = 2\pi \int_a^b x[f(x) - g(x)]dx$$

Summarizing:

VOLUME OF A SOLID OF REVOLUTION (Shell Method)

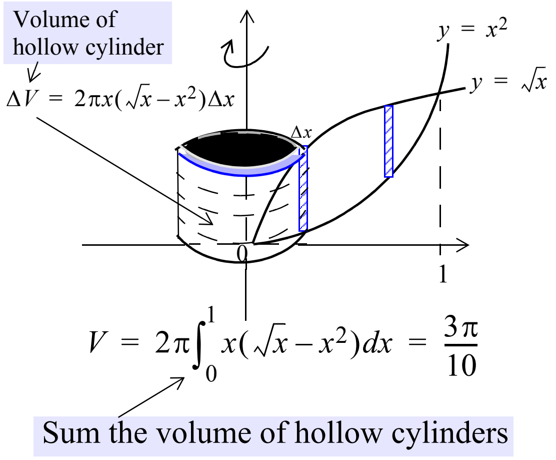
Let f and g be continuous over the interval $[a, b]$ with $f(x) \geq g(x)$. The volume of the solid obtained by rotating, about the y-axis, the region bounded above by the graph of the function f, below by the graph of g, and on the sides by $x = a$ and $x = b$, is given by:

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx$$

EXAMPLE 5.23 Use both the shell and the washer method to find the volume obtained by generating the finite region bounded by the graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = x^2$ about the y-axis.

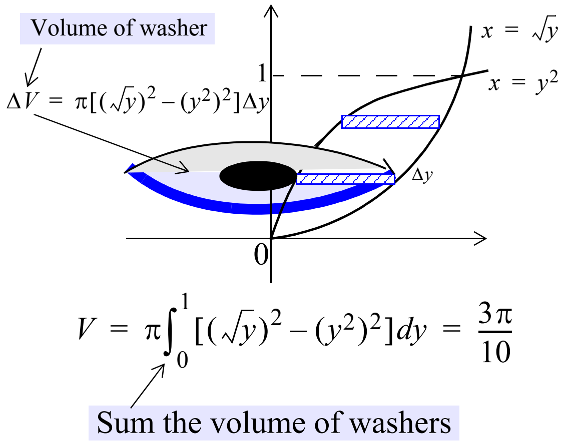
SOLUTION: (Without words)

$$\sqrt{x} = x^2 \Rightarrow x = x^4 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1$$



$$V = 2\pi\int_0^1 x(\sqrt{x} - x^2)dx = \frac{3\pi}{10}$$

Shell Method



$$V = \pi\int_0^1 [(\sqrt{y})^2 - (y^2)^2]dy = \frac{3\pi}{10}$$

Washer Method

CHECK YOUR UNDERSTANDING 5.22

Use both the shell and the washer method to find the volume obtained by rotating about the y-axis the finite region bounded above by the parabola $y = -x^2 + 4$, below by the line $y = 2$, on the left by the y-axis and on the right by the line $x = 1$.

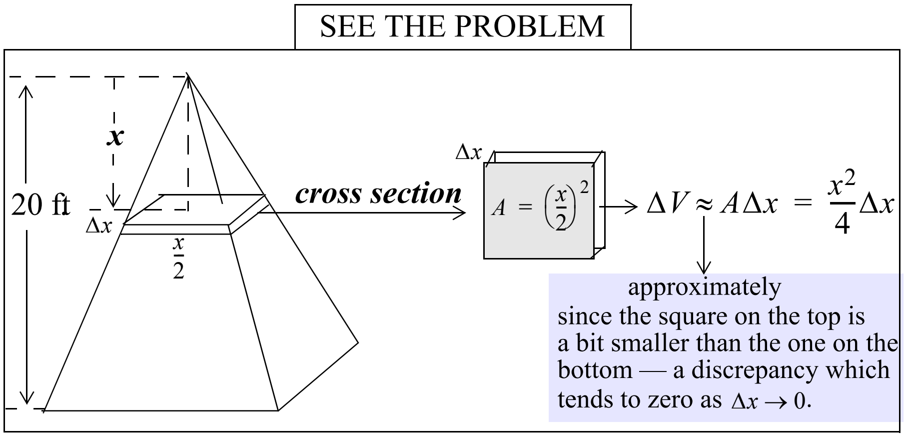
Answer: $\frac{3\pi}{2}$

VOLUMES BY SLICING

The cross sections of a solid need not be disks, but if you can find the area of its cross-sections, then you may still be able to determine its volume. Consider the following example.

EXAMPLE 5.24 A pyramid of height 20 feet is such that its cross-section perpendicular to its altitude a distance x feet from its vertex is a square with side of length $\frac{x}{2}$ feet. Find the volume of the pyramid.

SOLUTION:



approximately
since the square on the top is a bit smaller than the one on the bottom — a discrepancy which tends to zero as $\Delta x \to 0$.

For any given partition of the interval $[0, 20]$ we can obtain an approximation for the volume of the pyramid:

$$V \approx \sum_{0}^{20} \frac{x^2}{4}\Delta x \quad \text{(A Riemann sum!)}$$

By definition: $V = \lim\limits_{\Delta x \to 0} \sum_{0}^{20} \frac{x^2}{4}\Delta x = \int_0^{20} \frac{x^2}{4}dx \underset{\overset{\uparrow}{\text{margin}}}{=} \frac{2000}{3}\text{ ft}^3$

$$\int_0^{20} \frac{x^2}{4}dx = \frac{1}{4}\left(\frac{x^3}{3}\Big|_0^{20}\right) = \frac{1}{4}\cdot\frac{20^3}{3} = \frac{2000}{3}$$

In general:

DEFINITION 5.6 The volume of a solid with integrable cross-sections of area $A(x)$ from $x = a$ to $x = b$ is given by $\int_a^b A(x)dx$.

Note that Definition 5.5 is a special case of this more general definition. (How?)

CHECK YOUR UNDERSTANDING 5.23

Find the volume of a solid with a circular base of radius 1 if the cross-sections perpendicular to the base are equilateral triangles.

Answer: $\frac{4\sqrt{3}}{3}$

EXERCISES

Exercises 1-3. **(Area Between Graph and x-axis)** Find the area bounded by the x-axis and the graph of the function f, over the given interval.

1. $f(x) = -x^2+4;\ [-1, 2]$
2. $f(x) = x+\sqrt{x};\ [0, 1]$
3. $f(x) = \dfrac{x}{(x^2+1)^2};\ [0, 1]$

Exercises 4-27. (Area Between Curves) Find the area of the finite region bounded by the given functions and lines.

4. $f(x) = x^2,\ y = x$
5. $f(x) = x^2-2,\ y = x$
6. $f(x) = x^2-x,\ y = x$
7. $f(x) = x^2+x,\ g(x) = -x^2+1$
8. $f(x) = x^3,\ y = x$
9. $f(x) = x^3+1,\ g(x) = x^3+x^2$
10. $f(x) = x^4,\ g(x) = -x^4-2x^2+4$
11. $y = x-3,\ y = -x+3,\ x = -1$
12. $y = x-3,\ y = -x+3,\ y = -2x-3$
13. $y = x, y = -x+2, x = 0, x = 2$
14. $y = x,\ \ y = -x+1,\ y = 0$
15. $f(x) = \sqrt{x},\ y = -x+2, y = 0$
16. $f(x) = \dfrac{1}{x^2},\ y = -\dfrac{3}{4}x+\dfrac{7}{4}$
17. $y = x, y = -\dfrac{x}{2}, y = -x+2$
18. $f(x) = -\dfrac{1}{x^2},\ y = x, y = -x-\dfrac{9}{4}$
19. $f(x) = \dfrac{1}{x^2},\ y = x,\ x = 2$
20. $f(x) = x,\ g(x) = 4x,\ y = -x+2$
21. $f(x) = x^3,\ y = -x+2, y = x+6$
22. $f(x) = |2x|,\ y = -x+1$
23. $f(x) = |2x|,\ g(x) = x^2+1$
24. $f(x) = \dfrac{x}{(x^2+1)^2}, y = -x, x = 1$
25. $f(x) = x\sqrt{x^2+1}, y = -x, y = \sqrt{2}$
26. $f(x) = \sin x, g(x) = \cos x, x = -\dfrac{3\pi}{4}, x = \dfrac{\pi}{4}$
27. $f(x) = \sin x, g(x) = \cos x, x = \dfrac{\pi}{4}, x = \dfrac{\pi}{2}$

Exercise 28-36. (Rotation about the x-axis) Find the volume of the solid obtained by rotating, about the x-axis, the region bounded above by the graph of the function, below by the x-axis, and on the sides by vertical lines through the endpoints of the given interval.

28. $f(x) = 3;\ [1, 3]$
29. $f(x) = x;\ [1, 3]$
30. $f(x) = x^2;\ [1, 3]$
31. $f(x) = x^2 + 1;\ [0, 1]$
32. $f(x) = -x^2 + 2;\ [0, 1]$
33. $f(x) = -x^2 + x;\ [0, 1]$
34. $f(x) = x^2 + 2;\ [-2, 1]$
35. $f(x) = \sqrt{x};\ [0, 4]$
36. $f(x) = x^{1/3};\ [1, 8]$

37. **(Volume of Sphere)** Derive the formula for the volume of a sphere of radius r. (Equation of the circle of radius r and centered at the origin is given by: $x^2 + y^2 = r^2$.)

Exercise 38-51. (Rotation about the x-axis) Determine the volume of the solid obtained by rotating, about the x-axis, the finite region enclosed by the graphs of the given functions.

38. $f(x) = x^2,\ \ y = x$
39. $f(x) = x^4,\ \ y = x$
40. $f(x) = x^3 + 1,\ \ g(x) = x^3 + x^2$
41. $f(x) = x^4 + 1,\ \ g(x) = -x^2 + 3$
42. $f(x) = x^4,\ \ g(x) = -x^4 - 2x^2 + 4$
43. $f(x) = \frac{2}{x},\ \ y = -x + 3$
44. $y = x,\ \ y = -x + 2,\ \ y = 1, y = 2$
45. $f(x) = x^2,\ \ y = x + 2$
46. $f(x) = \frac{2}{x},\ \ y = x - 1,\ \ x = 4$
47. $f(x) = x^2 + x + 1,\ \ y = x + 2$
48. $f(x) = \sqrt{x},\ \ y = -x + 2,\ \ x = 2$
49. $y = 2x + 3, y = x + 4, y = -x$
50. $f(x) = \sec x, y = \frac{4\sqrt{2} - 4}{\pi}x + 1$
51. $f(x) = \sqrt{\sin x}, y = \frac{x}{2}, x = \frac{\pi}{2}$

Exercise 52-55. (Rotation about the y-axis) Find the volume of the solid obtained by rotating, about the y-axis, the finite region enclosed by the graphs of the given functions.

52. $f(x) = x^2,\ \ y = x$
53. $f(x) = x^4,\ \ y = x$
54. $f(x) = x^3,\ \ g(x) = 2x^2$
55. $f(x) = \frac{2}{x},\ \ y = -x + 3$

Exercise 56-63. (Rotation about a Line) Find the volume of the solid obtained by rotating, about the given line, the finite region enclosed by the graphs of the given functions and lines about the given line.

56. $f(x) = x^2, y = 2x$, about $y = -1$
57. $f(x) = x^2, y = 2x$, about $y = 4$

58. $f(x) = x^2, y = 2x$, about $x = -1$

59. $f(x) = x^2, y = 2x$, about $x = 3$

60. $f(x) = x^3, y = x$, about $y = -1$

61. $f(x) = x^3, y = x$, about $y = 1$

62. $y = x, y = -x + 2, y = -2x$ about $y = -1$

63. $y = x, y = -x + 2, y = -2x$ about $x = 1$

Exercise 64-65. (Rotation about the *y*-axis) Use both the shell and the washer method to find the volume obtained by revolving the region *S* about the *y*-axis; where

64. *S* is bounded on the left by the *y*-axis, on top by the line $y = -x + 2$, and on the right by the graph of $f(x) = x^2$.

65. *S* is bounded on the left by the line $x = 1$, on top by the line $y = -x + 6$, and on the right by the graph of $f(x) = x^2$.

Exercise 66-67. (Rotation about the *x*-axis) Use both the shell and the washer method to find the volume of the solid obtained by revolving the region *S* about the *x*-axis, where:

66. *S* is bounded on the left by the *y*-axis, on top by the line $y = -x + 2$, and on the right by the graph of $f(x) = x^2$.

67. *S* is bounded on the left by the line $x = 1$, on top by the line $y = -x + 6$, and on the right by the graph of $f(x) = x^2$.

Exercise 68-73. (Slicing) Determine the volume of the given solid.

68. The solid is a 25 foot pyramid whose base is a 10 foot square.

69. The solid is a pyramid of height *h* whose base is a square of side *l*.

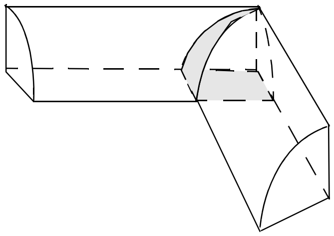
70. The solid is a A pyramid of height 25 feet whose base is a 5 foot by 10 foot rectangle.

71. The base of the solid is a circular disk of radius r and its cross-sections perpendicular to the base are squares.

72. The base of the solid is a circular disk of radius *r* and its cross-sections perpendicular to the base are equilateral triangles.

73. The base of the solid is the ellipse $x^2 + 4y^2 = 1$ and its cross-sections perpendicular to the base are squares.

74. Two right-circular cylinders of radius *r* have axes that intersect at right angles. Find the volume of the region common to the two cylinders.
Suggestion: Consider the adjacent figure depicting one-eighth of the solid in question.



§5. ADDITIONAL APPLICATIONS

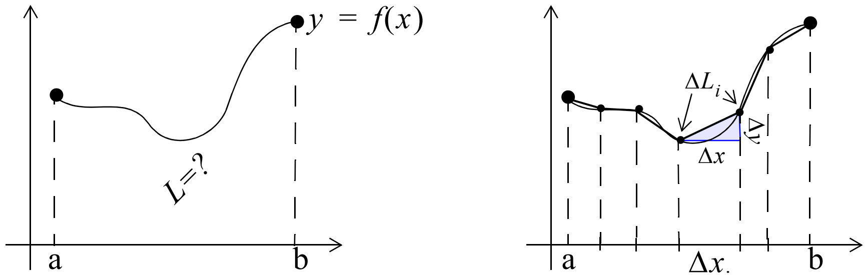
The concepts of arc length and work are addressed in this section. Additional applications are offered in the exercises.

We again acknowledge the fundamental theme for integral-applications:

FROM CONCEPT
TO RIEMANN SUM
TO DEFINITION
TO APPLICATION

ARC LENGTH

What is the length L of the curve $y = f(x)$ from $x = a$ to $x = b$ in Figure 5.8(a)? We again know what we are looking for, but still have to define it. And we are again essentially forced to mold our definition in accordance with pre-existing expectations. Specifically, we partition the interval $[a, b]$ into a number of pieces Δx_i and join their endpoints by the line segments of length ΔL_i, as is done in Figure 5.8(b), to obtain a polygonal path joining a to b of length $\sum_a^b \Delta L_i$ which appears to approximate the length we seek.



(a) (b)

Figure 5.8

Clearly, the smaller we make those Δx_i's the better $\sum_a^b \Delta L_i$ approximates that which we are trying to define; **forcing** us to define:

$$L = \lim_{\Delta x \to 0} \sum_a^b \Delta L_i \quad (*)$$

Applying the Pythagorean Theorem to the shaded right triangle in Figure 5.8(b) enables us to rewrite (*) in the form:

$$L = \lim_{\Delta x \to 0} \sum_a^b \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Which can be rewritten in the form (margin):

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2} \cdot (\Delta x)^2}$$
$$= \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}} \cdot \sqrt{(\Delta x)^2}$$
$$= \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}} \cdot \Delta x$$

A Riemann Sum!

$$L = \lim_{\Delta x \to 0} \sum_a^b \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}} \Delta x = \lim_{\Delta x \to 0} \sum_a^b \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$

In the event that the function $y = f(x)$ is differentiable on $[a, b]$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

Bringing us to:

DEFINITION 5.7
ARC LENGTH

The **length** of $y = f(x)$ from a to b is:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

(assuming that f is differentiable on $[a, b]$)

EXAMPLE 5.25 Express, in integral form, the length L of the graph of the function:

$$f(x) = x^3 + x^2$$

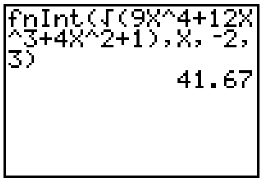
from $x = -2$ to $x = 3$.

SOLUTION: Turning to Definition 5.7:

$$L = \int_{-2}^{3} \sqrt{1 + [(x^3 + x^2)']^2}\, dx = \int_{-2}^{3} \sqrt{1 + [3x^2 + 2x]^2}\, dx$$

$$= \int_{-2}^{3} \sqrt{9x^4 + 12x^3 + 4x^2 + 1}\, dx$$

Alas, even with the techniques of integration introduced in subsequent sections you will not be able to evaluate the above integral; but all is not lost: ⟶



```
fnInt(√(9X^4+12X
^3+4X^2+1),X,-2,
3)
               41.67
```

CHECK YOUR UNDERSTANDING 5.24

Express, in integral form, the length L of the graph of the function:

$f(x) = \sqrt{x} + \frac{1}{x}$ from $x = 1$ to $x = 5$.

Answer:
$\int_1^5 \sqrt{1 + \frac{1}{4x} - \frac{1}{x^{5/2}} + \frac{1}{x^4}}\, dx$

EXAMPLE 5.26 Find the length L of the graph of the function $y = \frac{(x^2+2)^{3/2}}{3}$ over the interval $[1, 2]$.

SOLUTION: Turning to $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, we have:

$$y = \frac{(x^2+2)^{3/2}}{3}$$

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2}(x^2+2)^{1/2} \cdot 2x = x(x^2+2)^{1/2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^2(x^2+2)} = \sqrt{1 + x^4 + 2x^2}$$

$$= \sqrt{(x^2+1)^2} = |x^2+1| = x^2+1$$

In this case, we are able to finish the job by hand:

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 (x^2+1)\cdot d\cdot x = \left.\frac{x^3}{3}+x\right|_1^2 = \frac{10}{3}$$

CHECK YOUR UNDERSTANDING 5.25

Find the length L of the graph of the function $y = \frac{4\sqrt{2}}{3}x^{3/2}-1$ over the interval $[0, 1]$.

Answer: $\frac{13}{6}$

WORK

Work is a measure of the energy expended by a force in moving an object from one point to another. The work done when a **constant** force F causes an object to move a distance d in the direction of the force, is given by:

$$W = Fd \quad (\text{work} = \text{force} \times \text{distance})$$

As for units:

IN THE METRIC SYSTEM:

The unit of force is the newton, with one newton (N) being defined to be the force required to effect an acceleration of one meter-per-second-squared (m/s^2) on an object of mass one kilogram (kg): $1\ N = (1\ kg)\left(1\ \frac{m}{s^2}\right)$.

If F is measured in newtons and d in meters, then the unit for W is a **newton-meter**, or **joule** (J).

IN THE US (CUSTOMARY) SYSTEM:

The unit of force is the pound (lb).

If F is measured in pounds and d in feet, then the unit for W is a **foot-pounds** (ft-lb).

Conversion formulas that relate kilograms to pounds are really comparing apples to oranges:

> A pound is a measure of force while a kilogram is a measure of mass. Your weight on earth (a force) will differ from your weight on the moon, while your mass remains constant.

So, what is the unit for mass in the US system? The ***slug***, with:

$$1 pound = (1\text{ slug})\left(1\ \frac{ft}{s^2}\right)$$

Warning: While 50 newtons is a force, 50 kilograms is **NOT** a force. To convert 50-kilograms to newtons (on earth), you need to multiply it by $9.8\frac{m}{s^2}$:

$$\underset{\substack{\uparrow\\ \text{mass}}}{50kg} = \underset{\substack{\uparrow\\ \text{force}}}{50(9.8)N}$$

On the other hand: 50 pounds is already a force.

Assume now that a variable force $f(x)$ (**not** necessarily constant) is acting on an object in a linear direction from a point a to a point b. How should work be defined in this situation? Like this:

Partition the interval $[a, b]$ into a number of subintervals of length Δx_i. If Δx_i is relatively small, then we are justified in assuming that the force acting on the object throughout that small interval is essentially the **constant**: $f(x_i)$, where x_i is some chosen point in Δx_i (see Figure 5.9). It is therefore reasonable to stipulate that the work ΔW_i required to move the object through the interval Δx_i can be approximated by $\Delta W_i \approx f(x_i)\Delta x_i$.

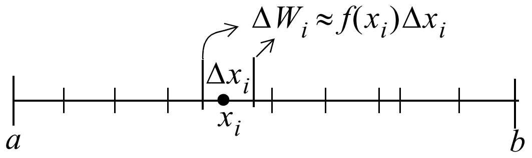


Figure 5.9

We can all agree that the approximation will improve as we make the partition finer and finer; forcing us to:

We can all agree that the approximation will improve as we make the partition finer and finer; forcing us to:

DEFINITION 5.8 The work done by a continuous force $f(x)$ along the x-axis from $x = a$ to $x = b$ is:

$$W = \lim_{\Delta x \to 0} \overset{\text{A Riemann Sum}}{\sum_{a}^{b} f(x_i)\Delta x_i} = \int_a^b f(x)dx$$

Hooke's law remains in effect providing x is not "too large."

In the following example we invoke Hooke's Law which asserts that the force required to maintain a spring's position when stretched or compressed x units beyond its natural length is proportional to x: $f(x) = kx$, where k, called the spring constant, is measured in force units per unit length.

EXAMPLE 5.27 A spring has a natural length of $\frac{1}{2}$ meter. Determine the amount of work it will take to stretch the spring to 1 meter, if a force of 25 newtons stretches the spring to a length of $\frac{3}{4}$ meters.

SOLUTION: We begin by finding the spring constant k:

$$\text{force} = k(\text{displacement}): \quad 25 = k \cdot \left(\frac{3}{4} - \frac{1}{2}\right) = \frac{1}{4}k$$

$$k = 100\ N/m$$

Turning to Definition 5.8 and Hooke's Law:

$$W = \int_0^{\frac{1}{2}} f(x)dx = \int_0^{\frac{1}{2}} 100x\,dx = 50x^2\Big|_0^{\frac{1}{2}} = \frac{50}{4} - 0 = \frac{50}{4}\ J$$

Note on units:

$$W = \lim_{\Delta x \to 0} \sum_{\frac{1}{2}}^{1} \underset{\frac{N}{m}}{\underset{\downarrow}{k}}\, \underset{m}{\underset{\downarrow}{x}}\, \underset{m}{\underset{\downarrow}{\Delta x_i}}$$

$$\frac{N}{m} \cdot m \cdot m = N \cdot m = J$$

CHECK YOUR UNDERSTANDING 5.26

A spring exerts a force of 1 pound when stretched $\frac{1}{2}$ foot beyond its natural length.

(a) What is the work done in stretching the spring $\frac{1}{4}$ ft beyond its natural length?

(b) What is the work done in stretching it an additional $\frac{1}{2}$ foot?

Answers: (a) $\frac{1}{16}$ ft-lb
(b) $\frac{1}{2}$ ft-lb

EXAMPLE 5.28 A 12 foot chain that weighs 2 pound per foot is lying on the ground. Determine the work done in lifting the chain so that it hangs from a beam that is 15 feet high.

SOLUTION: Cut the 12 foot chain into Δx pieces and lift those pieces to arrive at the hanging chain in Figure 5.10. Since the Δx piece weighs $(\Delta x \text{ ft})\left(2\frac{\text{lb}}{\text{ft}}\right) = 2\Delta x \text{ lb}$, the work done in lifting it a (vertical) distance of $d = 15 - x$ ft, is given by $\Delta W \approx 2\Delta x(15 - x)$ ft-lb.

"Approximately" since only the point in Δx that is exactly x units from the indicated end-point of the chain is lifted precisely $15 - x$ feet.

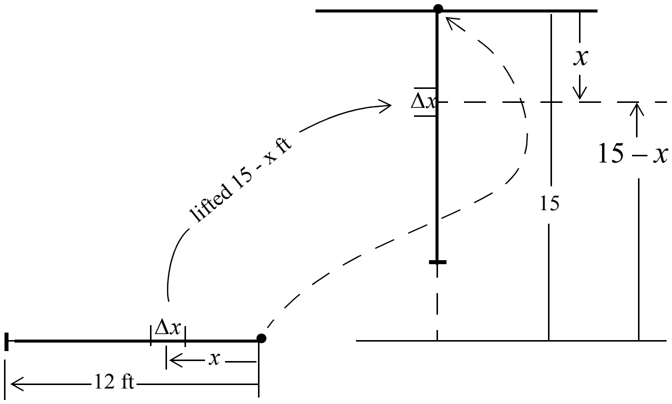


Figure 5.10

Bringing us to:

$$W = \lim_{\Delta x \to 0} \overset{\text{A Riemann Sum}}{\sum_{0}^{12} 2(15 - x)\Delta x} = \int_0^{12} (30 - 2x)dx = 216 \text{ ft-lb}$$

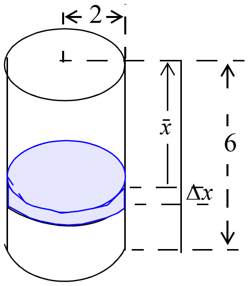
CHECK YOUR UNDERSTANDING 5.27

A 100 pound bag of sand is lifted for 8 seconds at the rate of 4 feet per second. Find the work done in lifting the bag if the sand leaks out at the rate of 1 pound per second.

Answer: 3072 ft-lb

EXAMPLE 5.29 A vertical cylindrical tank of radius 2 m and height 6 m is full of water. Find the work done in pumping out all of the water from an outlet at the top of the tank. (The density of water is $1000 \text{ kg}/\text{m}^3$.)

SOLUTION: The adjacent water-disk of volume $\pi \cdot 2^2 \cdot \Delta x = 4\pi\Delta x \text{ m}^3$ is to be lifted a distance of $\bar{x}$ m, where $\bar{x}$ is some chosen point in the interval Δx. Here is the mass of that disk:



$$4\pi\Delta x \text{ m}^3 \cdot 1000\frac{\text{kg}}{\text{m}^3} = 4000\pi\Delta x \text{ kg}$$

Here is the force needed to overcome the force of gravity in order to lift that mass (see margin):

As previously noted, while pound is a unit of force, gram is not — it is a unit of mass. Invoking Newton's Law: $F = ma$. Here, the acceleration is a consequence of the force of gravity which, near the surface of the earth, is approximately $9.8 \text{ m}/s^2$.

$$(4000\pi\Delta x \text{ kg})\left(9.8\frac{\text{m}}{s^2}\right) = 4000(9.8)\pi\Delta x \underset{\text{newton}}{\text{N}}$$

Here is the work done in lifting the indicated water-disk $\bar{x}$ meters:

$$\Delta W \approx (4000(9.8)\pi\Delta x)\bar{x} \underset{\text{joule}}{\text{J}}$$

And here is the total work done to empty the entire tank:

$$W = \lim_{\Delta x \to 0} \underbrace{\sum_0^6 4000(9.8)\pi x\Delta x}_{\text{A Riemann Sum}} = 39200\pi\int_0^6 x\,dx$$

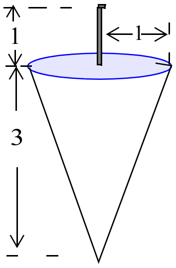
$$= 39200\pi\left(\frac{x^2}{2}\bigg|_0^6\right)$$

$$= 39200\pi\left(\frac{36}{2}\right) \approx 2.2 \times 10^6 \text{ J}$$

CHECK YOUR UNDERSTANDING 5.28

An inverted circular cone of height 3 m and radius 1 m is filled with water. Find the work done in pumping out all of the water from 1 m above the top of the tank.

(The density of water is $1000 \text{ kg}/\text{m}^3$.)



Answer: $17{,}150\pi$ J

EXERCISES

Exercises 1-6. (Arc Length) Express, in integral form, the length L of the graph of the function over the specified interval. Use a graphing calculator to approximate L to two decimal places.

1. $y = x^2, \quad 1 \le x \le 5$
2. $y = \frac{1}{2x+3}, \quad -1 \le x \le 3$
3. $y = \sqrt{2x-5}, 4 \le x \le 9$
4. $y = \sin x, 0 \le x \le 2\pi$
5. $y = 3x^2 + 4x - 5, -2 \le x \le 2$
6. $y = x\cos x, -\frac{\pi}{2} \le x \le 0$

Exercises 7-12. (Arc Length) Determine the length L of the graph of the function over the specified interval.

7. $f(x) = x^{3/2}, \quad [0, 8]$
8. $f(x) = (4 - x^{2/3})^{3/2}, \quad [0, 1]$
9. $y = \frac{x^3}{3} + \frac{1}{4x}, \quad [1, 2]$ Note: $1 + \left(\frac{dy}{dx}\right)^2$ turns out to be a perfect square
10. $y = \frac{x^4}{4} + \frac{1}{8x^2}, \quad [2, 3]$ Note: $1 + \left(\frac{dy}{dx}\right)^2$ turns out to be a perfect square
11. $y = \int_1^x \sqrt{t^2 - 1}\, dt, \quad [1, 2]$
    Suggestion: See Theorem 5.7, page 178.
12. $y = \int_1^x \sqrt{t - 1}\, dt, \quad [2, 3]$
    Suggestion: See Theorem 5.7, page 178.

13. **(Arc Length)** Express, in integral form, the length of the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

14. **(Arc Length)** Express, in integral form, the length of a cycle of the sine curve.

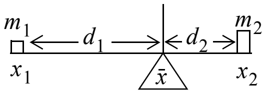
15. **(π)** Apply the arc length formula $L = \int_a^b \sqrt{1 + [f'(x)]^2}dx$ to the unit circle $x^2 + y^2 = 1$ to show that $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \pi$.

Exercises 16-30. (Work)

16. A spring is found to exert a force of 10 lb when stretched 4 in. beyond its natural length.
    (a) Find the work done in stretching the spring 1 ft from its natural length.
    (b) Find the work done in compressing the spring 5 in. from its natural length.
17. A spring is found to exert a force of 25 N when compressed 200 cm beyond its natural length.
    (a) Find the work done in stretching the spring 100 cm from its natural length.
    (b) Find the work done in compressing the spring 50 cm from its natural length.
    (c) How far will a force of 10N stretch the spring?
18. A spring has a natural length of 1 ft. A force of 8 oz. stretches the spring to a length of $\frac{3}{2}$ ft.
    (a) Find work required to stretch the spring to 1 ft beyond its natural length.
    (b) How far will a force of 1 lb stretch the spring?

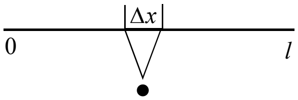
19. Find the natural length of a spring, given that the work done in stretching it from a length of 2 feet to a length of 3 feet is one-half the work done in stretching it from a length of 3 feet to a length of 4 feet.
20. A spring has a natural length of 1 m. A force of 12 N compresses the spring to a length of 0.7 m.
    (a) Find the work required to stretch the spring to 0.4 m beyond its natural length.
    (b) How far will a force of 10 N compress the spring?
21. Find the natural length of a spring, given that the work done in compressing it from a length of 1 m to a length of 75 cm is twice the work done in stretching it from a length of 1 m to a length of 2 m.
22. Find the natural length of a spring given that the work done in stretching it from a length of 1 ft to a length of 1.5 ft is half the work done in stretching it from a length of 1.5 ft to a length of 2 ft.
23. Given that a work W is needed to stretch a spring from its natural length l ft to a length of $l+a$ ft, find the work done in stretching the spring from a length of $l+a$ ft to a length of $l+2a$ ft.
24. A vertical cylindrical tank of radius 2 feet and height 6 feet is full of water. (Water weighs 62.5 pounds per cubic foot.) Find the work done in:
    (a) Pumping out all of the water from an outlet at the top of the tank.
    (b) Pumping out all of the water from an outlet that is 1 foot above the top of the tank.
    (c) Pumping out half of the water from an outlet at the top of the tank.
25. An inverted circular cone of height 3 ft and radius 1 ft is filled with a liquid weighing $6\ \text{oz}/\text{in}^3$. Find the work done in:
    (a) Pumping out all of the liquid from an outlet at the top of the tank.
    (b) Pumping out all of the liquid from an outlet that is 1 foot above the top of the tank.
    (c) Pumping out half of the liquid from an outlet at the top of the tank.
26. A chain lying on the ground is 5 m long and has a total mass of 50 kg. How much work is required to raise the chain to a height of 7 m?
27. A 25 foot rope weighs $4\ \text{oz}/\text{ft}$ is lying on the ground. How much work is required to raise the rope to a height of 30 ft?
28. A 40 ft cable weighing $2\ \text{lb}/\text{ft}$ hangs from a windlass. How much work is required in winding up 25 ft of the cable?
29. A bucket of sand that weighs 50 pounds hangs from a 20 foot cable that is attached to a beam that is 75 feet above the ground. Find the work done in lifting the bucket to the beam if the cable weighs 2 pounds per foot.
30. A bucket that weighs 50 lb is attached to the end of a 15 foot rope lying on the ground weighing $3\ \text{oz}/\text{ft}$. The rope is lifted and attached to a 30 ft beam. Initially the bucket contained 25 lb of liquid which is leaking out at a constant rate. How much work is done if:
    (a) All of the liquid finishes draining just when the bucked reaches its final destination?
    (b) All of the liquid finishes draining when the bucket is 10 ft from the ground?

Exercises 31-36. (Center of Mass (Gravity) [On Line]) The center of mass of an object (or system of objects) is the point at which the object (or system of objects) would balance if positioned at a the head of a pin positioned at that point.



Consider the seesaw in the adjacent figure with indicated masses m_1, m_2. In accordance with the lever law of physics, balance will occur if $m_1 d_1 = m_2 d_2$. Consequently, the center of mass occurs at the point $\bar{x}$ where $\sum_{i=1}^{2} m_i(x_i - \bar{x}) = 0$. More generally, if n masses, $m_1, m_2, \ldots, m_n$ are positioned along the x-axis at points $x_1, x_2, \ldots, x_n$, then the center of mass occurs at that point $\bar{x}$ satisfies the equation

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = 0 \Rightarrow \sum_{i=1}^{n} m_i x_i - \bar{x}\sum_{i=1}^{n} m_i = 0 \Rightarrow \boldsymbol{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}}$$



can be approximated by a point of mass $\delta(x)\Delta x$

Generalizing further, if $\delta(x)$ is the density function (mass per unit of length) of a rod of length l then its center of mass is given by

$$\lim_{\Delta x \to 0} \sum_{0}^{l} [\delta(x)\Delta x](x - \bar{x}) = \int_0^l (x - \bar{x})\delta(x)dx = 0$$

$$\Rightarrow \int_0^l x\delta(x)dx - \bar{x}\int_0^l \delta(x)dx = 0 \Rightarrow \bar{x} = \frac{\int_0^l x\delta(x)dx}{\int_0^l \delta(x)dx}$$

31. Find the center of mass of a system consisting of a 10 pound weight at $x = -5$ and a 15 pound weight at $x = 3$.
32. Find the center of mass of a system consisting of a 10 pound weight at $x = -5$, a 15 pound weight at $x = 3$, and a 2 pound weight at $x = 7$
33. A system consists of a 10 pound weight at $x = -5$ and a 15 pound weight at $x = 3$. What size weight needs to be positioned at $x = 7$ for the center of gravity of the system to be at $x = 0$?
34. A system consists of a 10 pound weight at $x = -5$ and a 15 pound weight at $x = 3$. Where should a 5 pounds weight be positioned in order for the center of gravity to occur at $x = 1$?
35. The density of a 10 foot rod, as measured from end-point A, is given by $\delta(x) = 1 + \frac{x}{10}\frac{\text{lb}}{\text{ft}}$.

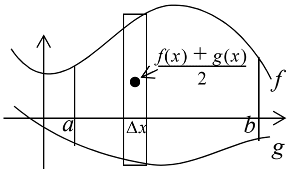
    Find the rod's center of mass.
36. The density of a 7 meter rod, as measured from end-point A, is given by $\delta(x) = 2 + \frac{x}{5}\frac{\text{kg}}{\text{m}}$.

    Find the rod's center of mass.

Exercises 37-39. (Center of Mass (Gravity) [In Plane]) Generalizing the mass-concept to a system of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the plane with respective masses $m_1, m_2, \ldots, m_n$ we find that the system has center of mass $(\bar{x}, \bar{y})$ is given by the equations:

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \qquad (*)$$

37. Determine the center of mass of a system consisting of ten pounds at $(1, 3)$, twenty pounds at $(-2, 2)$, and four pounds at $(-1, 8)$.
38. A system consists of ten pounds at $(1, 3)$, twenty pounds at $(-2, 2)$, and four pounds at $(-1, 8)$. What size weight needs to be positioned at the origin for the center of gravity of the system to be at the origin?
39. A system consists of ten pounds at $(1, 3)$, twenty pounds at $(-2, 2)$, and four pounds at $(-1, 8)$. Where should a five pound weight be positioned in order for the center of mass of the system to be at the origin?



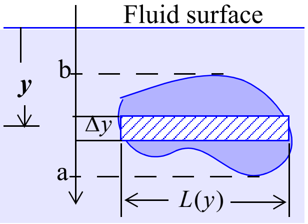
Exercises 40-45. (Center of Mass (Gravity)) Let $f(x) > g(x)$ over the interval $[a, b]$. To determine the center of mass of a region of uniform density ρ that is bounded above by the graph of f, below by the graph of g, and on the sides by the vertical lines $x = a$ and $x = b$, we proceed as follows. Partition the region with vertical strips of base Δx and height $f(x) - g(x)$. Next, concentrate the mass $[f(x) - g(x)]\Delta x \rho$ of that strip at (x, y), where $x \in \Delta x$ and $y = \frac{f(x) + g(x)}{2}$ (half-way up the strip). Returning to the formulas for $\bar{x}$ and $\bar{y}$ in (*) of Exercises 37-39 we see that:

$$\bar{x} = \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b ([f(x)]^2 - [g(x)]^2)dx}{\int_a^b [f(x) - g(x)]dx}$$

Find the center of mass of the finite region bounded by the graphs of the given functions and lines. Assume that the region is of uniform density.

40. $f(x) = x^2, y = 0, x = 0, x = 2$
41. $f(x) = x^2, y = 0, x = 0, x = 1$
42. $f(x) = x, g(x) = x^4$
43. $f(x) = x, g(x) = x^2$
44. $f(x) = 4 - x^2, g(x) = \frac{x^2}{4} - 1$
45. $f(x) = 2\sqrt{x}, g(x) = x$

Exercises 46-54. (Fluid Force) If a tank contains a fluid weighing w lb/ft^3, then the pressure exerted by the fluid at depth d is wd lb/ft^2 **in all directions**. In particular the fluid force on a horizontal surface of area A at depth d is $wd \cdot A$ pounds (equal to the weight of the column of fluid above the surface).

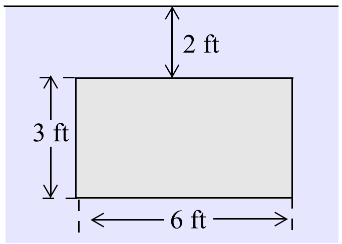


Consider, now, a vertical surface submerged in a fluid of constant weight-density w. Partitioning $[a, b]$ in the usual way we find the force on the indicated horizontal strip: $\Delta F \approx w\underset{\substack{\uparrow \\ \text{depth of strip}}}{y}\overbrace{L(y)\Delta y}^{\text{area of strip}}$; leading us to the formula for the fluid force exerted on the submerged surface:

$$F = \lim_{\Delta y \to 0} \sum_{a}^{b} F = w\int_a^b yL(y)dy$$

Determine the fluid force on the indicated vertical region when submerged in a liquid of weight-density w (in pounds per square feet).

46.



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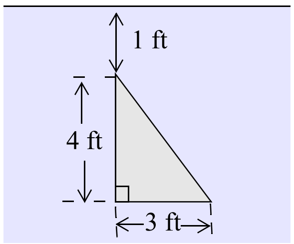

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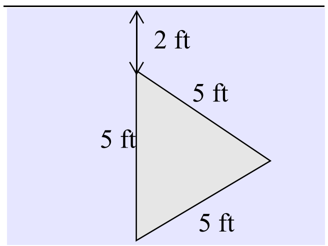

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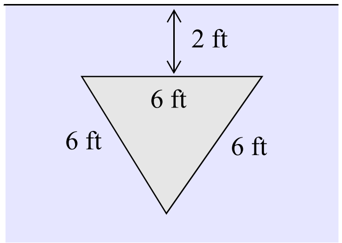

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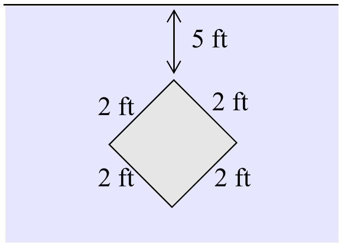

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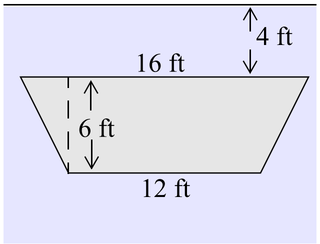
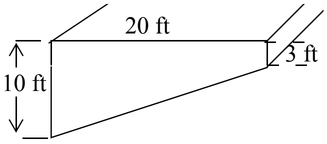


52. Find the force on a circular gate of diameter 4 ft in a vertical dam where the center of the gate is 20 ft below the surface if the water. $\left(\text{Weight density of water} \approx 62.4\frac{\text{lb}}{\text{ft}^3}\right)$

53. A swimming pool is 20 ft wide. The water is 3 ft deep at one end at 10 ft deep on the other end. Find the force of the water on one of the 20 ft sides. $\left(\text{Weight density of water} \approx 62.4\frac{\text{lb}}{\text{ft}^3}\right)$



54. Show that if a vertical surface descends vertically at a constant rate, then the fluid force on the surface increases at a constant rate.

CHAPTER SUMMARY

ANTIDERIVATIVE	An **antiderivative** of a function f is a function whose derivative is f.
INDEFINITE INTEGRAL	The collection of all antiderivatives of f is denoted by $\int f(x)dx$. In other words $$\int f(x)dx = g(x) + C$$
THEOREMS	$\int x^r dx = \frac{x^{r+1}}{r+1} + C$ (if $r \neq -1$) $\int \sin x\, dx = -\cos x + C$ $\int \cos x\, dx = \sin x + C$ $\int \sec^2 x\, dx = \tan x + C$ $\int \csc^2 x\, dx = -\cot x + C$ $\int \sec x \tan x\, dx = \sec x + C$ $\int \csc x \cot x\, dx = -\csc x + C$
DEFINITE INTEGRAL	A function f is said to be **integrable** over the interval $[a, b]$ if $\lim\limits_{\Delta x \to 0} \sum\limits_a^b f(x)\Delta x$ exists. In this case, we write: $$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x$$ and call the number $\int_a^b f(x)dx$ the **integral** of f over $[a, b]$. In addition: $\int_a^a f(x)dx = 0$ for any function containing a in its domain.
THE PRINCIPAL THEOREM OF CALCULUS	If f is continuous on $[a, b]$ then the function T given by $T(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b), with $T'(x) = f(x)$.

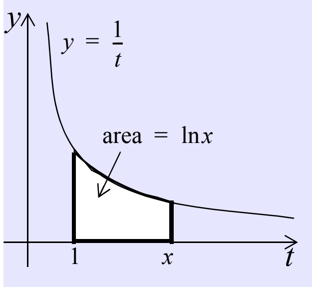
THE FUNDAMENTAL THEOREM OF CALCULUS	If f is continuous on $[a, b]$ and if $g'(x) = f(x)$, then: $\int_a^b f(x)dx = g(b) - g(a)$
THEOREMS	$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$ $\int_a^b cf(x)\,dx = c\int_a^b f(x)$ for any constant c. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (for $a < c < b$)
NET CHANGE FROM RATE OF CHANGE	The net-change from $x = a$ to $x = b$ is given by: Net- change $= \int_a^b f'(x)dx$
***U*-SUBSTITUTION METHOD**	If $F'(x) = f(x)$, then: $\int f[g(x)]g'(x)\,dx = F[g(x)] + C$ To soften the appearance of the above result, one makes the substitution: $u = g(x)$ to arrive at: $\int \underset{\downarrow}{f[g(x)]}\ \underset{\downarrow}{g'(x)\,dx} = F[\underset{\downarrow}{g(x)}] + C$ $\int f(u)\ dx = \boldsymbol{F}(\boldsymbol{u}) + C$
AREA BETWEEN CURVES	Let f and g be continuous over the interval $[a, b]$. The area between the graph of those functions between $x = a$ and $x = b$ is given by: $A = \int_a^b \lvert f(x) - g(x)\rvert dx$
VOLUME OF A SOLID OF REVOLUTION (DISK METHOD)	Let f be nonnegative and continuous over the interval $[a, b]$. The volume of the solid obtained by rotating, about the x-axis, the region bounded above by the graph of the function f, below by the x-axis, and on the sides by $x = a$ and $x = b$, is given by: $V = \pi \cdot \int_a^b [f(x)]^2 dx$

VOLUME OF A SOLID OF REVOLUTION (WASHER METHOD)	Let f and g be nonnegative and continuous over the interval $[a, b]$ with $f(x) \geq g(x)$. The volume of the solid obtained by rotating, about the x-axis, the region bounded above by the graph of the function f, below by the graph of g, and on the sides by $x = a$ and $x = b$, is given by: $V = \pi \cdot \int_a^b ([f(x)]^2 - [g(x)]^2)dx$
VOLUME OF A SOLID OF REVOLUTION (SHELL MEHTOD)	Let f and g be continuous over the interval $[a, b]$ with $f(x) \geq g(x)$. The volume of the solid obtained by rotating, about the y-axis, the region bounded above by the graph of the function f, below by the graph of g, and on the sides by $x = a$ and $x = b$, is given by: $V = 2\pi\int_a^b x[f(x) - g(x)]dx$
ARC-LENGTH	The **length** of $y = f(x)$ from a to b is: $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1 + [f'(x)]^2}dx$
WORK	The work done by a continuous force $f(x)$ along the x-axis from $x = a$ to $x = b$ is: $W = \int_a^b f(x)dx$

CHAPTER 6

ADDITIONAL TRANSCENDENTAL FUNCTIONS

§1. THE NATURAL LOGARITHMIC FUNCTION



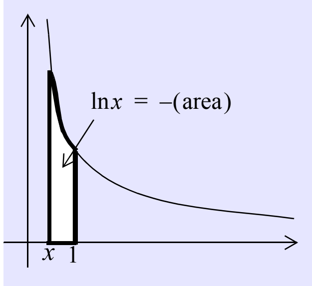
The familiar integration formula

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C$$

is a meaningless expression if $r = -1$ (why?). Fine, but since for any $x > 0$ the function $f(t) = \frac{1}{t}$ is continuous throughout its domain, we know that $\int_1^x \frac{1}{t}dt$ yields a number for any $x > 0$ (see margin). This number is denoted by $\ln x$. Formally:

DEFINITION 6.1 The **natural logarithmic function**, denoted by $\ln x$, has domain $(0, \infty)$ and range $(-\infty, \infty)$, and is given by:

$$\ln x = \int_1^x \frac{1}{t}dt$$



Thinking in terms of area, and recalling that $\int_a^b f(x)dx = -\int_b^a f(x)dx$, we note that $\ln x$ is negative for $0 < x < 1$ (see margin). Continuing to think in terms of area we obtain the (anticipated) graph of $\ln x$:

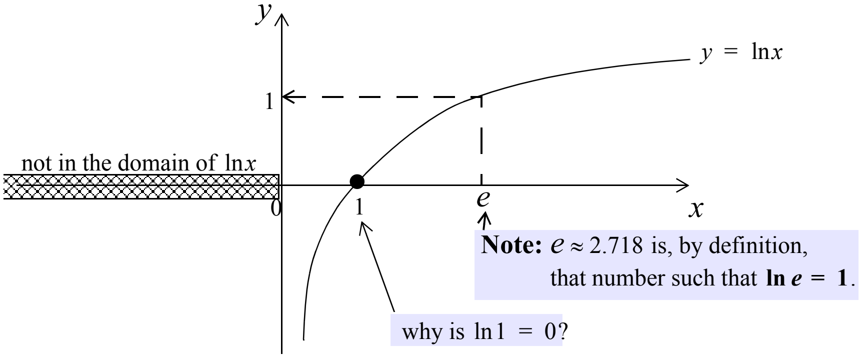


Figure 6.1

The above graph suggests that the derivative of the natural logarithmic function is positive throughout its domain. But what is $\frac{d}{dx}\ln x$? Here is the answer:

THEOREM 6.1

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

PROOF: A direct consequence of the Principal Theorem of Calculus (page 178).

As was anticipated, $(\ln x)' = \frac{1}{x}$ is positive throughout the domain of $\ln x$. Figure 6.1 also suggests that the graph of $\ln x$ is concave down throughout its domain; and it is:

$$(\ln x)'' = \left(\frac{1}{x}\right)' = (x^{-1})' = -1x^{-2} = -\frac{1}{x^2} < 0$$

EXAMPLE 6.1 Differentiate the given function.

(a) $f(x) = \ln(\sin x)$ (b) $g(x) = \frac{\sin(\ln x)}{x^2}$

SOLUTION:

(a) $$[f(x)]' = [\ln(\sin x)]' \underset{\uparrow}{=} \frac{1}{\sin x}(\sin x)' = \frac{1}{\sin x} \cdot \cos x = \cot x$$

the chain rule: $(\ln \square)' = \frac{1}{\square} \cdot \square'$

(b) $$\frac{d}{dx}\left[\frac{\sin(\ln x)}{x^2}\right] = \frac{x^2[\sin(\ln x)]' - [\sin(\ln x)](x^2)'}{x^4}$$

$$= \frac{x^2[\cos(\ln x) \cdot (\ln x)'] - [\sin(\ln x)] \cdot 2x}{x^4}$$

$$= \frac{x^2\left[\cos(\ln x) \cdot \frac{1}{x}\right] - [\sin(\ln x)] \cdot 2x}{x^4}$$

$$= \frac{\cos(\ln x) - 2\sin(\ln x)}{x^3}$$

CHECK YOUR UNDERSTANDING 6.1

Differentiate the given function:

(a) $x^3 \ln x^2$ (b) $\frac{\tan x}{\ln 2x}$ (c) $\ln(\ln x)$

Answer:
(a) $3x^2 \ln x^2 + 2x^2$
(b) $\frac{x \ln 2x \sec^2 x - \tan x}{x(\ln 2x)^2}$
(c) $\frac{1}{x \ln x}$

Does the expression $\ln|x|$ make sense? Sure, as long as $x \neq 0$ [for the natural logarithmic function is defined for all **positive** (real) numbers]. We can say more:

THEOREM 6.2 For any $x \neq 0$:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

PROOF: Case 1: $x > 0$.

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x \underset{\substack{\uparrow\\ \text{Theorem 6.1}}}{=} \frac{1}{x}$$

Case 2: $x < 0$.

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}\ln(-x) \underset{\substack{\uparrow\\ \text{chain rule}}}{=} \frac{1}{-x}\cdot(-x)' = \frac{1}{-x}(-1) = \frac{1}{x}$$

Turning Theorem 6.2 around, we have:

So, the "ugly duckling" x^{-1} can finally boast of having an antiderivative: $\ln|x|$.

THEOREM 6.3

$$\int\frac{1}{x}dx = \ln|x| + C$$

EXAMPLE 6.2 Determine:

(a) $\displaystyle\int\frac{x^2}{x^3+1}dx$ (b) $\displaystyle\int_1^e\frac{\ln x}{x}dx$

SOLUTION:

(a)
$$\int\frac{x^2}{x^3+1}dx \underset{\uparrow}{=} \frac{1}{3}\int\frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+1| + C$$

$$\begin{aligned} u &= x^3+1\\ du &= 3x^2dx\\ x^2dx &= \frac{1}{3}du \end{aligned}$$

(b)
$$\begin{aligned} u &= \ln x\\ du &= \boldsymbol{\frac{1}{x}dx} \end{aligned}$$

when $x = e, u = \ln e = 1$

$$\int_1^e\frac{\ln x}{\boldsymbol{x}}\,\boldsymbol{dx} = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$$

when $x = 1, u = \ln 1 = 0$

CHECK YOUR UNDERSTANDING 6.2

Perform the indicated operation:

(a) $\displaystyle\int\frac{7}{5x+2}dx$ (b) $\displaystyle\int_e^5\frac{dx}{x\ln x}$

Answer:
(a) $\frac{7}{5}\ln|5x+2| + C$
(b) $\ln(\ln 5)$

You were exposed to logarithmic functions of base b in precalculus, at which time you encountered the following logarithmic properties:

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b\frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^r = r\log_b x$$

That's all well and good, except for the fact that one needs the calculus to define the general logarithmic functions $\log_b x$. This we shall do in Section 3. For now:

THEOREM 6.4 For any positive numbers x and y and any real number r:

(a) $\ln xy = \ln x + \ln y$

(b) $\ln\frac{x}{y} = \ln x - \ln y$

(c) $\ln x^r = r\ln x$

A general proof of (c) is offered in Section 3, following the formal definition of x^r.

Note:

$5^{2/3} = (5^{1/3})^2 = (5^2)^{1/3}$

but what is 5^{π}?

PROOF: We establish (a), and ask you to prove (b) in CYU 6.3. You are invited to verify (c) for r a rational number in the exercises (see margin).

(a) Consider the function $\ln ax$, where a is an arbitrary positive constant. Note that $(\ln ax)' = (\ln x)'$:

$$(\ln ax)' = \frac{1}{ax}(ax)' = \frac{1}{ax}\cdot a = \boldsymbol{\frac{1}{x}} \quad \text{and} \quad (\ln x)' = \boldsymbol{\frac{1}{x}}$$

It follows from CYU 4.3, page 124, that $\ln ax$ and $\ln x$ can only differ by a constant:

$$\ln ax = \ln x + c \quad (*)$$

Evaluating the above equation at $x = 1$ we find that:

$$\ln a = \ln 1 + c \Rightarrow c = \ln a \text{ (since } \ln 1 = 0)$$

Returning to (*) we have: $\ln ax = \ln x + \ln a$.

Replacing the arbitrary positive constant a with the variable y brings us to: $\ln xy = \ln x + \ln y$.

CHECK YOUR UNDERSTANDING 6.3

Use Theorem 6.4(a) and (c) to verify Theorem 6.4(b).

Answer: See page A-33.

Here are four additional basic integration formulas for your consideration:

THEOREM 6.5

(a) $\int \tan x\,dx = \ln|\sec x| + C$

(b) $\int \cot x\,dx = -\ln|\csc x| + C$

(c) $\int \sec x dx = \ln|\sec x + \tan x| + C$

(d) $\int \csc x dx = -\ln|\csc x + \cot x| + C$

We establish (a) and (c) and invite you to verify (b) and (d) below.

PROOF: (a)

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{u}du = -\ln|u| + C$$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\ln|\cos x| + C$$

Theorem 6.4(c): $= \ln|\cos x|^{-1} + C$

$$= \ln|(\cos x)^{-1}| + C$$

$$= \ln|\sec x| + C$$

a clever 1

$u = \sec x + \tan x$

$du = (\sec^2 x + \sec\tan x)dx$

(c)

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}dx$$

$$= \int \frac{\sec^2 x + \sec\tan x}{\sec x + \tan x}dx = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

CHECK YOUR UNDERSTANDING 6.4

Establish:

(a) $\int \cot x\,dx = -\ln|\csc x| + C$ (b) $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

Answers: See page A-33.

EXAMPLE 6.3 Determine the area A of the region bounded above by the function $f(x) = \frac{x}{x^2+1}$, below by the line $y = -1$ and on the sides by the y-axis and the line $x = 1$.

SOLUTION: Since the function f is nonnegative throughout the interval $[0, 1]$:

dominant subordinate

$$A = \int_0^1 \left(\frac{x}{x^2+1} - (-1)\right)dx = \int_0^1 \left(\frac{x}{x^2+1}\right)dx + \int_0^1 1\,dx$$

$u = x^2 + 1$ when $x = 0, u = 1$

$du = 2x\,dx$ when $x = 1, u = 2$

$$\to = \frac{1}{2}\int_1^2 \frac{du}{u} + x\Big|_0^1$$

$$= \frac{1}{2}\left(\ln|u|\Big|_1^2\right) + \mathbf{1}$$

$$= \frac{1}{2}(\ln 2 - \ln 1) + \mathbf{1} = \frac{1}{2}\ln 2 + \mathbf{1}$$

$\ln 1 = 0$

CHECK YOUR UNDERSTANDING 6.5

Find the volume obtained by revolving the region that lies above the interval $[1, e]$ and below the graph of the function $f(x) = \frac{1}{\sqrt{x}}$, about the line $y = -1$.

Answer: $\pi(4\sqrt{e} - 3)$

EXAMPLE 6.4 Solve the second-order differential equation $f''(x) = \frac{1}{x^2} + x^2$, if $f'(1) = \frac{1}{3}$ and $f(1) = 3$.

SOLUTION:

$$f'(x) = \int\left(\frac{1}{x^2} + x^2\right)dx = \int(x^{-2} + x^2)dx = \frac{x^{-1}}{-1} + \frac{x^3}{3} + C$$

$$\text{since } f'(1) = \frac{1}{3}\text{:}\quad \frac{1}{3} = -1 + \frac{1}{3} + C \Rightarrow C = 1$$

At this point we have: $f'(x) = -\frac{1}{x} + \frac{x^3}{3} + 1$. Hence:

$$f(x) = \int\left(-\frac{1}{x} + \frac{x^3}{3} + 1\right)dx = -\ln|x| + \frac{x^4}{12} + x + C$$

$$\text{since } f(1) = 3\text{:}\quad 3 = -\underset{\substack{\uparrow \\ \mathbf{0}}}{\ln(1)} + \frac{1}{12} + 1 + C \ \Rightarrow C = \frac{23}{12}$$

Final answer: $f(x) = -\ln|x| + \frac{x^4}{12} + x + \frac{23}{12}$.

CHECK YOUR UNDERSTANDING 6.6

Solve the differential equation $f'(x) = \frac{1}{2x+1} + 2x + 1$ if $f(0) = 1$.

Answer:
$f(x) = \frac{1}{2}\ln|2x+1| + x^2 + x + 1$

EXERCISES

Exercise 1-18. (First Derivative) Differentiate.

1. $h(x) = x^3 \ln x$
2. $f(x) = x \ln x^3$
3. $h(x) = \ln x^2 + (\ln x)^2$
4. $g(x) = (x \ln x^3)^4$
5. $g(x) = \sin x \ln x$
6. $h(x) = \cos(\ln x)$
7. $f(x) = \ln(\cos x)$
8. $h(x) = \ln[\ln(x+1)]$
9. $f(x) = \dfrac{\sin x}{\ln x}$
10. $f(x) = \dfrac{\ln x}{\sin x}$
11. $f(x) = [\ln(x^2+1)]^2$
12. $h(x) = \sin^2 x \cdot \ln x$
13. $h(x) = \sqrt{x \ln x}$
14. $f(x) = \tan(\ln x)$
15. $f(x) = \ln(\sec x)$
16. $g(x) = \sqrt{\ln \sqrt{x}}$
17. $f(x) = \sin(\ln x^2)$
18. $f(x) = x \sin x \ln x$

Exercise 19-24. (Second Derivative) Determine $\dfrac{d^2y}{dx^2}$.

19. $y = \ln x$
20. $y = x \ln x$
21. $y = \dfrac{x}{\ln x}$
22. $y = x^2 (\ln x)^2$
23. $y = \sqrt{x \ln x}$
24. $y = \int_1^x \ln t \, dt$

Exercise 25-26. (Composite Functions) Determine the derivative of the composite function.

(a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$ (c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$ if:

25. $f(x) = 3x^2 + x$ and $g(x) = \ln x$
26. $f(x) = \dfrac{1}{2x}$ and $g(x) = \ln(x+1)$

Exercise 27-30. (Tangent Line) Determine the tangent line to the graph of the given function at the indicated point.

27. $y = 2 \ln x$ at $x = 1$
28. $y = x \ln x$ at $x = 1$
29. $y = \ln(\sin x)$ at $x = \dfrac{\pi}{2}$

30. (Implicit Differentiation) $x^2 \ln y + x^2 - 2y^3 = -1$ at $(1, 1)$

31. **(Point of Tangency)** Find the point on the graph of $f(x) = \ln x$ at which the tangent line passes through the origin.

Exercise 32-42. (Integration) Evaluate.

32. $\displaystyle\int \frac{x^2 - x + 5}{x^2} dx$
33. $\displaystyle\int \frac{x^2}{x^3 + 2} dx$
34. $\displaystyle\int \frac{\sin x}{\cos x + 1} dx$
35. $\displaystyle\int \frac{dx}{x \ln x^2}$
36. $\displaystyle\int \frac{\sin(\ln x)}{x} dx$
37. $\displaystyle\int \frac{\cos(\ln x)}{x} dx$

38. $\int x \tan x^2 dx$

39. $\int \frac{\cot\sqrt{x}}{\sqrt{x}} dx$

40. $\int \frac{\sec(\ln x)}{x} dx$

41. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x dx$

42. $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{1+\cos^2 x}$

Exercise 43-46. (Differential Equation) Solve for $f(x)$.

43. $f'(x) = \frac{x}{x^2+1}$ if $f(0) = 2$.

44. $f'(x) = \tan x$ if $f(0) = e$

45. $f'(x) = \frac{1}{x}\ln\frac{1}{x}$ if $f(e) = 1$.

46. $f''(x) = \frac{1}{x^2}$ if $f'(1) = 1$ and $f(1) = 0$

Exercise 47-48. (Graphing) Sketch the graph of the given function.

47. $f(x) = -\ln(-x-5)$

48. $f(x) = \ln(x^2-x-2)$

49. **(Area)** Determine the area A of the region bounded above by the graph of the function $y = \frac{1}{x}$, below by the line $y = -x$, and on the sides by the vertical line $x = 1$ and $x = e$.

50. **(Area)** Determine the area A of the region that lies above the interval $[e, 5]$ and below the graph of the function $f(x) = \frac{x^2}{x^3+1}$.

51. **(Volume)** Find the volume obtained by revolving the finite region bounded by the graphs of the functions $f(x) = x^2$, $g(x) = \frac{1}{\sqrt{x}}$, and the line $x = e$ about the x-axis.

52. **(Learning Curve)** A study has shown that the number, $N(t)$, of words per minute that an individual can type, after t hours of practice, is given by:

$$N(t) = 10 + 6\ln t, \quad 0 \le t \le 500$$

Determine the rate of change of $N(t)$ after:

(a) 10 hours of practice. (b) 100 hours of practice.

53. **(Work)** Determine the work done by a force of $\frac{1}{x}$ N along the x-axis from $x = 2$ to $x = 9$.

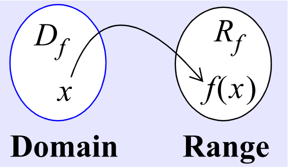
54. **(Arc Length)** Find the length L of the graph of the function $f(x) = x^2 - \frac{\ln x}{8}$ over the interval $[1, e]$.

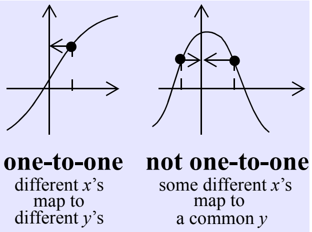
55. **(Maximum Velocity)** A particle moves on the x-axis in such a way that its velocity is given by $v = \frac{\ln t}{t}$ for $t > 1$. At what time will velocity be greatest?

56. **(Theory)** (a) Find a formula for the n^{th} derivative of $f(x) = \ln(cx)$, for $c > 0$.
(b) Use the Principle of Mathematical Induction to establish your answer in (a).

57. **(Theory)** Show that $\ln x^r = r\ln x$ for any rational number r and any $x > 0$.

§2. THE NATURAL EXPONENTIAL FUNCTION





We begin by reminding you that:

- The **domain** of a function f is the set D_f on which f "acts," and its **range** is the set R_f of the function values (see margin).
- A function f is **one-to-one** if for all a and b in D_f:

  $a \neq b \Rightarrow f(a) \neq f(b)$ (see margin and page 11)
- Let f be a one-to-one function with domain D_f and range R_f. The **inverse of *f***, denoted by f^{-1}, is that function with domain R_f and range D_f satisfying the following conditions (see page 13):

  $(f^{-1} \circ f)(x) = x$ for every x in D_f

  and $(f \circ f^{-1})(y) = y$ for every y in R_f

In other words:

$$f^{-1}[f(x)] = x \quad \text{and} \quad f[f^{-1}(y)] = y$$

MOVING ON:

The previously encountered graph of the natural logarithmic function $f(x) = \ln x$ displayed in Figure 6.3(a) reveals that it is a one-to-one function, with domain $D_f = (0, \infty)$ and range $R_f = (-\infty, \infty)$. The inverse of that function [with domain $(-\infty, \infty)$ and range $(0, \infty)$] is called the **natural exponential function**, and denoted by e^x. In particular, since $\ln x$ and e^x are inverses of each other:

THE NATURAL EXPONENTIAL FUNCTION

For any real number x: $\ln e^x = x$, and, for any $x \underset{\text{why?}}{>} 0$: $e^{\ln x} = x$

Employing Theorem 1.3, page 14, we arrive at the graph of $y = e^x$ in Figure 6.3(b), and highlight it exclusively in Figure 6.3(c).

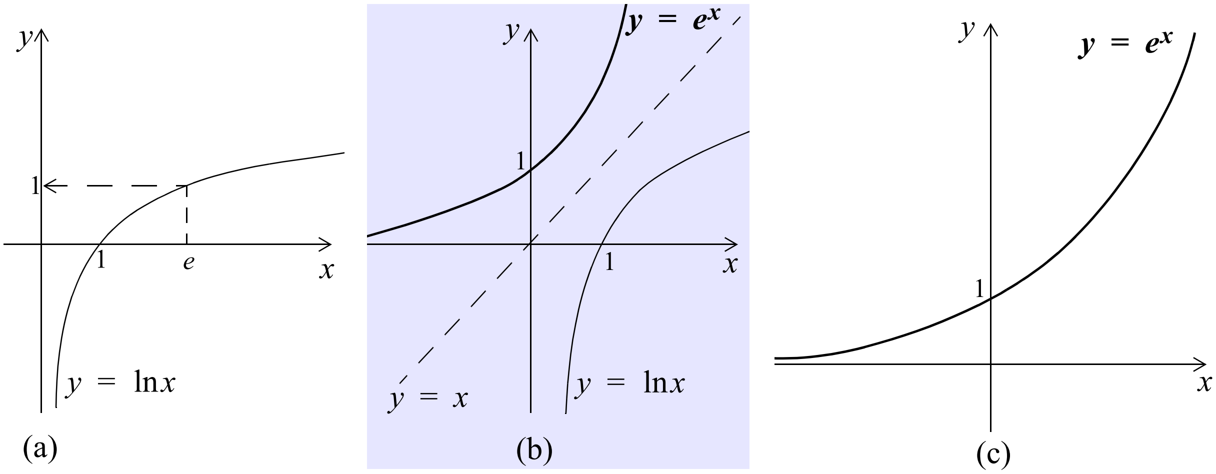


Figure 6.2

The exponential function is particularly pleasant in that both its derivative and integral are again itself:

THEOREM 6.6

$$\frac{d}{dx}e^x = e^x \quad \text{and} \quad \int e^x dx = e^x + C$$

PROOF: Accepting the fact that the exponential function is differentiable throughout its domain we can easily establish the first part of the theorem:

$$\begin{aligned}
\text{Since } e^x \text{ is the inverse of } \ln x\text{:} \quad \ln(e^x) &= x \\
[\ln(e^x)]' &= x' \\
\text{Chain Rule:} \quad \frac{1}{e^x} \cdot (e^x)' &= 1 \\
\text{Multiply both sides by } e^x\text{:} \quad (e^x)' &= e^x
\end{aligned}$$

And the second part is even easier:

$$\text{Since } e^x \text{ is an antiderivative of } e^x\text{: } \int e^x dx = e^x + C$$

EXAMPLE 6.5 Differentiate the given function.

(a) $f(x) = e^x \sin x$ (b) $g(x) = e^{x^2 \ln x}$

SOLUTION:

(a) $$(e^x \sin x)' \underset{\substack{\uparrow \\ \text{product rule}}}{=} e^x(\sin x)' + \sin x(e^x)' = e^x \cos x + \sin x \cdot e^x$$
$$= e^x(\sin x + \cos x)$$

(b) $$\frac{d}{dx}(e^{x^2 \ln x}) \overset{\substack{(e^x)' = e^x \Rightarrow [e^{g(x)}]' = e^{g(x)}g'(x) \text{ (the Chain rule)} \\ \downarrow}}{=} e^{x^2 \ln x}\frac{d}{dx}(x^2 \ln x)$$
$$= e^{x^2 \ln x}\left[x^2\frac{d}{dx}\ln x + \ln x\frac{d}{dx}(x^2)\right]$$
$$= e^{x^2 \ln x}\left(\frac{x^2}{x} + \ln x \cdot 2x\right) = e^{x^2 \ln x}(x + 2x\ln x)$$

CHECK YOUR UNDERSTANDING 6.7

Differentiate the given function.

(a) $f(x) = \dfrac{e^x}{\ln x} + 1$ (b) $g(x) = \sqrt{xe^x}$ (c) $h(x) = xe^{\sqrt{x}}$

Answer:
(a) $\dfrac{e^x(x\ln x - 1)}{x(\ln x)^2}$
(b) $\dfrac{\sqrt{e^x}(x+1)}{2\sqrt{x}}$
(c) $\frac{1}{2}e^{\sqrt{x}}(\sqrt{x} + 2)$

EXAMPLE 6.6 Perform the indicated operation.

(a) $\int xe^{x^2+1}dx$ (b) $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

Alternative substitution:
$u = e^{x^2+1}$
$du = e^{x^2+1}\cdot 2xdx$
etc.

SOLUTION: (a) $\int xe^{x^2+1}dx \underset{\substack{u = x^2+1 \\ du = 2xdx}}{=} \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+1} + C$

(b) $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx \underset{\substack{u = \sqrt{x}:\ x = 1 \Rightarrow u = 1,\ x = 4 \Rightarrow \boldsymbol{u = 2} \\ du = \frac{1}{2\sqrt{x}}dx}}{=} 2\int_1^2 e^u du = 2e^u\Big|_1^2 = 2(e^2 - e)$

CHECK YOUR UNDERSTANDING 6.8

Perform the indicated operation.

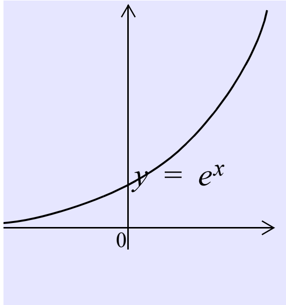
(a) $\int e^x \sin e^x dx$ (b) $\int_0^\pi \cos x e^{\sqrt{\sin x}}dx$

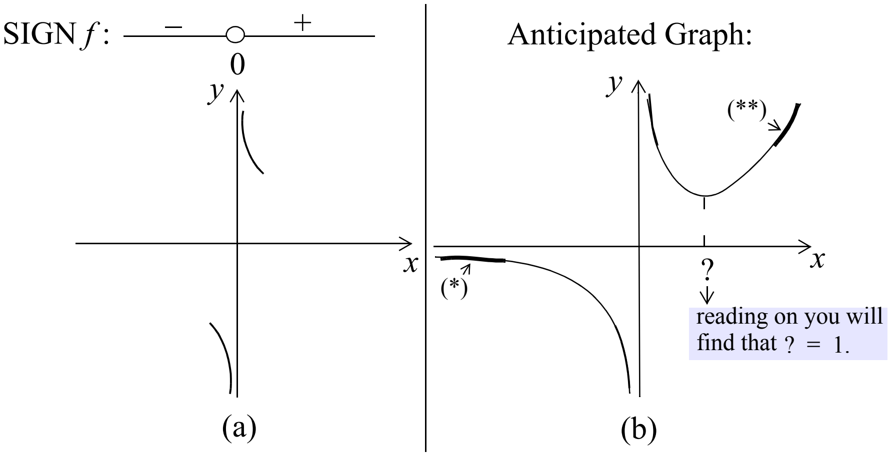
Answers:
(a) $-\cos e^x + C$ (b) 0

EXAMPLE 6.7 Sketch the graph of the function:

$$f(x) = \frac{e^x}{x}$$

SOLUTION: As is our custom, we will first attempt to get a sense of the graph of f directly. Noting that e^x is always positive (margin) and that a vertical asymptote occurs at $x = 0$ takes us to Figure 6.4(a).





reading on you will find that ? = 1.

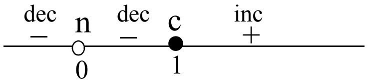
Figure 6.3

Since e^x tends to zero as $x \to -\infty$, so must $\frac{e^x}{x}$ (and even at a faster rate since the denominator is getting bigger and bigger) [see (*) of Figure 6.4(b)]. Moreover, since e^x is increasing faster than x to the right of the origin, we expect that the graph of *f* will head back up towards ∞ as $x \to \infty$ [see (**) of Figure 6.4(b)]. These observations bring us to the anticipated graph of *f* in Figure 6.2(b).

If our anticipated graph is on target, then the first derivative of *f* must be negative for $x \in (-\infty, 0)$, as well as between 0 and some positive x ("?" in Figure), at which a minimum occurs; after which the derivative will be positive. In addition, the second derivative must be negative on $(-\infty, 0)$, and positive from zero on. This is indeed the case:

$$f'(x) = \left(\frac{e^x}{x}\right)'$$

$$= \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$



SIGN $f'(x)$

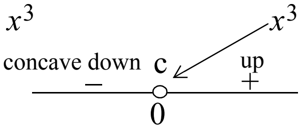
$$f''(x) = \left(\frac{xe^x - e^x}{x^2}\right)'$$

$$= \frac{x^2(\boldsymbol{xe^x} - e^x)' - (xe^x - e^x)(x^2)'}{x^4}$$

$$= \frac{x^2[(\boldsymbol{e^x + xe^x}) - e^x] - (xe^x - e^x)2x}{x^4}$$

$$= \frac{x^2e^x - 2xe^x + 2e^x}{x^3} = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

SIGN $f''(x)$:



(note that e^x and $x^2 - 2x + 2$ are always positive)

CHECK YOUR UNDERSTANDING 6.9

Sketch the graph of the function $f(x) = e^{x^2 - 1}$.

Answer: See page A-34.

As might be expected:

The general exponent formulas appear in the next section, following the definition of $f(x) = a^x$ for any $a > 0$.

THEOREM 6.7 For all real numbers a and b:

(a) $e^a e^b = e^{a+b}$

(b) $\frac{e^a}{e^b} = e^{a-b}$

(c) $(e^a)^b = e^{ab}$

(d) $e^{-x} = \frac{1}{e^x}$

You are asked to establish (b) in CYU 6.11; and (c) and (d) in the exercises.

PROOF: [Proof of (a)] We observe that $\ln(e^a e^b) = \ln(e^{a+b})$:

$$\ln(e^a e^b) \underset{\uparrow}{=} \ln e^a + \ln e^b \underset{\uparrow}{=} a + b \underset{\uparrow}{=} \ln(e^{a+b})$$

Theorem 6.4(a), page 326 e^x is the inverse of $\ln x$

Since $\ln(e^a e^b) = \ln(e^{a+b})$ and since the natural logarithmic function is one-to-one: $e^a e^b = e^{a+b}$.

CHECK YOUR UNDERSTANDING 6.10

Prove: Theorem 6.7(b).

Answer: See Page A-35

The irrational number $e \approx 2.718$ is the solution of the equation $\ln x = 1$ (see Figure 6.1, page 223). Taking another approach:

An alternate form:

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$$

(How?)

THEOREM 6.8

$$e = \lim_{x \to 0}(1 + x)^{1/x}$$

PROOF: Let $f(x) = \ln x$. Replacing "*h*" with "*x*" in the Definition 3.1, page 66, we have:

$$\begin{aligned} f'(1) = \lim_{x \to 0}\frac{f(1+x) - f(1)}{x} &= \lim_{x \to 0}\frac{\ln(1+x) - \ln 1}{x} \\ &= \lim_{x \to 0}\frac{\ln(1+x) - 0}{x} \\ &= \lim_{x \to 0}\frac{1}{x}\ln(1+x) \underset{\uparrow}{=} \lim_{x \to 0}\ln(1+x)^{1/x} \end{aligned}$$

Theorem 6.4(c), page 226

Since $f'(1) = 1$ (recall that $f'(x) = \frac{1}{x}$):

$$\lim_{x \to 0}\ln(1+x)^{1/x} = 1$$

$$e^{\lim_{x \to 0}\ln(1+x)^{1/x}} = e^1$$

Theorem 2.5, page 58: $$\lim_{x \to 0} e^{\ln(1+x)^{1/x}} = e$$

$$\lim_{x \to 0}(1+x)^{1/x} = e$$

EXPONENTIAL GROWTH AND DECAY

Certain quantities, like the number of living organisms in a population, or the mass of a radioactive substance, vary at a rate proportional to the amount present at any given time. The following result reveals the exponential nature of such quantities.

Note that at $t = 0$,

$A(t) = A_0$

THEOREM 6.9

EXPONENTIAL GROWTH/DECAY FORMULA

If the rate of change of the amount of a substance $A(t)$ is proportional to its size: $\frac{dA(t)}{dt} = kA(t)$, then the amount present t units of time before or after an established initial time $(t = 0)$ is given by the formula:

$$A(t) = A_0e^{kt}$$

where A_0 denotes the initial amount of the substance.

PROOF: From the given information $A'(t) = kA(t)$ we have:

$$\frac{A'(t)}{A(t)} = k \text{ or } [\ln A(t)]' = (kt)'$$

Recalling that if two functions have the same derivative then they can only differ by a constant (CYU 4.3, page 124), we conclude that:

$$\ln A(t) = kt + c$$

Applying the exponential function gives us:

$$e^{\ln A(t)} = e^{kt + c}$$

$e^{\ln x} = x$: $$A(t) = e^{kt + c}$$

Theorem 6.7(a): $$A(t) = e^c e^{kt} \quad (*)$$

Evaluating (*) at $t = 0$, we have:

$$A(0) = e^c e^0 \Rightarrow A(0) = e^c$$

Replacing $e^c = A(0)$ in (*) with the more compact symbol A_0 to denote the initial amount $A(0)$, we arrive at the formula $A(t) = A_0e^{kt}$.

RADIOACTIVE DECAY

By emitting alpha and beta particles and gamma rays, the radioactive mass of a substance decreases at a rate proportional to the amount present. By definition, the **half-life** of a radioactive substance is the time required for half of its original mass to decay.

Organic substances contain both carbon-14 and non-radioactive carbon in known proportions. A living organism absorbs no more carbon when it dies. The carbon-14 decays, thus changing the proportions of the two kinds of carbon in the organism. By comparing the present proportion of carbon-14 with the assumed original proportion, one can determine how much of the original carbon-14 is present, and therefore how long the organism has been dead; hence how old it is. The next example illustrates this method, called **carbon-14 dating**.

EXAMPLE 6.8 A skeleton is found to contain one-sixth of its original amount of carbon-14. How old is the skeleton, given that carbon-14 has a half-life of 5730 years?

It is the value of k that distinguishes one exponential growth or decay situation from another. Generally, the first task in solving an exponential growth or decay problem is to determine the value of e^k for the situation at hand.

SOLUTION: Let $t = 0$ denote the time of demise and $A(t)$ the amount of carbon-14 present t years later, then:

$$A(t) = A_0 e^{kt}$$

$A(t) = \frac{A_0}{2}$ when $t = 5730$:

$$\frac{1}{2}A_0 = A_0 e^{5730k}$$

divide both sides by A_0:

$$\frac{1}{2} = e^{5730k}$$

$$\ln\left(\frac{1}{2}\right) = 5730k$$

This brings us to the exponential decay formula for carbon-14:

$$A(t) = A_0 e^{\left[\frac{\ln(1/2)}{5730}\right]t} \qquad (*)$$

We are told that the skeleton contains one-sixth of its original amount of carbon-14, that is: $A(t) = \frac{A_0}{6}$. To find the skeleton's age, we substitute $\frac{A_0}{6}$ for $A(t)$ in (*), and solve for t:

$$\frac{A_0}{6} = A_0 e^{\left[\frac{\ln(1/2)}{5730}\right]t}$$

Divide both sides by A_0:

$$e^{\left[\frac{\ln(1/2)}{5730}\right]t} = \frac{1}{6}$$

Apply ln to both sides:

$$\left[\frac{\ln(1/2)}{5730}\right]t = \ln\frac{1}{6}$$

$$t = \frac{5730\ln(1/6)}{\ln(1/2)} \approx 14{,}812$$

We conclude that the skeleton is approximately 14,812 years old.

CHECK YOUR UNDERSTANDING 6.11

A certain radioactive substance loses $\frac{1}{3}$ of its original mass in four days. How long will it take for the substance to decay to $\frac{1}{10}$ of its original mass?

Answer: Approximately 22.7 days.

POPULATION GROWTH

In an ideal environment, the rate of change of a population of living organisms (humans, rabbits, bacteria, etc.) increases at a rate proportional to the amount present. By definition, the **doubling time** of the organism is the time required for a population to double.

The number of individuals in a population can only take on integer values. A sufficiently large population, however, can safely be described by a continuous function.

EXAMPLE 6.9 The doubling time of E.coli bacteria is 20 minutes. If a culture of the bacteria contains one million cells, determine how long it will take before the culture increases to 9 million cells?

SOLUTION: We turn to the formula of Theorem 6.8, where $A(t)$ now denotes the number of cells (in millions) present at time t (in minutes):

From Theorem 6.9: $A(t) = A_0e^{kt}$

Since $A(t) = 2A_0$ when $t = 20$: $2A_0 = A_0e^{20k}$

$$\begin{aligned} 2 &= e^{20k} \\ \ln 2 &= 20k \\ k &= \frac{\ln 2}{20} \end{aligned}$$

We now have the exponential growth formula for E.coli bacteria:

$$A(t) = A_0e^{\left[\frac{\ln 2}{20}\right]t}$$

In particular:

Since there are 1 (million) initially, $A_0 = 1$: $A(t) = 1 \cdot e^{\left[\frac{\ln 2}{20}\right]t}$

To determine how long it will take for there to be 9 million cells, set $A(t) = 9$ and solve for t: $9 = e^{\left[\frac{\ln 2}{20}\right]t}$

$$\begin{aligned} \ln 9 &= \left[\frac{\ln 2}{20}\right]t \\ t &= \frac{20\ln 9}{\ln 2} \approx 63 \end{aligned}$$

We conclude that it will take approximately 63 minutes for the culture to increase from 1 million cells to 9 million cells. In other words, in an ideal situation, E.coli bacteria will increase by a factor of 9 approximately every 63 minutes.

CHECK YOUR UNDERSTANDING 6.12

The population of a town grows at a rate proportional to its population. The initial population of 500 increased by 15% in 9 years. How long will it take for the population to triple?

Answer: ≈ 70.75 years

EXERCISES

Exercise 1-22. (First Derivative) Differentiate.

1. $f(x) = e^{2x}$
2. $g(x) = e^{x^2}$
3. $f(x) = x^3 e^x$
4. $g(x) = x^2 e^x$
5. $g(x) = e^{2x} - 2e^x$
6. $f(x) = x^2 e^{x^2}$
7. $g(x) = \dfrac{e^{2x}}{2x}$
8. $f(x) = \dfrac{x^3 + 2}{e^x - 1}$
9. $g(x) = (x + e^x)^5$
10. $g(x) = (x^2 e^{2x})^5$
11. $f(x) = (x^2 + e^2)^5$
12. $h(t) = (2te^{2t} + t^2)^5 + 500$
13. $g(x) = e^{\sin x}$
14. $f(x) = \sin e^x$
15. $f(x) = e^x \sin x^2$
16. $\cos(\sin e^x)$
17. $g(x) = \dfrac{\tan x^2}{e^{x^2}}$
18. $f(x) = \dfrac{\ln(\sin x)}{\cos e^x}$
19. $f(x) = \ln(x^2 + e^{x^2})$
20. $f(x) = \sqrt{e^x \ln x}$
21. $f(x) = e^{e^x}$
22. $f(x) = e^x e^x$

Exercise 23-24. (Implicit Differentiation) Determine $\dfrac{dy}{dx}$.

23. $xe^y + \ln y - x^2 = 1$
24. $\ln(e^y + 1) - xy = x$

Exercise 25-30. (Second Derivative) Determine $\dfrac{d^2y}{dx^2}$.

25. $y = xe^x$
26. $y = e^{\sin x}$
27. $y = \ln x e^{x^2}$
28. $y = \sin e^x$
29. $y = e^{2x} \cos 3x$
30. $y = \int_1^x e^{\sqrt{t^3 + 1}}\, dt$

Exercise 31-32. (Composite Functions) Determine the derivative of the composite function.

(a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$ (c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$ if:

31. $f(x) = 3x^2 + x$ and $g(x) = e^x$
32. $f(x) = e^{x^2}$ and $g(x) = e^x$

Exercise 33-36. (Tangent Line) Determine the tangent line to the graph of the given function at the indicated point.

33. $y = e^{x^2}$ at $x = 2$
34. $y = xe^x + \ln x$ at $x = 1$
35. $y = e^x \sin x$ at $x = \dfrac{\pi}{2}$
36. (Implicit Differentiation) $xe^y + ye^x - xy = 3$ at $(0, 3)$

37. **(Point of Tangency)** Find the points on the graph of the function $f(x) = e^{x^2}$ where the slope of the tangent line to the graph equals the function value.

38. **(Point of Tangency)** Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

Exercise 39-50. (Integration) Evaluate.

39. $\int x^2 e^{x^3}\, dx$

40. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

41. $\int \frac{e^{1/x}}{x^2} dx$

42. $\int \frac{e^x}{e^x - 1} dx$

43. $\int \frac{e^x + 1}{e^x} dx$

44. $\int \frac{e^{-1/x^2}}{x^3} dx$

45. $\int e^x \cos e^x dx$

46. $\int e^x \sin e^x dx$

47. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

48. $\int_0^{\ln 2} e^{3x} dx$

49. $\int_0^{\frac{\pi}{4}} e^{\sin x} \cos x dx$

50. $\int_0^{\sqrt{\ln 2}} x e^{x^2} dx$

Exercise 51-58. (Differential Equation) Solve.

51. $f'(x) = xe^{x^2}$ if $f(0) = 2$.

52. $f'(x) = e^{\tan x}\sec^2 x$ if $f\left(\frac{\pi}{4}\right) = 0$

53. $f''(x) = e^{2x}$ if $f(0) = f'(0) = 1$.

54. $f'''(x) = e^x$ if $f(1) = 1$, $f'(1) = 2$ and $f''(1) = 2$

55. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the equation $y'' + 2y' + y = 0$ for all real numbers A and B.

56. For what values of a does the function e^{ax} satisfy the equation $y'' + 6y' + 8y = 0$?

57. For what values of a does the function e^{ax} satisfy the equation $y'' - 5y' + 6y = 0$?

58. For what values of a does the function e^{ax} satisfy the equation $y'' - y' - y = 0$?

Exercise 59-62. (Graphing) Sketch the graph of the given function.

59. $f(x) = xe^x$

60. $f(x) = \frac{e^x}{x}$

61. $f(x) = \frac{\ln x}{x}$

62. $f(x) = x^2 e^x$

63. **(Area)** Determine the area A of the region bounded above by the graph of the function $y = e^{2x}$, below by the graph of $y = e^{-2x}$, and on the sides by the vertical lines $x = 0$ and $x = \ln 2$.

64. **(Area)** Find the positive number a such that the area lying below the graph of the function $y = e^x$ and above the x-axis over the interval $[-a, 0]$ is equal to that over the interval $[0, 1]$.

65. **(Volume)** Find the volume obtained by rotation about the x-axis the region in the first quadrant that lies below the line $y = e$ and above the graph of the function $y = e^x$.

66. **(Related Rate)** The vertices of a rectangle are at $(0, 0)$, $(0, e^x)$, $(x, 0)$, and (x, e^x). If x is increasing at a rate of 1 unit per second, at what rate is the:

(a) Area increasing when $x = 5$? (b) Perimeter increasing when $x = 5$?

67. **(Optimization)** Show that the rectangle of greatest area bounded below by the x-axis and above by the graph of the function $f(x) = e^{-x^2}$, has two of its vertices at the inflection points of that graph.

Exercise 68-69. (Continuous Compound Interest) If an amount A_0 is invested at an annual interest rate of r, and the interest is compounded continuously, then the amount $A(t)$ accumulated after t years is given by:

$$A(t) = A_0 e^{rt}$$

68. Determine the annual interest rate r required for capital to double in 10 years, when interest is compounded continuously.

69. How much should be invested at an annual rate of 4% compounded continuously in order to have a total of $10,000 at the end of 5 years?

70. **(Radioactive Substance)** A certain radioactive substance loses 20% of its original mass in two days. How long will it take for the substance to decay to 90% of its original mass?

71. **(Population)** The world population was 5.28 billion in 1990, and 6.37 billion in 2004. Assuming that, at any given time, the population increases at a rate proportional to the population at that time, determine:

    (a) The population in the year 2010.

    (b) The year in which the population will reach 9 billion.

72. **(Dead Sea Scrolls)** Approximately 20% of the original carbon-14 remains in the Dead Sea Scrolls. How old are they? (See Example 6.9)

73. **(Theory)** Prove Theorem 6.7(c).

74. **(Theory)** Prove Theorem 6.7(d).

75. **(Theory)** Prove that if $f'(x) = f(x)$ for all x in $(-\infty, \infty)$ then $f(x) = ce^x$ for some constant c. Suggestion: consider the derivative of the function $g(x) = e^{-x}f(x)$.

76. **(Theory)** (a) Find a formula for the n^{th} derivative of $f(x) = e^{cx}$, for $c > 0$.

    (b) Use the Principle of Mathematical Induction to establish your answer in (a).

§3. a^x AND $\log_a x$

While e^x is meaningful for any number x, the same cannot, as yet, be said for an expression of the form 2^x — but only "as yet:"

Note that the expression $e^{x\ln a}$ to the right of the equal sign is meaningful for any positive a and any x whatsoever. We also know that $e^{x\ln a} = (e^{\ln a})^x$. and that $e^{\ln a} = a$. In a way, then, this definition is kind of forced on us.

DEFINITION 6.2 For any $a > 0$ and any x:

$$a^x = e^{x\ln a}$$

For example: $2^{\pi} = e^{\pi\ln 2}$ — a well-defined expression.

In the exercises you are invited to show that the following familiar laws of exponents hold in the current general setting:

THEOREM 6.10 For $a > 0$ and any x and y:

(a) $a^x a^y = a^{x+y}$

(b) $\dfrac{a^x}{a^y} = a^{x-y}$

(c) $(a^x)^y = a^{xy}$

(d) $a^{-x} = \dfrac{1}{a^x}$

Way back on page 78 [Theorem 3.2(b)] we noted that for any real number r, $(x^r)' = rx^{r-1}$, but have only established that result for integer exponents (Example 3.10, page 84, and CYU 3.10). We now come to the end of the power-rule journey:

THEOREM 6.11 For any $x > 0$ and any real number r:

$$(x^r)' = rx^{r-1}$$

PROOF:

$$(\boldsymbol{x^r})' = (e^{r\ln x})' \underset{\substack{\uparrow \\ (e^{\square})' = e^{\square}\cdot(\square)'}}{=} e^{r\ln x}\cdot(r\ln x)' = \underbrace{e^{r\ln x}\cdot\frac{r}{x} = x^r}\cdot\frac{r}{x} = \boldsymbol{rx^{r-1}}$$

For example: $\left(x^{\frac{2}{\sqrt{5}}}\right)' = \dfrac{2}{\sqrt{5}}x^{\frac{2}{\sqrt{5}}-1}$ (for $x > 0$),

and $\dfrac{d}{dx}(\sin x)^{\pi+1} = (\pi+1)(\sin x)^{\pi}$ (for $\sin x > 0$).

CHECK YOUR UNDERSTANDING 6.13

Determine $\dfrac{d^2y}{dx^2}$ for $y = x^{e+1}$.

Answer: $(e^2 + e)x^{e-1}$

We now know that $(x^a)' = ax^{a-1}$ and that $(e^x)' = e^x$, but what is the derivative of a^x? This:

THEOREM 6.12 For $a > 0$, $(a^x)' = a^x \ln a$.

PROOF:

$$(a^x)' \underset{\uparrow}{=} (e^{x\ln a})' \overset{\substack{\text{chain rule}\\ \downarrow}}{=} e^{x\ln a}\cdot(x\ln a)' = e^{x\ln a}\cdot\ln a \underset{\uparrow}{=} a^x\ln a$$

Definition 6.2

EXAMPLE 6.10 Differentiate:

(a) $y = 4^{\sin x}$ (b) $f(x) = x^x$ for $x > 0$.

SOLUTION: (a) $$\frac{d}{dx}4^{\sin x} \underset{\uparrow}{=} 4^{\sin x}\ln 4\cdot\frac{d}{dx}\sin x = \ln 4\cdot 4^{\sin x}\cdot\cos x$$

Theorem 6.11 and the Chain Rule

(b) $$\begin{aligned}(x^x)' &\underset{\uparrow}{=} (e^{x\ln x})' = e^{x\ln x}(x\ln x)' = e^{x\ln x}[\ln x\cdot x' + x(\ln x)'] \\ &= e^{x\ln x}(\ln x + 1) \\ &\underset{\rightarrow}{=} x^x(\ln x + 1)\end{aligned}$$

Definition 6.2

CHECK YOUR UNDERSTANDING 6.14

Differentiate $f(x) = x^{\sin x}$ (for $x > 0$).

Answer:
$x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$

The integral formula for a^x is just a tad more complicated than that for e^x:

THEOREM 6.13 For any $a > 0$ distinct from 1:

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Why must "1" be eliminated?

PROOF: We simply show that $\frac{a^x}{\ln a}$ is an antiderivative of a^x:

$$\left(\frac{a^x}{\ln a}\right)' = \frac{1}{\ln a}(a^x)' = \frac{1}{\ln a}(a^x\ln a) = a^x$$

EXAMPLE 6.11 Perform the indicated operation:

(a) $\int \cos x 9^{\sin x} dx$ (b) $\int_1^2 x3^{x^2}\cdot d\cdot x$

SOLUTION: (a) $\int \cos x 9^{\sin x} dx \underset{\substack{u = \sin x \\ du = \cos x dx}}{\uparrow}= \int 9^u du = \frac{9^u}{\ln 9} + C = \frac{9^{\sin x}}{\ln 9} + C$

(b) $\int_1^2 x 3^{x^2} dx \underset{\substack{u = x^2 \text{ at } x = 1, u = 1 \text{ and at } x = 2, u = 4 \\ du = 2x dx}}{\uparrow}= \frac{1}{2}\int_1^4 3^u du = \frac{1}{2}\left(\frac{3^u}{\ln 3}\Bigg|_1^4\right) = \frac{1}{2\ln 3}(3^4 - 3) = \frac{39}{\ln 3}$

CHECK YOUR UNDERSTANDING 6.15

Perform the indicated operation:

(a) $\int \frac{2^{\sqrt{x}}}{\sqrt{x}}$ (b) $\int_1^e \frac{5^{\ln x}}{x} dx$

Answers: (a) $\frac{2^{\sqrt{x}+1}}{\ln 2} + C$
(b) $\frac{4}{\ln 5}$

THE FUNCTION $\log_a x$

The one-to-one property of $\ln x$ led us to the definition of e^x. Reversing the process will now bring us to the definition of $\log_a x$. We first observe that:

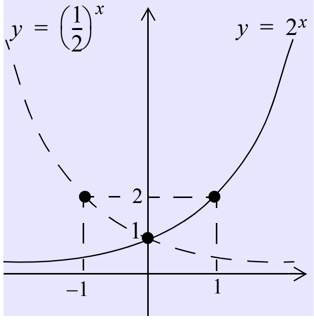
THEOREM 6.14 For any positive number a distinct from 1, the function $f(x) = a^x$ is one-to-one.

PROOF: Consider:

$$(a^x)' = (e^{x\ln a})' = e^{x\ln a} \cdot \ln a$$

Since the natural exponential function only takes on positive values, $e^{x\ln a} > 0$ for all x. Noting that $\ln a < 0$ if $0 < a < 1$ and that $\ln a > 0$ if $a > 1$ (see Figure 6.1, page 223), we conclude that $(a^x)' < 0$ if $0 < a < 1$ and that $(a^x)' > 0$ if $a > 1$. To put it another way:

$$a^x \text{ is } \begin{cases} \text{a decreasing function if } 0 < a < 1 \\ \text{an increasing function if } a > 1 \end{cases} \text{(see margin)}$$

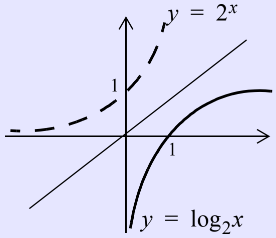


In either case, $f(x) = a^x$ is one-to-one.

Bringing us to:

DEFINITION 6.3 For any positive number $a \neq 1$: $\log_a x$, read "log base a of x," is the inverse of the function a^x.

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ about the line $y = x$. In particular:



Note that since the exponential function a^x has domain $(-\infty, \infty)$ and range $(0, \infty)$, its inverse, the logarithmic function $\log_a x$, has domain $(0, \infty)$ and range $(-\infty, \infty)$.

NOTE: The logarithmic function $\log_e x$ is our friend: the natural logarithmic function $\ln x$.
The logarithmic function $\log_{10} x$ also has its own notation and name. It is called the **common logarithmic function**, and is denoted by $\log x$.

Since $\log_a x$ and a^x are inverses of each other, we have:

$$a^{\log_a x} = x \quad (\text{for } x > 0)$$
$$\log_a a^x = x \quad (\text{for all } x)$$

In addition:

THEOREM 6.15 For any positive numbers x and y and any positive number $a \neq 1$:
(a) $\log_a xy = \log_a x + \log_a y$
(b) $\log_a \frac{x}{y} = \log_a x - \log_a y$
(c) $\log_a x^r = r\log_a x$

PROOF: We establish (c), and invite you to verify (a) and (b) below.

$$\text{Since } a^{\log_a x^r} = x^r \text{ and } a^{r\log_a x} \overset{\text{Theorem 6.9(c)}}{=} (a^{\log_a x})^r = x^r, \text{ and}$$
since the function a^x is one-to-one: $\log_a x^r = r\log_a x$.

CHECK YOUR UNDERSTANDING 6.16

Prove Theorem 6.15(a) and (b).

Answer: See page A-36.

As previously noted:

$$(e^x)' = e^x \text{ and, for any } a > 0\text{: } (a^x)' = a^x \ln a.$$

There is also little difference between the derivative formula $(\ln x)' = \frac{1}{x}$ and that for the general logarithmic function. Specifically:

THEOREM 6.16 For any positive number $a \neq 1$:

$$(\log_a x)' = \frac{1}{x \ln a}$$

PROOF: Accepting the fact that the logarithmic function is differentiable throughout its domain, we have:

$$\begin{aligned} a^{\log_a x} &= x \\ (a^{\log_a x})' &= x' \\ (a^{\square})' = \ln a \cdot a^{\square}(\square)':\quad \ln a \cdot a^{\log_a x}(\log_a x)' &= 1 \\ \ln a \cdot x(\log_a x)' &= 1 \\ (\log_a x)' &= \frac{1}{\ln a}\cdot\frac{1}{x} \end{aligned}$$

Compare with Example 6.1(a), page 224.

EXAMPLE 6.12 Differentiate

$$f(x) = \log_2(\sin x)$$

SOLUTION:

$$[\log_2(\sin x)]' \underset{\uparrow}{=} \frac{1}{\ln 2\cdot\sin x}(\sin x)' = \frac{1}{\ln 2\cdot\sin x}\cdot\cos x = \frac{\cot x}{\ln 2}$$

the chain rule: $(\log_2\square)' = \frac{1}{\ln_2\square}\cdot\square'$

Answers: (a) $\frac{2}{x\ln 3}$

(b) $\frac{\cos(\log_5 x)}{2x\ln 5\sqrt{\sin(\log_5 x)}}$

CHECK YOUR UNDERSTANDING 6.17

Differentiate the given function:

(a) $f(x) = \log_3 x^2$ (b) $g(x) = \sqrt{\sin(\log_5 x)}$

EXERCISES

Exercise 1-20. (First Derivative) Differentiate.

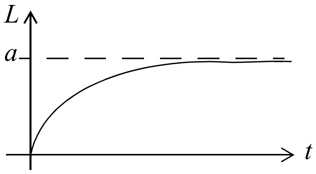
1. $f(x) = 5^{2x}$
2. $g(x) = 2^{x^2}$
3. $f(x) = x^3 3^x$
4. $f(x) = x^2 2^{x^2}$
5. $g(x) = \dfrac{2^{2x}}{2x}$
6. $f(x) = \dfrac{x^3+2}{5^x-1}$
7. $g(x) = 5^{\sin x}$
8. $f(x) = \sin 5^x$
9. $f(x) = 5^x \sin x^2$
10. $h(x) = \cos(\sin 5^x)$
11. $g(x) = \dfrac{\ln x}{3^x}$
12. $f(x) = \dfrac{\log_2 x}{3^x}$
13. $f(x) = 2^x \log_2 x$
14. $f(x) = \sqrt{2^x \ln x}$
15. $f(x) = \ln(\log_2 x)$
16. $f(x) = \log_2(\ln x)$
17. $f(x) = \log_2(\log_2 x)$
18. $f(x) = 3^{\log_2 3^2}$
19. $f(x) = (3x)^x$
20. $f(x) = x^{\cos x}$

Exercise 21-26. (Integration) Evaluate.

21. $\int x 5^{x^2}\, dx$
22. $\int \dfrac{2^{\sqrt{x}}}{\sqrt{x}}\, dx$
23. $\int \dfrac{5^{1/x}}{x^2} dx$
24. $\int_1^4 \dfrac{4^{\ln x}}{x} dx$
25. $\int_1^{\sqrt{2}} x 2^{x^2} dx$
26. $\int_0^{\pi/2} 3^{\cos x} \sin x\, dx$

27. **(Half Life)** Prove that in an exponential decay situation, if the half-life of a substance is H, then the amount of substance present at time t is given by $A(t) = A_0 \cdot 2^{-t/H}$ where A_0 denotes the initial amount present.

28. **(Doubling Time)** Prove that in an exponential growth situation, if the doubling time of a substance is D, then the amount of substance present at time t is given by $A(t) = A_0 \cdot 2^{t/D}$ where A_0 denotes the initial amount of the substance.



Exercise 29-30. (Learning Curve) Learning curves are graphs of exponential functions of the form $L(t) = a(1 - b^t)$ where $a > 0$ and $0 < b < 1$. As you can see from the adjacent figure, while initially rapid, the learning process levels off with time.

29. Practicing one hour a day, it took Bill 9 days to learn to type 30 words per minute. How many days of practice will he need in order to get his speed up to 60 words per minute, assuming that an average experienced typist can type 73 words per minute?

30. (a) Find the learning curve formula for Mary's riveting abilities if it took her 5 days before she could do 27 rivets per hour, given that the average experienced riveter can do 43 rivets per hour.

    (b) In how many more days will she be able to do 30 rivets per hour?

    (c) How long will it take before she can be expected to do 40 rivets per hour?

31. **(Theory: Change of Base Formula)** Prove that for any $a > 0, b > 0$: $\log_a x = \dfrac{\log_b x}{\log_b a}$

32. **(Theory)** Prove: (a) Theorem 6.9(a) (b) Theorem 6.9(b)
(c) Theorem 6.9(c) (d) Theorem 6.9(d)

COMMON LOGARITHMS

Common logarithms are logarithms to the base 10 and are typically denoted by $\log x$ rather than by $\log_{10} x$. These logarithms appear in many scientific formulas, a few of which are featured below.

Exercise 33-35. (Sound Intensity) The intensity level L (in bels) of sound is defined in terms of the common logarithm (base 10) of the intensity I (i.e. energy density) of a sound-wave when it hits your eardrums. It is measured in bels: $L = \log\frac{I}{I_0}$ where I is measured in Watts per square meter, and I_0 is the constant intensity of 10^{-12} Watts per square meter (roughly the intensity of the faintest audible sound). In this logarithmic scale, when the energy of a sound is $I = 10I_0$, its intensity level, L, is 1 bel. When $I = 100I_0$, its intensity level is 2 bels, and so on. Every time the energy density increases by a factor of 10, the intensity level increases by one bel.

Because the bel is a large unit, it is customary to express the intensity level in decibels [db], where $10 \text{ db} = 1 \text{ bel}$. To summarize:

The **intensity level** of sound, in decibels, is given by:

$$L = 10\log\frac{I}{I_0}$$

where I is the sound intensity in Watts per square meter, and $I_0 = 10^{-12}\frac{\text{Watts}}{m^2}$.

33. Find the intensity of the given sound.

    (a) Heavy city traffic at 90 db. (b) Dripping faucet at 30 db. (c) Rustle of leaves at 10 db.

34. What is the difference in the intensity level of two sounds, if the intensity of one sound is 70 times that of the other?

35. It is known that the sound intensity due to independent sources is the sum of the individual intensities. Given that the intensity level of the average whisper is 20 db, how many students would have to be whispering simultaneously in order to produce an intensity level of 60 db, which approximates the intensity level of ordinary conversation?

Exercise 36-38. (Richter Scale) Like the intensity level L of sound, the intensity of an earthquake, as measured by the Richter magnitude scale, is also defined in terms of the common logarithm: The magnitude R (on the Richter scale) of an earthquake of intensity I is given by:

$$R = \log\frac{I}{I_0}$$

where I_0 is a "minimum" intensity used for comparison.

36. The 1985 earthquake in Mexico City measured 8.1 on the Richter scale, while the 1989 California earthquake measured 7.0. How much more intense was the Mexico City earthquake?

37. On August 16, 1999, an earthquake measuring 7.4 on the Richter scale struck Turkey. The following day, an earthquake measuring 5.0 occurred in California. How much more intense was the earthquake in Turkey?

38. If an earthquake has an intensity which is 300 times the intensity of a smaller earthquake, how much larger would its Richter scale measurement be?

Exercise 39-40. (Chemistry-pH) The pH (hydrogen potential) of a solution is given by $pH = -\log[H^+]$, where $[H^+]$ is the hydrogen ion concentration in moles per liter. The pH values vary from 0 (very acidic) to 14 (very basic, alkaline). Pure water has a pH of 7.0 and is neutral, neither acidic nor alkaline.

39. Find the pH value of sea water, given that $[H^+] = 6.31 \times 10^{-9}$.

40. (a) Find the $[H^+]$ value of lemon juice, given that its pH value is 2.3.

    (b) Find the $[H^+]$ value of milk, given that its pH value is 6.6.

    (c) How much greater is the hydrogen ion concentration of lemon juice than that of milk?

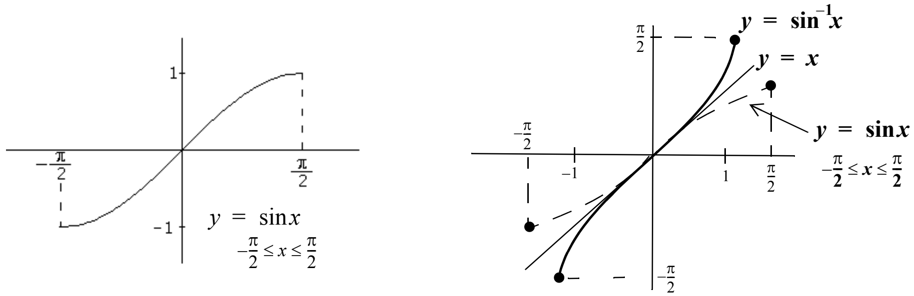
§4. INVERSE TRIGONOMETRIC FUNCTIONS

Since trigonometric functions are not one-to-one, they do not have inverses (see page 12). We can, however, restrict the domain of each trigonometric function to an interval on which it is one-to-one, and then consider the inverse of the resulting **restricted** function.

SPECIFICALLY:

Restricting the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ produces a one-to-one function [see Figure 6.4(a)]. The inverse of that restricted function is called the **inverse sine** function (or arc-sine function), and is denoted by $\sin^{-1}x$ (or **arcsin *x***). Reflecting the graph of the restricted sine function about the line $y = x$ yields the graph of the inverse sine function in Figure 6.4(b) (see Theorem 1.3, page 14).

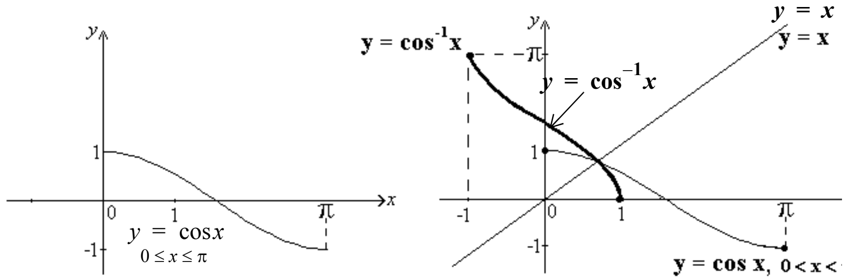
In spite of its notation and name, $\sin^{-1}x$ is **not** the inverse of the sine function. It can't be, since the sine function, not being one-to-one, has no inverse; it is the inverse of the sine function **restricted** to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



(a) (b)

Figure 6.4

Restricting the cosine function to the interval $[0, \pi]$ produces a one-to-one function [see Figure 6.5(a)]. The inverse of that restricted function is called the **inverse cosine** function (or **arc-cosine** function), and is denoted by $\cos^{-1}x$ (or **arccos *x***). Reflecting the graph of the restricted cosine function about the line $y = x$ yields the graph of the inverse cosine function in Figure 6.5(b).



(a) (b)

Figure 6.5

Restricting the tangent function to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ produces a one-to-one function [see Figure 6.6(a)]. The inverse of that restricted function is called the **inverse tangent** function (or **arc-tangent**), and is denoted by $\tan^{-1}x$ (or **arctan *x***). Reflecting the graph of the restricted tangent function about the line $y = x$ yields the graph of the inverse tangent function in Figure 6.6(b).

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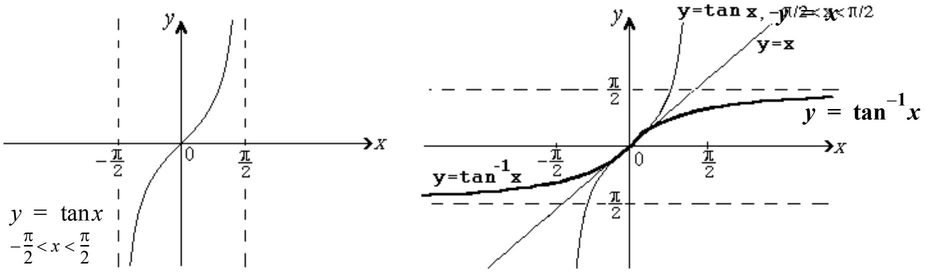


(a) (b)

Figure 6.6

In summary:

DEFINITION 6.4

INVERSE SINE

(a) The **inverse sine** function has domain $[-1, 1]$ and is given by:

$$y = \sin^{-1} x \text{ if } \sin y = x \text{ with } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

INVERSE COSINE

(b) The **inverse cosine** function has domain $[-1, 1]$ and is given by:

$$y = \cos^{-1} x \text{ if } \cos y = x \text{ with } 0 \le y \le \pi.$$

INVERSE TANGENT

(c) The **inverse tangent** function has domain $(-\infty, \infty)$ and is given by:

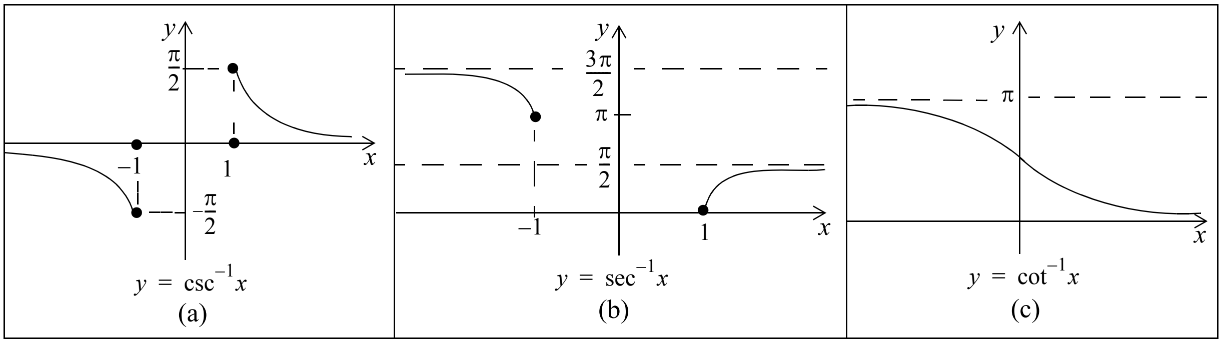
$$y = \tan^{-1} x \text{ if } \tan y = x \text{ with } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The remaining three inverse trigonometric functions are similarly defined:

The **inverse cosecant** function has domain $(-\infty, -1] \cup [1, \infty)$ and is given by: $y = \csc^{-1} x$ if $\csc y = x$ with $-\frac{\pi}{2} \le y < 0$ or $0 < y \le \frac{\pi}{2}$ [see Figure 6.7(a)].

The **inverse secant** function has domain $(-\infty, -1] \cup [1, \infty)$ and is given by: $y = \sec^{-1} x$ if $\sec y = x$ with $0 \le y < \frac{\pi}{2}$ or $\pi \le y < \frac{3\pi}{2}$ [see Figure 6.7(b)].

The **inverse cotangent** function has domain $(-\infty, \infty)$ and is given by: $y = \cot^{-1} x$ if $\cot y = x$ with $0 < y < \pi$ [see Figure 6.7(c)].



$y = \csc^{-1} x$ (a)

$y = \sec^{-1} x$ (b)

$y = \cot^{-1} x$ (c)

Figure 6.7

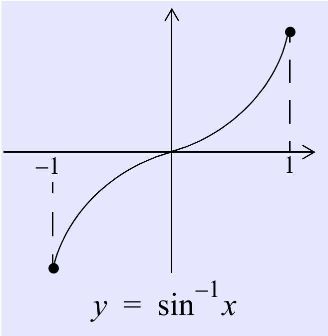
THEOREM 6.17

(a) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ (b) $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$

(c) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ (d) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

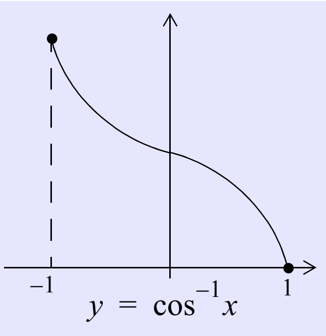
(e) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ (f) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

We will establish (a) and invite you to verify the rest on your own. First, however, we want to point out how the derivative formulas of (a), (c), and (e) are in total harmony with the graphs of their corresponding inverse trigonometric functions of Figure 6.4(b), Figure 6.5(b), and Figure 6.6(b), respectively:



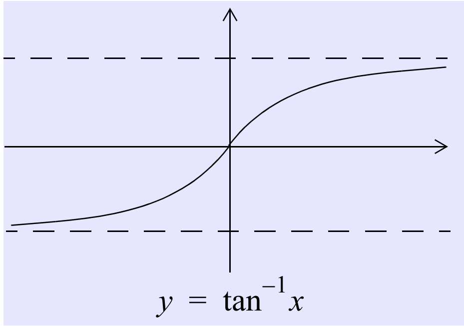
$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$ **:**

Derivative is positive in $(-1, 1)$ and the graph does increase over that interval. The derivative does not exist at ±1 where the tangent lines to the graph appear to be vertical.



$(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$ **:**

Derivative is negative in $(-1, 1)$ and the graph does decrease over that interval. The derivative does not exist at ±1 where the tangent lines to the graph appear to be vertical.



$(\tan^{-1}x)' = \frac{1}{1+x^2}$ **:**

Derivative is positive everywhere and the graph does increase throughout the domain $(-\infty, \infty)$.

Note that $\frac{1}{1+x^2} \to 0$ as $x \to \pm\infty$, reflecting the fact that the slope of the tangent lines tend to zero as $x \to \pm\infty$

PROOF OF THEOREM 6.17(a): Accepting the fact that the inverse sine function is differentiable throughout the interval $(-1, 1)$ (Theorem 3.10, page 97), we start with the identity $\sin(\sin^{-1}x) = x$ [see Definition 6.4(a)], and differentiate both sides:

$$[\sin(\sin^{-1}x)]' = x'$$

$[\sin\square]' \underset{\uparrow}{=} \cos\square\cdot\square'$: chain rule

$$\cos(\sin^{-1}x)\cdot(\sin^{-1}x)' = 1$$

$$(\sin^{-1}x)' = \frac{1}{\cos(\sin^{-1}x)} \quad (*)$$

Now comes a tricky-trig maneuver:

Since $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$, $\cos(\sin^{-1}x) \ge 0$. It follows that:
$\sqrt{\cos^2(\sin^{-1}x)} = \cos(\sin^{-1}x)$

Theorem 1.5(a), page 37: $\cos^2(\sin^{-1}x) + \sin^2(\sin^{-1}x) = 1$

$$\cos^2(\sin^{-1}x) = 1 - [\sin(\sin^{-1}x)]^2$$

$$\sqrt{\cos^2(\sin^{-1}x)} = \sqrt{1 - [\sin(\sin^{-1}x)]^2}$$

(see margin): $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$

Returning to (*): $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}$

CHECK YOUR UNDERSTANDING 6.18

Verify that:

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

Answer: See page A-36

EXAMPLE 6.13 Determine the domain of the given function and find its derivative.

(a) $f(x) = \sin^{-1}(x^2 - 1)$ (b) $g(x) = \tan^{-1}e^{2x}$

SOLUTION: (a) Definition 6.4(a) tells us that x is in the domain of $f(x) = \sin^{-1}(x^2-1)$ if and only if $-1 \le x^2 - 1 \le 1$, which is to say:

$$\begin{array}{ccc} -1 \le x^2 - 1 & \text{and} & x^2 - 1 \le 1 \\ x^2 \ge 0 & & x^2 \le 2 \\ (-\infty, \infty) & & \sqrt{x^2} \le \sqrt{2} \\ & & |x| \le \sqrt{2} \\ & & -\sqrt{2} \le x \le \sqrt{2} \end{array}$$

Conclusion: $D_f = [-\sqrt{2}, \sqrt{2}]$

As for the derivative:

since $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}$: $[\sin^{-1}\square]' = \dfrac{1}{\sqrt{1-\square^2}} \cdot \square'$

$$f'(x) = [\sin^{-1}(x^2-1)]' \overset{\downarrow}{=} \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot (x^2-1)'$$

$$= \frac{2x}{\sqrt{1-(x^2-1)^2}} = \frac{2x}{\sqrt{-x^4+2x^2}}$$

(b) Since e^{2x} is defined for all x, and since the inverse tangent function is also defined for all x, the domain of $g(x) = \tan^{-1}e^{2x}$ is $(-\infty, \infty)$. As for its derivative:

since $(\tan^{-1}x)' = \frac{1}{1+x^2}$: $[\tan^{-1}\square]' = \frac{1}{1+\square^2}\cdot\square'$

$$g'(x) = (\tan^{-1}e^{2x})' \overset{\downarrow}{=} \frac{1}{1+(e^{2x})^2}\cdot(e^{2x})' = \frac{2e^{2x}}{1+e^{4x}}$$

CHECK YOUR UNDERSTANDING 6.19

Determine the domain of the given function and that of its derivative.

(a) $f(x) = \cos^{-1}(\ln x)$ (b) $g(x) = \ln(\tan^{-1}x)$

Answers:

(a) $\left[\frac{1}{e}, e\right], \left(\frac{1}{e}, e\right)$

(b) $(0, \infty)$, $(0, \infty)$

It's time to turn the derivative formulas of Theorem 6.16 around into integral formulas. Actually, since the derivatives of $\sin^{-1}x$ and $\cos^{-1}x$ differ only by a negative sign we will just turn around $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$. For the same reason, turning around $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ and $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$ will accommodate the rest:

THEOREM 6.18

(a) $\int\frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C$

(b) $\int\frac{1}{1+x^2}dx = \tan^{-1}x + C$

(c) $\int\frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + C$

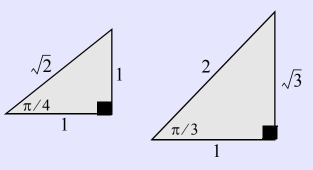
EXAMPLE 6.14 Evaluate:

(a) $\int_{1/\sqrt{2}}^{\sqrt{3}/2}\frac{1}{\sqrt{1-x^2}}dx$ (b) $\int\frac{e^x}{\sqrt{1-e^{2x}}}dx$

(c) $\int\frac{x}{x^4+1}dx$ (d) $\int\frac{1}{x\sqrt{x^2-9}}dx$

SOLUTION: (a) $\int_{1/\sqrt{2}}^{\sqrt{3}/2}\frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x\Big|_{1/\sqrt{2}}^{\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}}$

(see margin): $= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$



(b) $$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx = \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C$$

$$u = e^x \qquad du = e^x dx$$

$$= \sin^{-1}e^x + C$$

(c) $$\int \frac{\boldsymbol{x}}{x^4+1}\boldsymbol{dx} = \int \frac{\boldsymbol{x}}{1+(x^2)^2}\boldsymbol{dx} = \frac{1}{2}\int\frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C$$

$$u = x^2 \qquad du = 2\boldsymbol{xdx}$$

$$= \frac{1}{2}\tan^{-1}x^2 + C$$

(d) The integral $\int \frac{1}{x\sqrt{x^2-\mathbf{9}}}dx$ looks like $\int \frac{1}{x\sqrt{x^2-\mathbf{1}}}dx = \sec^{-1}x + C$ except for that "**9**" which we "turn into a **1**" by dividing the numerator and denominator by $\sqrt{9} = 3$:

$$\int \frac{1}{x\sqrt{x^2-9}}dx = \int \frac{\frac{1}{3}}{x\sqrt{\frac{x^2-9}{9}}}dx = \frac{1}{3}\int \frac{\mathbf{1}}{\boldsymbol{x}\sqrt{\left(\frac{\boldsymbol{x}}{\mathbf{3}}\right)^{\mathbf{2}}-\mathbf{1}}}\boldsymbol{dx}$$

$$u = \frac{x}{3} \Rightarrow du = \frac{dx}{3} \Rightarrow \boldsymbol{dx = 3du} \qquad \boldsymbol{x = 3u}$$

$$\rightarrow \; = \frac{1}{3}\int \frac{1}{\boldsymbol{3u}\sqrt{u^2-1}}\boldsymbol{3du}$$

$$= \frac{1}{3}\sec^{-1}u + C = \frac{1}{3}\sec^{-1}\frac{x}{3} + C$$

Check:

$$\left(\frac{\mathbf{1}}{\mathbf{3}}\sec^{-1}\frac{x}{3}\right)' = \frac{\mathbf{1}}{\mathbf{3}}\cdot\frac{1}{\frac{x}{3}\sqrt{\left(\frac{x}{3}\right)^2-1}}\cdot\left(\frac{x}{\mathbf{3}}\right)' = \frac{1}{\mathbf{9}}\frac{1}{\frac{x}{3}\sqrt{\left(\frac{x}{3}\right)^2-1}} = \frac{1}{3x\sqrt{\left(\frac{x}{3}\right)^2-1}}$$

$$= \frac{1}{x\sqrt{9\left[\left(\frac{x}{3}\right)^2-1\right]}} = \frac{1}{x\sqrt{x^2-9}}$$

CHECK YOUR UNDERSTANDING 6.20

Evaluate:

(a) $\int_0^1 \frac{1}{x^2+1}dx$

(b) $\int \frac{1}{\sqrt{1-(2x+1)^2}}dx$

Answers: (a) $\frac{\pi}{4}$

(b) $\frac{1}{2}\sin^{-1}(2x+1)+C$

EXERCISES

Exercise 1-6. (Domain) Determine the domain of the given function.

1. $f(x) = \sin^{-1}(\ln x)$
2. $f(x) = \ln(\sin^{-1}x)$
3. $f(x) = e^{\tan^{-1}x}$
4. $f(x) = \tan^{-1}e^{x}$
5. $f(x) = \sin(\sin^{-1}x)$
6. $\cos(\sin^{-1}x)$

Exercise 7-24. (Derivative) Differentiate the given function.

7. $f(x) = \sin^{-1}(x^2)$
8. $f(x) = \dfrac{1}{\cos^{-1}x}$
9. $f(x) = \cos^{-1}(x^2)$
10. $f(x) = (\cos^{-1}x)^2$
11. $f(x) = \tan^{-1}(\cos x)$
12. $f(x) = \cos(\tan^{-1}x)$
13. $f(x) = \sec^{-1}(e^x)$
14. $f(x) = e^{\sec^{-1}x}$
15. $f(x) = \sin^{-1}(e^{2x})$
16. $f(x) = \dfrac{x}{\sin^{-1}(x^2)}$
17. $f(x) = \dfrac{\tan^{-1}x}{x}$
18. $f(x) = (\sin^{-1}x)(\cos^{-1}x)$
19. $f(x) = \sqrt{\tan^{-1}2x}$
20. $f(x) = x\sec^{-1}x$
21. $f(x) = \sin^{-1}\left(\dfrac{x}{x+1}\right)$
22. $f(x) = \cot^{-1}\sqrt{x+1}$
23. $f(x) = \csc^{-1}(x^2+1)$
24. $f(x) = \dfrac{\sin^{-1}(x^2)}{(\cos^{-1}x)^2}$

Exercise 25-28. (Tangent Line) Determine the tangent line to the graph of the given function at the indicated point.

25. $y = \sin^{-1}x$ at $x = 0$
26. $y = x\tan^{-1}x$ at $x = 1$
27. $y = \sec^{-1}(2x)$ at $x = 1$

28. (Implicit Differentiation) $y(\tan^{-1}x) + x^2y^2 - 4 = \dfrac{\pi}{2}$ at $(1, 2)$

Exercise 29-46. Evaluate

29. $\displaystyle\int\frac{dx}{\sqrt{9-x^2}}$
30. $\displaystyle\int\frac{dx}{\sqrt{5-x^2}}$
31. $\displaystyle\int\frac{dx}{\sqrt{3-4x^2}}$
32. $\displaystyle\int\frac{dx}{x\sqrt{9x^2-1}}$
33. $\displaystyle\int\frac{e^x}{\sqrt{1-e^{2x}}}dx$
34. $\displaystyle\int\frac{dx}{1+16x^2}$
35. $\displaystyle\int\frac{dx}{(\sqrt{1-x^2})\sin^{-1}x}$
36. $\displaystyle\int\frac{x}{\sqrt{1-x^4}}dx$
37. $\displaystyle\int\frac{dx}{\sqrt{x}(1+x)}$
38. $\displaystyle\int\frac{dx}{x\sqrt{5x^2-3}}$
39. $\displaystyle\int\frac{\cos x}{\sin^2x+9}dx$
40. $\displaystyle\int\frac{dx}{e^x+e^{-x}}$

41. $\int_0^{1/4} \frac{dx}{\sqrt{1-4x^2}}$

42. $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}dx$

43. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}dx$

44. $\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$

45. $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

46. $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

Exercise 47-48. (Differential Equation) Solve for $f(x)$.

47. $f'(x) = \frac{1}{1+3x^2}$ if $f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$.

48. $f'(x) = \frac{1}{x\sqrt{x^2-9}}$ if $f(3\sqrt{2}) = \pi$

49. **(Area)** Determine the area of the region bounded above by the graph of the function $f(x) = \frac{x}{x^4+9}$, below by the x-axis, and on the sides by the y-axis and the vertical line $x = 3^{3/2}$.

50. **(Volume)** Find the volume obtained by revolving about the x-axis the region in the first quadrant bounded by the graph of the function $f(x) = \frac{1}{\sqrt{1+x^2}}$ over the interval $[0, 4]$.

51. **(Angle of Depression)** A boat is pulled toward a dock by a rope attached to the bow of the boat and passing through a ring on the dock that is 12 feet higher than the bow of the boat. How fast is the angle of depression of the rope changing when there are still 20 feet of rope out, if the rope is being pulled in at a rate of 1 ft/sec?

52. **(Maximum Inclination)** Determine the maximum angle of elevation of the tangent lines to the graph of the function $f(x) = \frac{x}{x^2+1}$.

Exercise 53-56. (Theory) Establish the following differentiation formulas:

53. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

54. $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$

55. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

56. $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

Exercise 57-60. (Theory) Establish the following integration formulas by:

(a) The u-substitution method. (b) Differentiating the right side of the equation.

57. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

58. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

59. $\int \frac{dx}{a^2+(x+b)^2} = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$

60. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

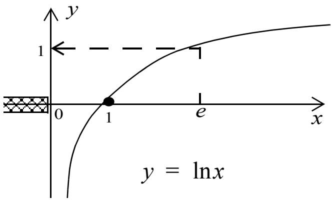
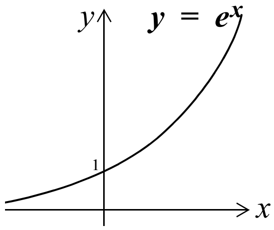
Exercise 61-63. (Theory) The Mean Value Theorem of page 121 assures us that if f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) for which $f'(c) = \frac{f(b)-f(a)}{b-a}$. Find such a c in $[0, 1]$ for the given function.

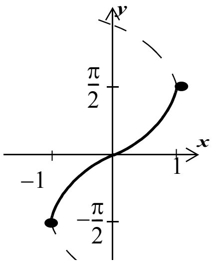
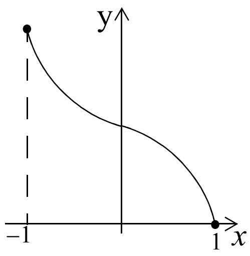
61. $f(x) = \sin^{-1}(x)$

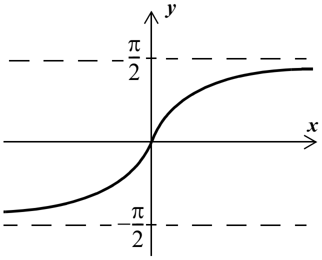
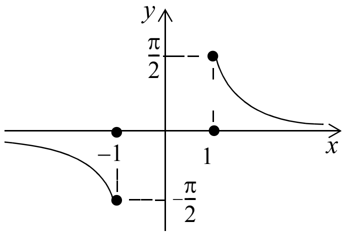
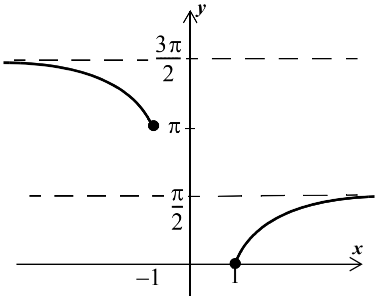
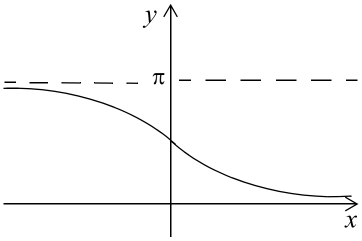
62. $f(x) = \cos^{-1}(x)$

63. $f(x) = \tan^{-1}(x)$

CHAPTER SUMMARY

THE NATURAL LOGARITHMIC FUNCTION	The **natural logarithmic function**, denoted by $\ln x$, has domain $(0, \infty)$, range $(-\infty, \infty)$, and is given by: $\ln x = \int_1^x \frac{1}{t}\,dt$ 
THEOREMS	$\frac{d}{dx}\ln x = \frac{1}{x}$ For any $x \neq 0$: $\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$ $\int \frac{1}{x}dx = \ln\lvert x\rvert + C$ For any positive numbers x and y and any real number r: (a) $\ln xy = \ln x + \ln y$ (b) $\ln\frac{x}{y} = \ln x - \ln y$ (c) $\ln x^r = r\ln x$
THE NATURAL EXPONENTIAL FUNCTION	The **natural exponential function**, denoted by e^x, is the inverse of the natural logarithmic function. As such, its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$. 
THEOREMS	$\frac{d}{dx}e^x = e^x$ and $\int e^x dx = e^x + C$ For all real numbers a and b : (a) $e^a e^b = e^{a+b}$ (b) $\frac{e^a}{e^b} = e^{a-b}$ (c) $(e^a)^b = e^{ab}$ (d) $e^{-x} = \frac{1}{e^x}$
GENERAL EXPONENTIAL FUNCTIONS	For any $a > 0$, $a \neq 1$, and any x: $a^x = e^{x\ln a}$

THEOREMS	For any real number r: $(x^r)' = rx^{r-1}$ For a positive and distinct from 1: $(a^x)' = a^x \ln a$ $\int a^x dx = \frac{a^x}{\ln a} + C$ For any x and y: (a) $a^x a^y = a^{x+y}$ (b) $\frac{a^x}{a^y} = a^{x-y}$ (c) $(a^x)^y = a^{xy}$ (d) $a^{-x} = \frac{1}{a^x}$
GENERAL LOGARITHMIC FUNCTIONS	For any positive number $a \neq 1$ $\log_a x$ is the inverse of the function a^x.
THEOREMS	For any positive number $a \neq 1$: $(\log_a x)' = \frac{1}{x \ln a}$ For any x and y: (a) $\log_a xy = \log_a x + \log_a y$ (b) $\log_a \frac{x}{y} = \log_a x - \log_a y$ (c) $\log_a x^r = r \log_a x$
INVERSE SINE FUNCTION	The **inverse sine** function, denoted by $\sin^{-1} x$, has domain $[-1, 1]$ and is given by: $y = \sin^{-1} x$ if $\sin y = x$ with $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. 
INVERSE COSINE FUNCTION	The **inverse cosine** function, denoted by $\cos^{-1} x$, has domain $[-1, 1]$ and is given by: $y = \cos^{-1} x$ if $\cos y = x$ with $0 \le y \le \pi$. 

INVERSE TANGENT FUNCTION	The **inverse tangent** function, denoted by $\tan^{-1}x$, has domain $(-\infty, \infty)$ and is given by: $y = \tan^{-1}x$ if $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$. 
INVERSE COSECANT FUNCTION	The **inverse cosecant** function has domain $(-\infty, -1] \cup [1, \infty)$ and is given by: $y = \csc^{-1}x$ if $\csc y = x$ with $0 < y \le \frac{\pi}{2}$ or $-\frac{\pi}{2} \le y < 0$. 
INVERSE SECANT FUNCTION	The **inverse secant** function, denoted by $\sec^{-1}x$, has domain $(-\infty, -1] \cup [1, \infty)$ and is given by: $y = \sec^{-1}x$ if $\sec y = x$ with $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$. 
INVERSE COTANGENT FUNCTION	The **inverse cotangent** function has domain $(-\infty, \infty)$ and is given by: $y = \cot^{-1}x$ if $\cot y = x$ with $0 < y < \pi$. 
THEOREMS	$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad (\tan^{-1}x)' = \frac{1}{1+x^2}$ $(\sec^{-1}x)' = \frac{1}{x\sqrt{x^2-1}}$ $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$ $\quad \int \frac{dx}{1+x^2} = \tan^{-1}x + C$ $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C$

CHAPTER 7
TECHNIQUES OF INTEGRATION

§1. INTEGRATION BY PARTS

Shifting the chain rule into reverse brought us to the u-substitution method (Theorem 5.12, page 189). Turning the derivative product rule around takes us to another important technique of integration, called **integration by parts**:

For f and g differentiable: $$[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$$

Integrate both sides: $$\int [f(x)g(x)]'dx = \int f(x)g'(x)dx + \int g(x)f'(x)dx$$

Rearrange: $$\int f(x)g'(x)dx = \int [f(x)g(x)]'dx - \int g(x)f'(x)dx$$

$f(x)g(x)$ is an antiderivative of $[f(x)g(x)]'$: $$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

As we did in the development of the u-substitution method, we can soften the appearance of the last equation by letting $u = f(x)$, $v = g(x)$ and by **symbolically** replacing $f'(x)dx$ and $g'(x)dx$ with du and dv, respectively:

$$\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{v\,du} \quad (*)$$

Basically, the above **Integration by Parts Formula** should be invoked when you can't evaluate $\int \boldsymbol{u\,dv}$ but can evaluate $\int \boldsymbol{v\,du}$. The first obstacle, of course, is to be able to go from "$\boldsymbol{dv}$" to "$\boldsymbol{v}$"; which is to say, that the "dv - expression" must itself be integrable.

TO ILLUSTRATE:

$$\int x\,\boxed{\sin x\,dx}$$

dv - expression is integrable

$$\boxed{\begin{aligned} u &= x \\ du &= dx \end{aligned}} \qquad \boldsymbol{dv} = \boxed{\sin x dx} : \ \boldsymbol{v} = \int \sin x dx = -\cos x + C$$

So: $$\begin{aligned} u &= x \\ du &= dx \end{aligned} \qquad \begin{aligned} dv &= \sin x dx \\ v &= -\cos x \end{aligned}$$

And: $$\int x \sin x dx = \int u dv = uv - \int v du = x(-\cos x) - \int (-\cos x) dx$$
$$= -x\cos x + \sin x + C$$

Check: $$(-x\cos x + \sin x)' = -x(\cos x)' + \cos x(-x)' + (\sin x)'$$
$$= -x(-\sin x) - \cos x + \cos x = x\sin x$$

In the above, of all the antiderivatives of $\sin x$; namely $-\cos x + C$, we chose the simplest: $v = -\cos x$. Suppose you are particularly fond of the number 7, and decide to do this:

$$u = x \qquad dv = \sin x dx$$
$$\mathbf{du = dx} \qquad \mathbf{v} = -\cos x + 7$$

No problem:

$$\int x \sin x\, dx = \int u dv = uv - \int v du = x(-\cos x + 7) - \int(-\cos x + 7)dx$$
$$= -x\cos x + 7x - (-\sin x + 7x) + C$$
$$= -x\cos x + \sin x + C \longleftarrow \boxed{\text{same result}}$$

EXAMPLE 7.1 Evaluate: $\int x^3 e^{x^2} dx$

SOLUTION:

we needed to get to a "manageable dv"

$$\int x^3 e^{x^2} dx = \int x^2 (xe^{x^2})dx$$

$$dv = xe^{x^2}dx \Rightarrow v = \int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

$$u = x^2 \qquad dv = xe^{x^2}dx$$
$$du = 2xdx \qquad v = \frac{1}{2}e^{x^2}$$

And: $$\int x^3 e^{x^2} dx = \int u dv = uv - \int v du = x^2\left(\frac{1}{2}e^{x^2}\right) - \int\left(\frac{1}{2}e^{x^2}\right)2xdx$$
$$= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2}dx$$
$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$$
$$= \frac{1}{2}e^{x^2}(x^2 - 1) + C$$

Check: $$\left[\frac{1}{2}e^{x^2}(x^2 - 1)\right]' = \frac{1}{2}[e^{x^2}(x^2 - 1)]'$$
$$= \frac{1}{2}[e^{x^2} \cdot 2x + (x^2 - 1) \cdot 2xe^{x^2}]$$
$$= \frac{1}{2}(2xe^{x^2} + 2x^3 e^{x^2} - 2xe^{x^2}) = x^3 e^{x^2}$$

CHECK YOUR UNDERSTANDING 7.1

Determine:

$$\int x \cos x\, dx$$

Answer: $x \sin x + \cos x + C$

You already know that:

$$(e^x)' = e^x,\ (\ln x)' = \frac{1}{x}, \text{ and } \int e^x dx = e^x + C$$

How does one integrate the logarithmic function? Like this:

THEOREM 7.1

$$\int \ln x\,dx = x\ln x - x + C$$

$$(x\ln x - x)'$$
$$= x(\ln x)' + \ln x\cdot x' - x'$$
$$= x\cdot\frac{1}{x} + \ln x - 1 = \ln x$$

PROOF: We could, of course, simply verify that $(x\ln x - x)' = \ln x$ (margin), but inquiring minds may want to know where the formula came from in the first place. Alright then, have it your way:

$$\int \ln x\,\boxed{dx}$$

$$u = \ln x \qquad du = \frac{1}{x}dx$$

$$dv = dx \qquad v = x$$

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int x\cdot\frac{1}{x}dx = x\ln x - x + C$$

EXAMPLE 7.2 Evaluate:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\ln(\sin x)dx$$

SOLUTION:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\ln(\sin x)dx \underset{\uparrow}{=} \int_{\frac{1}{2}}^{1} \ln u\,du = (u\ln u - u)\Big|_{\frac{1}{2}}^{1}$$

$$u = \sin x$$
$$du = \cos x dx$$
$$x = \frac{\pi}{6} \Rightarrow u = \sin\frac{\pi}{6} = \frac{1}{2}$$
$$x = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} = 1$$

$$= (\ln 1 - 1) - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right)$$

$$= (0 - 1) - \left(\ln\sqrt{\frac{1}{2}} - \frac{1}{2}\right)$$

$$= -\frac{1}{2} - \ln\sqrt{\frac{1}{2}}$$

CHECK YOUR UNDERSTANDING 7.2

Evaluate:

(a) $\int_0^1 x\ln(2x^2 + 1)dx$ (b) $\int \tan^{-1}x dx$

Suggestion: $u = \tan^{-1}x$, $dv = dx$

Answers: (a) $-\frac{1}{2} + \frac{3}{4}\ln 3$

(b) $x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C$

The integration by parts procedure may need to be employed more than once to evaluate a given integral. Consider the following examples:

EXAMPLE 7.3 Evaluate:
(a) $\int x^2 \cos x dx$ (b) $\int e^x \sin x dx$

SOLUTION: (a) $\int x^2 \cos x dx$

$$\begin{aligned} u &= x^2 \\ du &= 2x dx \end{aligned} \qquad \begin{aligned} dv &= \cos x\, dx \\ v &= \sin x \end{aligned}$$

$$\int x^2 \cos x\, dx = \int u dv = uv - \int v du = x^2 \sin x - \int \boldsymbol{2x \sin x dx} \qquad (*)$$

Then $\int \boldsymbol{2x \sin x dx}$:

$$\begin{aligned} u &= 2x \\ du &= 2dx \end{aligned} \qquad \begin{aligned} dv &= \sin x\, dx \\ v &= -\cos x \end{aligned}$$

$$\int \boldsymbol{2x \sin x\, dx} = \overbrace{-2x\cos x}^{uv} - \overbrace{\int(-\cos x)2dx}^{\int v du} = \boldsymbol{-2x\cos x + 2\sin x + C}$$

Returning to (*):

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x dx = x^2 \sin x + 2x \cos x - 2\sin x + C$$

(b) $\int e^x \sin x dx$

$$\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned} \qquad \begin{aligned} dv &= \sin x dx \\ v &= -\cos x \end{aligned}$$

$$\int e^x \sin x\, dx = -e^x \cos x - \int -\cos x (e^x\, dx) = -e^x \cos x + \int \boldsymbol{e^x \cos x dx} \quad (*)$$

Continuing the good fight: $\int \boldsymbol{e^x \cos x dx}$

$$\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned} \qquad \begin{aligned} dv &= \cos x\, dx \\ v &= \sin x \end{aligned}$$

$$\int \boldsymbol{e^x \cos x\, dx} = e^x \sin x - \int e^x \sin x dx$$

Returning to (*):

$$\begin{aligned} \int e^x \sin x\, dx &= -e^x \cos x + \int \boldsymbol{e^x \cos x dx} \\ &= -e^x \cos x + \boldsymbol{e^x \sin x} - \int \boldsymbol{e^x \sin x dx} \end{aligned}$$

Adding $\int e^x \sin x\, dx$ to both sides of the above equation brings us to:

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C$$

$$\int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Just in case you're wondering: To say that $\frac{C}{2}$ is an arbitrary constant is the same as saying that C is an arbitrary constant.

CHECK YOUR UNDERSTANDING 7.3

Determine:

$$\int e^x \cos x dx$$

Answer: $\frac{1}{2}e^x(\sin x + \cos x) + C$

The integration by parts formula for indefinite integrals:

$$\int u dv = uv - \int v du$$

can be modified to accommodate definite integrals. Specifically:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du$$

EXAMPLE 7.4 Employ the above formula to Evaluate:

$$\int_0^\pi x \sin x dx$$

SOLUTION:

$$\int_0^\pi x \sin x dx$$

$u = x$

$du = dx$

$dv = \sin x dx$

$v = -\cos x$

$$\begin{aligned}\int_0^\pi x\sin x dx = uv|_0^\pi - \int_0^\pi v du &= x(-\cos x)|_0^\pi - \int_0^\pi (-\cos x) dx\\ &= -x\cos x|_0^\pi + \sin x|_0^\pi\\ &= -(\pi\cos\pi - 0\cos 0) + (\sin\pi - \sin 0)\\ &= -[\pi(-1)] + 0 = \pi\end{aligned}$$

If you prefer, you may use the indefinite integral formula, as we did on page 261 to arrive at: $\int x \sin x dx = -x\cos x + \sin x + C$

Then: $\int_0^\pi x\sin x dx = (-x\cos x + \sin x)|_0^\pi$

CHECK YOUR UNDERSTANDING 7.4

Evaluate:

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx$$

Answer: $\frac{\pi}{2} - 1$

The following theorem illustrates a procedure that can be used to integrate powers of the sine or cosine functions.

THEOREM 7.2

REDUCTION FORMULAS

For any integer $n \geq 2$:

$$\int \sin^n x dx = -\frac{\cos x}{n}\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x dx = \frac{\sin x}{n}\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

PROOF: We derive the sine-formula

$$\int \sin^n x dx = -\frac{\cos x}{n}\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

and invite you verify the cosine-formula in the exercises:

$$\int \sin^n x dx$$

$u = \sin^{n-1}x$

$du = (n-1)(\sin^{n-2}x)\cos x dx$

$dv = \sin x dx$

$v = -\cos x$

$$\int \sin^n x\,dx = \int u dv = uv - \int v du = \sin^{n-1}x(-\cos x) - \int -\cos x[(n-1)(\sin^{n-2}x)\cos x dx]$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \mathbf{\cos^2 x}\sin^{n-2}x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (\mathbf{1 - \sin^2 x})(\sin^{n-2}x dx)$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx - (n-1)\int \sin^n x dx$$

Adding $(n-1)\int \sin^n x dx$ to both sides of the above equation brings us to:

$$\int \sin^n x dx + (n-1)\int \sin^n x dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$n\int \sin^n x dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$\int \sin^n x dx = -\frac{\cos x}{n}\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

EXAMPLE 7.5 Use a reduction formula to determine:

$$\int \cos^4 x\,dx$$

SOLUTION: Appealing to the cosine-reduction formula a couple of times:

$$\begin{aligned}\int \cos^4 x\,dx &= \frac{\sin x}{4}\cos^3 x + \frac{3}{4}\int \cos^2 x\,dx\\ &= \frac{\sin x}{4}\cos^3 x + \frac{3}{4}\left[\frac{\sin x}{2}\cos x + \frac{1}{2}\int \cos^0 x\,dx\right]\\ &= \frac{\sin x}{4}\cos^3 x + \frac{3\sin x}{8}\cos x + \frac{3}{8}\int dx\\ &= \frac{\sin x}{4}\cos^3 x + \frac{3\sin x}{8}\cos x + \frac{3\boldsymbol{x}}{8} + C\end{aligned}$$

CHECK YOUR UNDERSTANDING 7.5

Use a reduction formula to determine:

$$\int \sin^3 x\,dx$$

Answer:
$-\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C$

EXERCISES

Exercise 1-47. Evaluate.

1. $\int xe^{-x}dx$
2. $\int xe^{2x}dx$
3. $\int x\sin 3x\,dx$
4. $\int -2x\cos\frac{x}{2}dx$
5. $\int x\sin ax\,dx$
6. $\int x\cos ax\,dx$
7. $\int x^2\ln x\,dx$
8. $\int xe^{3x}dx$
9. $\int x^2e^{3x}dx$
10. $\int x^2e^{x}dx$
11. $\int \frac{x^3}{\sqrt{1+x^2}}dx$
12. $\int x^3\ln x\,dx$
13. $\int x^2e^{-x}dx$
14. $\int \sin^{-1}x\,dx$
15. $\int \sqrt{x}\ln x\,dx$
16. $\int (\ln x)^2dx$
17. $\int \ln\frac{1}{x}dx$
18. $\int \ln(x+c)dx$
19. $\int x\ln(x+c)dx$
20. $\int \frac{xe^{2x}}{(1+2x)^2}dx$
21. $\int \cos(\ln x)dx$
22. $\int \sin(\ln x)dx$
23. $\int \cos x\ln(\sin x)dx$
24. $\int (\sin^{-1}x)^2dx$
25. $\int (x^2-5x)e^{x}dx$
26. $\int x(\ln x)^2dx$
27. $\int x\tan^2x\,dx$
28. $\int \frac{\ln x}{\sqrt{x}}dx$
29. $\int \cos^3x\,dx$
30. $\int \sin^4x\,dx$
31. $\int \frac{x^2}{(1+x^2)^2}dx$
32. $\int \frac{\tan^{-1}x}{x^2}dx$
33. $\int x^2\tan^{-1}x\,dx$
34. $\int \sin 3x\sin 2x\,dx$
35. $\int \sin 3x\cos 2x\,dx$
36. $\int \cos x\cos 3x\,dx$
37. $\int_0^1 x^2e^{x}dx$
38. $\int_0^{\pi} x\sin 2x\,dx$
39. $\int_0^1 x^2e^{-3x}dx$
40. $\int_1^2 \frac{\ln x}{x^2}dx$
41. $\int_0^1 \tan^{-1}x\,dx$
42. $\int_1^e x^2\ln x\,dx$
43. $\int_{-2}^{2} \ln(x+3)dx$
44. $\int_0^{\frac{\pi}{2}} \sin x\ln\cos x\,dx$
45. $\int_0^{\frac{\pi}{2}} x\sin 4x\,dx$
46. $\int_0^{\frac{\pi}{2}} \sin^4 2x\cos^3 2x\,dx$
47. $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx$

48. **(Area)** Find the area of the region enclosed by the graph of $f(x) = x\cos x$ and the x axis for $0 \le x \le \frac{3\pi}{2}$.

49.**(Area)** Find the area of the region enclosed by the graph of $f(x) = x\sin x$ and the x axis for $0 \le x \le 3\pi$.

50. **(Area)** Find the area of the region enclosed by the graph of $f(x) = x\sin x$, and the lines $y = x$, $x = 0$, and $x = \frac{\pi}{2}$.

51. **(Volume)** Find the volume obtained by revolving the region bounded by the graph of the function $f(x) = \ln x$, the line $x = e$ and the x-axis about the x-axis.

52. **(Volume)** Find the volume obtained by revolving the finite region enclosed by the graphs of the sine and cosine functions, the y-axis and the vertical line $y = \pi$ about the x-axis.

53. **(Velocity)** A particle moving along a line has velocity $v(t) = te^t$ feet per second. How far will it travel during the first 3 seconds?

54. **(Velocity)** A particle moving along a line has velocity $v(t) = e^t\sin\pi t$ meters per second. How far will it travel during the first 2 seconds?

Exercise 55-59. (Reduction Formulas) Derive the given reduction formula, where n is an integer greater than 1.

55. $\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

56. $\int \cos^n x dx = \frac{\sin x}{n}\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\, dx$

57. $\int \sec^n x dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x dx$

58. $\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx$

59. $\int x^m(\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx$

Exercise 60-64. (Integral Formulas) Derive the given integral equation.

60. $\int x^n\sin^{-1}x dx = \frac{x^{n+1}}{n+1}(\sin^{-1}x) - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$

61. $\int \ln(x+\sqrt{x^2+a^2})dx = x\ln(x+\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + C$

62. $\int \ln(x+a)dx = (x+a)\ln(x+a) - x + C$

63. $\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

64. $\int e^{ax}\cos bx dx = \frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2} + C$

65. **(Theory)** Show that $uv - \int vdu = u(v+C) - \int (v+C)du$ for any constant C.

66. **(Theory)** Show that $\int f(x)dx = xf(x) - \int xf'(x)dx$.

§2. COMPLETING THE SQUARE AND PARTIAL FRACTIONS

Once we rewrite the quadratic polynomial x^2+4x+8 in the form $(x+2)^2+4$, we will be able to evaluate $\int \frac{dx}{x^2+4x+8}$:

$$\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{(x+2)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x+2}{2}\right)^2+1}$$

$u = \frac{x+2}{2}, du = \frac{1}{2}dx \Rightarrow dx = 2du$:

$$= \frac{1}{2}\int \frac{du}{u^2+1}$$

Theorem 6.17(b), page 255:

$$= \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$$

But how does one go from x^2+4x+8 to $(x+2)^2+4$? By using the **completing the square method**, which we now describe:

When polynomials of the form:

$(x+a)^2$ and $(x-a)^2$ (called **perfect squares**)

are expanded, they take on the following form:

$$(x+a)^2 = x^2+2ax+\boldsymbol{a^2}$$
$$(x-a)^2 = x^2-2ax+\boldsymbol{a^2}$$

The square of $\frac{1}{2}$ the coefficient of x

Note that for this technique to work the coefficient of x^2 **must be 1**.

In particular, to be a perfect square, the question mark in the expression:

$$x^2+4x+?$$

must be replaced by the square of one-half the coefficient of x:

$$x^2+4x+4 = (x+2)^2$$

$$\left(\frac{4}{2}\right)^2 = 4$$

In particular:

turn this piece into a perfect square

$$\boxed{x^2+4x}+8 = \boxed{x^2+4x+4}-4+8 = (x+2)^2+4$$

since we added 4, we must subtract 4

The above completing the square method can be used to evaluate certain integrals involving ax^2+bx+c. Consider the following examples.

EXAMPLE 7.6 Evaluate:

(a) $\int \frac{dx}{\sqrt{-x^2+10x-21}}$ (b) $\int \frac{x-5}{x^2+2x+2}dx$

SOLUTION: (a) We begin by molding $-x^2+10x-21$ into a form that contains a perfect square:

add the square of one-half the coefficient of x

since we subtracted 25, we added it back

$$-x^2+10x-21 = -(1x^2-10x)-21 = -(x^2-10x+25)-21+25$$

coefficient of x^2 must be 1

$$= -(x-5)^2+4$$

Then:

$$\int\frac{dx}{\sqrt{-x^2+10x-21}} = \int\frac{dx}{\sqrt{-(x-5)^2+4}} = \int\frac{dx}{\sqrt{4\left[-\frac{(x-5)^2}{4}+1\right]}}$$

$\boxed{\text{motivated by } \int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C}$

$$= \frac{1}{2}\int\frac{dx}{\sqrt{1-\left(\frac{x-5}{2}\right)^2}}$$

$u = \frac{x-5}{2}$

$du = \frac{dx}{2}$

$$= \int\frac{du}{\sqrt{1-u^2}}$$

Theorem 6.17(a), page 255: $= \sin^{-1}u + C$

$$= \sin^{-1}\left(\frac{x-5}{2}\right) + C$$

(b) It would be nice if the numerator in the integral $\int\frac{x-5}{x^2+2x+2}dx$ were a $\mathbf{2x+2}$, for we could then let $u = x^2+2x+2$, $du = 2x+2$ and then go on from there. But it isn't. So we first focus on getting the $2x$ of $\mathbf{2}x+2$ into the numerator: $\frac{1}{2}\int\frac{\mathbf{2}x-10}{x^2+2x+2}dx$; and then squeeze in the $+2$: $\frac{1}{2}\int\frac{\mathbf{2x+2}-12}{x^2+2x+2}dx$. Breaking the integral into two integrals we have:

$$\int\frac{x-5}{x^2+2x+2}dx = \frac{1}{2}\int\frac{\mathbf{2x+2}-12}{x^2+2x+2}dx$$
$$= \frac{1}{2}\int\frac{\mathbf{2x+2}}{x^2+2x+2}dx - \frac{1}{2}\int\frac{12}{x^2+2x+2}dx$$

No problem with the first integral:

$$\frac{1}{2}\int\frac{\mathbf{2x+2}}{x^2+2x+2}dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+2x+2) + C$$

$u = x^2+2x+2$, $du = (2x+2)dx$

note that $x^2+2x+2>0$ for all x

The second integral takes a bit more work:

$$\frac{1}{2}\int\frac{12}{x^2+2x+2}dx = 6\int\frac{dx}{(x^2+2x+\underset{\uparrow}{1})+2-\underset{\uparrow}{1}}$$

$$\boxed{\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}} \rightarrow = 6\int\frac{dx}{1+(x+1)^2}$$

$$= 6\int\frac{du}{1+u^2} = 6\tan^{-1}u + C = 6\tan^{-1}(x+1)+C$$

Putting it all together we have:

$$\int\frac{x-5}{x^2+2x+2}dx = \frac{1}{2}\ln(x^2+2x+2) - 6\tan^{-1}(x+1)+C$$

CHECK YOUR UNDERSTANDING 7.6

Evaluate:

$$\int\frac{dx}{\sqrt{3x-x^2}}$$

Answer:

$$\sin^{-1}\left(\frac{2}{3}x-1\right)+C$$

PARTIAL FRACTIONS

As you know, to perform the sum $\frac{2}{x+3}+\frac{1}{x-2}$ you first find the least common denominator, and then go on from there:

$$\frac{2}{x+3}+\frac{1}{x-2} = \frac{2x-4+x+3}{(x+3)(x-2)} = \frac{3x-1}{(x+3)(x-2)}$$

In this section you will need to go the other way:

Go from the rational expression $\frac{3x-1}{(x+3)(x-2)}$

to its so-called **partial fractions** form $\frac{3x-1}{(x+3)(x-2)} = \frac{2}{x+3}+\frac{1}{x-2}$.

ax^2+bx+c is **irreducible** if it cannot be expressed as a product of linear factors with real coefficients; equivalently: if its discriminant b^2-4ac is negative.

The first step toward obtaining a partial fraction decomposition of a rational expression (with the degree of the numerator **LESS** than that of the denominator) is to factor its denominator into a product of powers of distinct linear factors, $ax+b$, and powers of irreducible quadratic polynomials, ax^2+bx+c. Next, obtain the **general decomposition** of the given rational expression by writing it as a sum of rational expressions of the form:

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

where A and B denote real numbers.

The following table reveals the terms to be included in the general decomposition to accommodate each factor in the denominator of the given expression:

Powers of linear factors	**Terms in the decomposition**
(i) $ax + b$	$\frac{A}{ax+b}$
(ii) $(ax + b)^2$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
(iii) $(ax + b)^3$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$
Powers of Irreducible quadratic factors	**Terms in the decomposition**
(iv) $ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$
(v) $(ax^2 + bx + c)^2$	$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$
(vi) $(ax^2 + bx + c)^3$	$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3}$

Figure 7.1

To illustrate:

see (ii) in above table

$$\frac{1}{(2x-1)^2(x^2+3)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{Cx+D}{x^2+3}$$

irreducible—see (iv)

Answer:

$$\frac{A}{x-3} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2} + \frac{Dx+E}{x^2+5} + \frac{Fx+G}{(x^2+5)^2}$$

CHECK YOUR UNDERSTANDING 7.7

Find the general decomposition for:

$$\frac{x-4}{(x-3)(2x+1)^2(x^2+5)^2}$$

The following example illustrates a technique that can be used to find the final partial fraction decomposition of a rational expression.

EXAMPLE 7.7 Find the partial fraction decomposition of:

(a) $\frac{-4x+9}{2x^2+5x-3}$ (b) $\frac{2x^2+3}{x(x-1)^2}$

(c) $\frac{x^2-2x+1}{(x^2+1)^2}$

SOLUTION: (a) Express the rational expression in general decomposition form:

$$\frac{-4x+9}{2x^2+5x-3} = \frac{-4x+9}{(2x-1)(x+3)} \overset{\overset{\text{Figure 7.1(i)}}{\downarrow}}{=} \frac{A}{2x-1}+\frac{B}{x+3}$$

Clear denominators by multiplying both sides of the equation:

$$\frac{-4x+9}{2x^2+5x-3} = \frac{A}{2x-1}+\frac{B}{x+3}$$

by $(2x-1)(x+3)$:

$$-4x+9 = A(x+3)+B(2x-1) \qquad (*)$$

Here are two methods that can be used to find the values of A and B:

By setting x equal to $\mathbf{-3}$ in (*), the term $A(\boldsymbol{x+3})$ in (*) will drop out, and this will enable us to easily find the value of B:

$$-4(\mathbf{-3})+9 = A(-\mathbf{3}+3)+B[2(\mathbf{-3})-1]$$

$$21 = -7B \Rightarrow B = -3$$

By setting x equal to $\frac{1}{2}$ in (*), the term $B(2x-1)$ in (*) will drop out, and we can solve for A:

$$-4\left(\frac{1}{2}\right)+9 = A\left(\frac{1}{2}+3\right) \Rightarrow 7 = \frac{7}{2}A \Rightarrow A = 2$$

Rewrite the right side of (*) in polynomial form:

$$-4x+9 = (A+2B)x+(3A-B)$$

Equate the like coefficients of the polynomials on the left and right sides of the equation:

$$-4 = A+2B$$

$$9 = 3A-B$$

You can the solve the above system of equations; and, if you do, you will again find that:

$$A = 2, B = -3$$

Note: Unlike the "easier" method on the left, this method can be used to find the final partial fraction decomposition of any rational expression.

Decomposition: $\dfrac{-4x+9}{2x^2+5x-3} = \dfrac{2}{2x-1}+\dfrac{-3}{x+3} = \dfrac{2}{2x-1}-\dfrac{3}{x+3}$

(b)

$$\frac{2x^2+3}{x(x-1)^2} \overset{\overset{\text{Figure 7.1 (i) and (iii)}}{\downarrow}}{=} \frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Clear denominators: $2x^2+3 = A(x-1)^2+Bx(x-1)+Cx$ (*)

Setting $x = 1$: $2(1)^2+3 = A\cdot 0+B\cdot 0+C\cdot 1 \Rightarrow C = 5$

Setting $x = 0$: $2(0)^2+3 = A+B\cdot 0+C\cdot 0 \Rightarrow A = 3$

We still have to find the value of B and could do it in many ways.

One way is to replace A with 3 and C with 5 in (*):

$$2x^2+3 = 3(x-1)^2+Bx(x-1)+5x$$

Then substitute any value for x other than 0 and 1, say $x = 2$, and solve for B:

$$2\cdot\mathbf{2}^2+3 = 3(\mathbf{2}-1)^2+B\cdot\mathbf{2}(2-1)+5\cdot 2$$

$$11 = 3+2B+10$$

$$B = -1$$

Decomposition: $\dfrac{2x^2+3}{x(x-1)^2} = \dfrac{3}{x}-\dfrac{1}{x-1}+\dfrac{5}{(x-1)^2}$

(c) $$\frac{x^2-2x+1}{(x^2+1)^2} \overset{\substack{\text{Figure 7.1(v)}\\ \downarrow}}{=} \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

Clear denominators: $x^2-2x+1 = (Ax+B)(x^2+1)+(Cx+D)$.

Since no value of x will make a term drop out, we proceed by expanding the right side (details omitted) to come to:

$$x^2-2x+1 = Ax^3+Bx^2+(A+C)x+(B+D) \quad (*)$$

Equating the coefficients of like powers of x we have:

x^3-coefficients: $A = 0$

x^2-coefficients: $B = 1$

x-coefficients: $A+C = -2$ → $C = -2$

constant coefficient: $B+D = 1$ → $D = 0$

Decomposition: $$\boldsymbol{\frac{x^2-2x+1}{(x^2+1)^2} = \frac{1}{x^2+1} - \frac{2x}{(x^2+1)^2}}$$

CHECK YOUR UNDERSTANDING 7.8

Find the partial fraction decomposition of:

$$\frac{1}{x(x^2+x+1)}$$

Answer: $\frac{1}{x} - \frac{x+1}{x^2+x+1}$

Back to the calculus:

EXAMPLE 7.8

Evaluate: (a) $\int \frac{-4x+9}{2x^2+5x-3}dx$

(b) $\int \frac{2x^2+3}{x(x-1)^2}dx$ (c) $\int \frac{x^2-2x+1}{(x^2+1)^2}dx$

SOLUTION:

(a) $$\int \frac{-4x+9}{2x^2+5x-3}dx \overset{\substack{\text{Example 7.6(a)}\\ \downarrow}}{=} \int\left(\frac{2}{2x-1}-\frac{3}{x+3}\right)dx$$

$$= 2\int\frac{dx}{2x-1} - 3\int\frac{dx}{x+3}$$

$$= \underset{\substack{\uparrow\\ \text{see margin}}}{\ln}|2x-1| - 3\ln|x+3| + C$$

$$\boxed{u = 2x-1,\ du = 2dx}$$

$$2\int\frac{dx}{2x-1} \overset{\downarrow}{=} 2\cdot\frac{1}{2}\int\frac{du}{u} = \ln|u| + C = \ln|2x-1| + C$$

(b) $$\int \frac{2x^2+3}{x(x-1)^2}dx \overset{\substack{\text{Example 7.6(b)}\\ \downarrow}}{=} \int\left(\frac{3}{x}-\frac{1}{x-1}+\frac{5}{(x-1)^2}\right)dx$$

$$= 3\int\frac{dx}{x} - \int\frac{dx}{x-1} + 5\int\frac{dx}{(x-1)^2}$$

$$= 3\ln|x| - \ln|x-1| - \underset{\substack{\uparrow\\ \text{see margin}}}{\frac{5}{x-1}} + C$$

$$\boxed{u = x-1,\ du = dx}$$

$$5\int\frac{dx}{(x-1)^2} \overset{\downarrow}{=} 5\int\frac{du}{u^2} = 5\int u^{-2}du = -5u^{-1} + C = -\frac{5}{x-1} + C$$

$$\boxed{u=x^2+1,\ du=2xdx}$$

$$\begin{aligned}\int\frac{2x}{(x^2+1)^2}dx &\overset{\downarrow}{=}\int\frac{du}{u^2}\\ &=\int u^{-2}du\\ &=-\frac{1}{u}+C\\ &=-\frac{1}{x^2+1}+C\end{aligned}$$

(c)
$$\begin{aligned}\int\frac{x^2-2x+1}{(x^2+1)^2}dx &\overset{\substack{\text{Example 7.6(c)}\\ \downarrow}}{=}\int\left(\frac{1}{x^2+1}-\frac{2x}{(x^2+1)^2}\right)dx\\ &=\int\frac{dx}{1+x^2}-\int\frac{2x}{(x^2+1)^2}dx\\ &=\underset{\substack{\uparrow\\ \text{Theorem 6.17(b), page 255}}}{\tan^{-1}x}+\underset{\substack{\uparrow\\ \text{see margin}}}{\frac{1}{x^2+1}}+C\end{aligned}$$

Answer:
$$\ln|x|-\frac{1}{2}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C$$

CHECK YOUR UNDERSTANDING 7.9

Evaluate:
$$\int\frac{dx}{x(x^2+x+1)}$$

We remind you that the decomposition procedure summarized in Figure 7.1 can only be invoked when the degree of the numerator of the given rational expression is **LESS** than the degree of the denominator. What if that is not the case? The answer surfaces in the next example.

EXAMPLE 7.9

Evaluate:
$$\int\frac{x^3}{x^3-3x+2}dx$$

SOLUTION: Since the degree of the polynomial in the numerator is not less than that in the denominator, we divide (see margin) to arrive at:

$$\frac{x^3}{x^3-3x+2}=1+\frac{3x-2}{x^3-3x+2}$$

$$\begin{array}{rl} & 1\\ x^3-3x+2 & \overline{)\,x^3}\\ & x^3-3x+2\\ \hline -: & 3x-2\end{array}$$

Observing that 1 is a zero of x^3-3x+2 we apply Theorem 1.4, page 19, to arrive at a factorization of the cubic polynomial (see margin):

$$x^3-3x+2=(x-1)(x^2+x-2)=(x-1)(x-1)(x+2)$$

$$\begin{array}{rl} & x^2+x-2\\ x-1 & \overline{)\,x^3+0x^2-3x+2}\\ & x^3-x^2\\ \hline -: & x^2-3x+2\\ & x^2-x\\ \hline -: & -2x+2\\ & -2x+2\\ \hline -: & 0\end{array}$$

Bringing us to:

$$\begin{aligned}\int\frac{x^3}{x^3-3x+2}dx &=\int\left(1+\frac{3x-2}{(x-1)^2(x+2)}\right)dx\\ &=\int dx+\int\frac{3x-2}{(x-1)^2(x+2)}dx\quad(*)\end{aligned}$$

Referring to Figure 7.1, we have

$$\frac{3x-2}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$$

$$\mathbf{3x-2=A(x-1)(x+2)+B(x+2)+C(x-1)^2}$$

$$\text{Setting } x=1:\ 1=3B\Rightarrow B=\frac{1}{3}$$

$$\text{Setting } x=-2:\ -8=9C\Rightarrow C=-\frac{8}{9}$$

$$\text{Equating the } x^2 \text{ coefficients: } 0=A+C\Rightarrow A=-C\Rightarrow A=\frac{8}{9}$$

Returning to (*):

$$\int\frac{x^3}{x^3-3x+2}dx = \int dx + \int\left(\frac{\frac{8}{9}}{x-1}+\frac{\frac{1}{3}}{(x-1)^2}-\frac{\frac{8}{9}}{x+2}\right)dx$$

margin: $$= x+\frac{8}{9}\ln|x-1|-\frac{1}{3(x-1)}-\frac{8}{9}\ln|x+2|+C$$

$$\int\frac{8/9}{x-1} \overset{\uparrow}{=} \frac{8}{9}\int\frac{du}{u} = \frac{8}{9}\ln|u|+C$$

$u = x-1$

$du = dx$

$$= \frac{8}{9}\ln|x-1|+C$$

$$\begin{aligned}\int\frac{1/3}{(x-1)^2} &\overset{\downarrow}{=} \frac{1}{3}\int\frac{du}{u^2}\\ &= \frac{1}{3}\int u^{-2}du\\ &= -\frac{1}{3u}+C\\ &= -\frac{1}{3(x-1)}+C\end{aligned}$$

CHECK YOUR UNDERSTANDING 7.10

Evaluate:

$$\int\frac{2x^2-4x+3}{x^2-4x+4}dx$$

Answer:

$$2x+4\ln|x-2|-\frac{3}{x-2}+C$$

EXERCISES

Exercise 1-16. (Completing the Square) Evaluate.

1. $\int \frac{dx}{x^2+2x+5}$
2. $\int \frac{dx}{x^2+4x+5}$
3. $\int \frac{dx}{x^2+6x+25}$
4. $\int \frac{dx}{x^2-8x+17}$
5. $\int \frac{dx}{\sqrt{6x-x^2}}$
6. $\int \frac{dx}{\sqrt{10x-x^2}}$
7. $\int \frac{dx}{4x^2+8x+29}$
8. $\int \frac{dx}{9-2x-x^2}$
9. $\int \frac{dx}{x^2+3x+5}$
10. $\int \frac{dx}{\sqrt{6x^2-24x+32}}$
11. $\int \frac{x}{\sqrt{3+2x-x^2}}dx$
12. $\int_0^1 \frac{dx}{x^2+4x+5}$
13. $\int_{\frac{3}{2}}^{1+\frac{\sqrt{3}}{2}} \frac{dx}{-5+8x-4x^2}$
14. $\int_0^1 \sqrt{4x-x^2}\,dx$
15. $\int_1^2 \frac{dx}{\sqrt{4x-x^2}}$
16. $\int \sqrt{\frac{x+a}{x+b}}\,dx$ Suggestion: Multiply top and bottom by $\sqrt{x+a}$.

Exercise 17-52. (Partial Fractions) Evaluate.

17. $\int \frac{dx}{x^2-x-2}$
18. $\int \frac{dx}{x^2+3x-4}$
19. $\int \frac{\boldsymbol{x}}{x^2-3x+2}dx$
20. $\int \frac{2\boldsymbol{x}}{x^2-2x-8}dx$
21. $\int \frac{x^2}{(x-1)^2(x+1)}dx$
22. $\int \frac{x^3-1}{x^4-3x^3}dx$
23. $\int \frac{x}{6x^2-x-2}dx$
24. $\int \frac{x^2+x+2}{(x^2+1)^2}dx$
25. $\int \frac{7x+3}{x^3-2x^2-3x}dx$
26. $\int \frac{1}{x^2(x-3)^2}dx$
27. $\int \frac{5x^2+18x-1}{(x+4)^2(x-3)}dx$
28. $\int \frac{1}{(x-1)^2(x-2)^3}dx$
29. $\int \frac{4x^2+x+2}{x^3(x+2)}dx$
30. $\int \frac{x-1}{x^4+x^2}dx$
31. $\int \frac{x^3-x}{(x^2+1)^2}dx$
32. $\int \frac{x^2+2x}{(x^2+1)^2}dx$
33. $\int \frac{x-1}{x(x^2+1)}dx$
34. $\int \frac{x^2+2}{x^4+x^2}dx$
35. $\int \frac{7x^3-3x^2+9x-6}{x^4+3x^2+2}dx$
36. $\int \frac{x^3+2}{x^3-3x^2+2x}dx$
37. $\int \frac{x^2+2}{x^2+2x}dx$
38. $\int \frac{x^3+8}{x(x^2+4)}dx$
39. $\int \frac{x^3-3x^2+2x-3}{x^2+1}dx$
40. $\int \frac{x^5}{(x^2+1)^2}dx$

41. $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x - 6}dx$ 42. $\displaystyle\int \frac{\cos x}{\sin^2 x - 5\sin(x) + 6}dx$ 43. $\displaystyle\int \frac{e^x}{e^{2x} - 1}dx$

44. $\displaystyle\int \frac{e^x}{e^{2x} + 3e^x + 2}dx$ 45. $\displaystyle\int_0^1 \frac{dx}{x^2 - 9}$ 46. $\displaystyle\int_1^2 \frac{dx}{16 - x^2}$

47. $\displaystyle\int_1^2 \frac{22}{6x^2 + 5x - 4}dx$ 48. $\displaystyle\int_1^2 \frac{2x + 1}{x^2 + x}dx$ 49. $\displaystyle\int_{-\sqrt{3}}^1 \frac{2x^3 + 4x^2 + 2x - 3}{x^2(x^2 + 1)}dx$

50. $\displaystyle\int_2^3 \frac{2x^3 - x^2 + 2x + 1}{x^4 - 1}$ 51. $\displaystyle\int_1^2 \frac{2x^2 - x + 2}{x^3 + x}dx$ 52. $\displaystyle\int_0^1 \frac{x^2 + 3}{x^4 + 3x^2 + 2}dx$

Exercise 53-54. (Differential Equation) Solve the given differential equation.

53. $f'(x) = \dfrac{1}{x^2 - 3x + 2}$ if $f(3) = 0$. 54. $f'(x) = \dfrac{1}{x^3 + x}$ if $f(2) = 2$

55. **(Area)** Find the area of the region below the graph of $f(x) = \dfrac{7x - 5}{4x^2 - 7x - 2}$ that lies above the interval $[3, 5]$.

56. **(Area)** Find the area of the region below the graph of $f(x) = \dfrac{1}{x^3 - 3x^2 + 2x}$ tat lies above the interval $[3, 5]$.

57. **(Volume)** Find the volume obtained by generating, about the x-axis, the region bounded by the graph of the function $f(x) = \dfrac{x}{x^2 - 4}$ above the interval $[3, 5]$.

58. **(Volume)** Find the volume obtained by generating, about the x-axis, the finite region bounded by the graph of the function $f(x) = \dfrac{3}{\sqrt{3x - x^2}}$ and the lines $y = 0$, $x = \dfrac{1}{2}$, $x = \dfrac{5}{2}$.

Exercise 59-60. (Formulas) Derive the given integral formula.

59. $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C, \quad a > 0$ 60. $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x + a}{x - a}\right| + C, \quad a > 0$

§3. POWERS OF TRIGONOMETRIC FUNCTIONS AND TRIGONOMETRIC SUBSTITUTION

The Pythagorean identity $\sin^2 x + \cos^2 x = 1$ can be employed to evaluate integrals of the form $\int \sin^n x \cos^m x\,dx$ when at least one of the positive integer exponents, n and m, is **odd**. Consider the following example.

EXAMPLE 7.10 Evaluate:

(a) $\int \sin^3 x\,dx$ (b) $\int \sin^2 x \cos^5 x\,dx$

SOLUTION: (a) We could use a reduction formula, Theorem 7.2 (page 266) to evaluate $\int \sin^3 x\,dx$ (see CYU 7.5). Here is another approach:

Note that we "saved" one of the three sines to accommodate the subsequent u-substitution.

$$\int \sin^3 x\,dx = \int \underset{\substack{\downarrow\\ \text{see margin}}}{\boldsymbol{\sin x}}\sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx$$

$$= \int \sin x - \int \sin x \cos^2 x\,dx$$

$$= -\cos x - \int \sin x \cos^2 x\,dx$$

$u = \cos x$
$du = -\sin x dx$:

$$= -\cos x - \left[-\int u^2 du\right]$$

$$= -\cos x + \frac{u^3}{3} + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

Check: $\left(-\cos x + \frac{\cos^3 x}{3}\right)' = \sin x + \frac{3\cos^2 x(-\sin x)}{3} = \sin x + (1-\sin^2 x)(-\sin x) = \sin^3 x$

Note how the reserved $\boldsymbol{\cos x}$ serves us well in the subsequent u-substitution.
Why would this method fail had we used $\cos^{\mathbf{6}} x$ rather than $\cos^{\mathbf{5}} x$?

(b) $$\int \sin^2 x \cos^5 x\,dx = \int \sin^2 x\,\boldsymbol{\cos x}\cos^4 x\,dx$$

$$= \int \sin^2 x \cos x[\cos^2 x]^2 dx$$

$$= \int \sin^2 x \cos x[1-\sin^2 x]^2 dx$$

$$= \int \sin^2 x \cos x(1-2\sin^2 x + \sin^4 x)dx$$

$$= \int \sin^2 x \cos x\,dx - 2\int \cos x \sin^4 x\,dx + \int \cos x \sin^6 x\,dx$$

$u = \sin x$
$du = \cos x\,dx$

$$= \int u^2 du - 2\int u^4 du + \int u^6 du = \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

CHECK YOUR UNDERSTANDING 7.11

Evaluate: (a) $\int_0^{\frac{\pi}{2}}\cos^3 x\,dx$ (b) $\int \sin^3 x\cos^2 x\,dx$

Answers: (a) $\frac{2}{3}$

(b) $-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$

The above method for evaluating $\int \sin^n x\cos^m x\,dx$ will not work when n and m are both even. In that case, you can turn to the identities:

$$(*)\ \sin^2 x = \frac{1-\cos 2x}{2} \qquad (**)\ \cos^2 x = \frac{1+\cos 2x}{2}$$

See Theorem 1.5, page 37

EXAMPLE 7.11

Evaluate:

(a) $\int \cos^4 x\,dx$ (b) $\int \sin^2 x\cos^2 x\,dx$

SOLUTION: (a) We could use a reduction formula, Theorem 7.2 (page 266), to evaluate $\int \cos^4 x\,dx$ (see Example 7.5, page 267). Here is a direct approach:

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx \overset{(**)}{\overset{\downarrow}{=}} \int\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \int\left(\frac{1}{4}+\frac{2\cos 2x}{4}+\frac{\cos^2 2x}{4}\right)dx$$

$$= \int\frac{dx}{4}+\frac{1}{2}\int\cos 2x\,dx+\frac{1}{4}\int\underset{\underset{(**)}{\downarrow}}{\cos^2 2x}\,dx$$

$$= \frac{x}{4}+\frac{1}{2}\frac{\sin 2x}{2}+\frac{1}{4}\int\frac{1+\cos 4x}{2}dx$$

$$= \frac{x}{4}+\frac{1}{4}\sin 2x+\frac{1}{8}\left(x+\frac{\sin 4x}{4}\right)+C$$

$$= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

(b)

$$\int \sin^2 x\cos^2 x\,dx = \int\frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}dx = \frac{1}{4}\int(1-\underset{\swarrow}{\cos^2 2x})dx$$

$$= \frac{1}{4}\int\left(1-\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{8}\int(1-\cos 4x)dx$$

$$= \frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C$$

$$= \frac{1}{8}x-\frac{1}{32}\sin 4x+C$$

Answers: (a) $\frac{3\pi}{16}$

(b) $\frac{x^2}{16} - \frac{\sin 4x}{64} + C$

CHECK YOUR UNDERSTANDING 7.12

Evaluate:

(a) $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$ (b) $\int (x\sin^2 x^2)(\cos^2 x^2)\,dx$

INTEGRALS OF THE FORM $\int \tan^n x \sec^m x\,dx$ AND $\int \cot^n x \csc^m x\,dx$

Integrals of the above form can be evaluated with the help of the identities:

$$\sec^2 x = 1 + \tan^2 x \text{ and } \csc^2 x = 1 + \cot^2 x$$

Which follow directly from the Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \begin{cases} \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x \\ \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \cot^2 x = \csc^2 x \end{cases}$$

EXAMPLE 7.12 Evaluate:

(a) $\int \tan^3 x \sec^3 x\,dx$ (b) $\int \cot^3 x\,dx$ (c) $\int \tan^6 x\,dx$

SOLUTION: (a) Motivated by $(\sec x)' = \mathbf{\sec x \tan x}$ we have:

$$\begin{aligned} \int \tan^3 x \sec^3 x\,dx &= \int \tan^2 x \sec^2 x\,\mathbf{\sec x \tan x}\,dx \\ &= \int (\sec^2 x - 1)\sec^2 x\,\mathbf{\sec x \tan x}\,dx \\ &= \int \sec^4 x\,\mathbf{\sec x \tan x}\,dx - \int \sec^2 x\,\mathbf{\sec x \tan x}\,dx \end{aligned}$$

$u = \sec x$
$du = \sec x \tan x\,dx$

$$= \int u^4 du - \int u^2 du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

(b) Motivated by $(\cot x)' = -\csc^2 x$ we have:

$$\begin{aligned} \int \cot^3 x\,dx = \int \cot x \cot^2 x\,dx &= \int \cot x(\csc^2 x - 1)dx \\ &= \int \cot x \csc^2 x\,dx - \int \cot x\,dx \end{aligned}$$

$u = \cot x$
$du = -\csc^2 x\,dx$

$$= -\int u\,du - \int \frac{\cos x}{\sin x}dx$$

$v = \sin x$
$dv = \cos x\,dx$

$$\begin{aligned} &= -\frac{u^2}{2} - \int \frac{dv}{v} \\ &= -\frac{\cot^2 x}{2} - \ln|v| + C \\ &= -\frac{\cot^2 x}{2} - \ln|\sin x| + C \end{aligned}$$

(c) Motivated by $(\tan x)' = \sec^2 x$ we have

$$\int \tan^6 x\,dx = \int \tan^4 x \tan^2 x\,dx = \int \tan^4 x(\sec^2 x - 1)\,dx$$

$$= \int (\tan^4 x\sec^2 x - \tan^4 x)dx$$

$$= \int (\tan^4 x\sec^2 x - \tan^2 x\tan^2 x)dx$$

$$= \int [\tan^4 x\sec^2 x - \tan^2 x(\sec^2 x - 1)]dx$$

$$= \int (\tan^4 x\sec^2 x - \tan^2 x\sec^2 x + \tan^2 x)dx$$

$$= \int (\tan^4 x\sec^2 x - \tan^2 x\sec^2 x + \sec^2 x - 1)dx$$

$u = \tan x$

$du = \sec^2 x dx$

$$= \int u^4 du - \int u^2 du + \int \sec^2 x dx - \int 1 dx$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + \tan x - x + C$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$$

CHECK YOUR UNDERSTANDING 7.13

Evaluate:

$$\int (\tan x)^{-3/2}\sec^4 x dx$$

Answer:

$$-\frac{2}{\sqrt{\tan x}} + \frac{2(\tan x)^{3/2}}{3} + C$$

INTEGRALS INVOLVING $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, and $\sqrt{x^2 - a^2}$

Here is some good advice:

If the integral involves	use the substitution	and the identity	to replace
(1) $\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\cos^2\theta = 1 - \sin^2\theta$	$a^2 - x^2$ **with** $a^2\cos^2\theta$ Note that $\cos\theta$ is positive in the specified range.
(2) $\sqrt{x^2 + a^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sec^2\theta = 1 + \tan^2\theta$	$x^2 + a^2$ **with** $a^2\sec^2\theta$ Note that $\sec\theta$ is positive in the specified range.
(3) $\sqrt{x^2 - a^2}$	$x = a\sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2} \\ \text{or} \\ \pi \le \theta < \frac{3\pi}{2} \end{cases}$	$\tan^2\theta = \sec^2\theta - 1$	$x^2 - a^2$ **with** $a^2\tan^2\theta$ Note that $\tan\theta$ is positive in the specified range.

Figure 7.2

Consider the following example.

EXAMPLE 7.13 Evaluate:

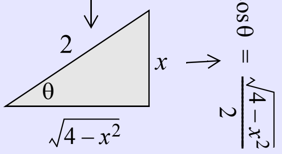
(a) $\int\sqrt{4-x^2}dx$ (b) $\int\frac{dx}{\sqrt{x^2+4}}$ (c) $\int\frac{dx}{x^2\sqrt{x^2-4}}$

SOLUTION: (a)

$x = 2\sin\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\boldsymbol{dx = 2\cos\theta d\theta}$

$\sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}\frac{x}{2}$



Note that $\cos\theta > 0$ when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int\sqrt{4-x^2}\,\boldsymbol{dx} \underset{\uparrow}{=} \int\sqrt{4\cos^2\theta}\,\boldsymbol{2\cos\theta d\theta} = \int 4\cos^2\theta d\theta$$

See (1) of Figure 7.2, and margin

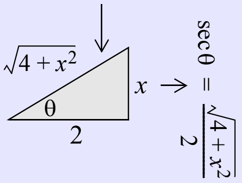
$$= 4\int\frac{1+\cos 2\theta}{2}d\theta$$
$$= 2\int(1+\cos 2\theta)d\theta$$
$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$
$$= 2\theta + \sin 2\theta + C$$
$$= 2\theta + 2\sin\theta\cos\theta + C$$
$$= 2\left(\sin^{-1}\frac{x}{2}\right) + 2\left[\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}\right] + C$$
$$= 2\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C$$

(b)

$x = 2\tan\theta$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2\theta d\theta$

$\theta = \tan^{-1}\frac{x}{2} \Rightarrow \tan\theta = \frac{x}{2}$



See (2) of Figure 7.2, and margin

$$\int\frac{dx}{\sqrt{x^2+4}}\boldsymbol{dx} \underset{\downarrow}{=} \int\frac{\boldsymbol{2\sec^2\theta}}{\sqrt{4\sec^2\theta}}\boldsymbol{d\theta} = \int\sec\theta d\theta$$

Theorem 6.4(c), page 227: $= \ln|\sec\theta + \tan\theta| + C$

See the triangle in the margin: $= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

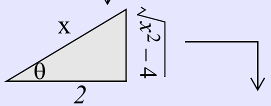
$$= \ln\left|\frac{x+\sqrt{4+x^2}}{2}\right| + C$$
$$= \ln\left|x+\sqrt{4+x^2}\right| + C$$

(where did the 2 go?)

$x = 2\sec\theta$, with $0 < \theta < \frac{\pi}{2}$

$\boldsymbol{dx = 2\sec\theta\tan\theta d\theta}$

$\theta = \sec^{-1}\frac{x}{2} \Rightarrow \sec\theta = \frac{x}{2}$



$\sin\theta = \frac{\sqrt{x^2-4}}{x}$

See (3) of Figure 7.2, and margin

$$\text{(c)} \int\frac{\boldsymbol{dx}}{x^2\sqrt{x^2-4}} \underset{\downarrow}{=} \int\frac{\boldsymbol{2\sec\theta\tan\theta}}{(2\sec\theta)^2\sqrt{4\tan^2\theta}}\boldsymbol{d\theta}$$
$$= \int\frac{2\sec\theta\tan\theta}{4\sec^2\theta\,2\tan\theta}d\theta = \frac{1}{4}\int\frac{1}{\sec\theta}d\theta$$
$$= \frac{1}{4}\int\cos\theta d\theta = \frac{1}{4}\sin\theta + C$$

See the triangle in the margin: $= \frac{\sqrt{4-x^2}}{4x} + C$

CHECK YOUR UNDERSTANDING 7.14

Evaluate:

$$\int \frac{\sqrt{9-x^2}}{x^2}dx$$

Answer:
$-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

We offer some additional examples for your consideration:

EXAMPLE 7.14 Evaluate:

$$\text{(a)}\int \frac{dx}{x^2\sqrt{4-x^2}} \qquad \text{(b)} \int \frac{dx}{x\sqrt{x^4-4}}$$

SOLUTION:

(a)

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta d\theta}{(4\sin^2\theta)(2\cos\theta)} = \int \frac{d\theta}{4\sin^2\theta}$$

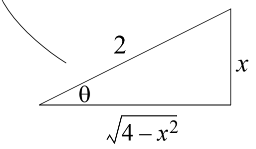
$$= \frac{1}{4}\int \csc^2\theta d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

$x = 2\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$dx = 2\cos\theta d\theta$
$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$
$= 2\sqrt{1-\sin^2\theta}$
$= 2\cos\theta$

From $x = 2\sin\theta$: $\sin\theta = \frac{x}{2}$



(b) Our first step is to mold the $\sqrt{x^4-4}$ into the form $\sqrt{u^2-4}$ and then hope for the best:

$u = x^2 \Rightarrow x = \sqrt{u}, \quad du = 2xdx \Rightarrow dx = \frac{du}{2x} = \boldsymbol{\frac{du}{2\sqrt{u}}}$

$$\int \frac{dx}{x\sqrt{x^4-4}} = \int \frac{\boldsymbol{du}}{\sqrt{u}\sqrt{u^2-4}\cdot \boldsymbol{2\sqrt{u}}}$$

$$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-4}} = \frac{1}{2}\int \frac{2\sec\theta\tan\theta}{2\sec\theta\sqrt{4\tan^2\theta}}d\theta$$

(3) Figure 7.2

$u = 2\sec\theta, \ du = 2\sec\theta\tan\theta d\theta$
$\sec\theta = \frac{u}{2} \Rightarrow \theta = \sec^{-1}\frac{u}{2}$

$$= \frac{1}{4}\int d\theta = \frac{\theta}{4} + C$$

$$= \frac{1}{4}\sec^{-1}\frac{u}{2} + C$$

$$= \frac{1}{4}\sec^{-1}\frac{x^2}{2} + C$$

CHECK YOUR UNDERSTANDING 7.15

Evaluate:

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

Answer:
$\frac{(x^2-8)\sqrt{x^2+4}}{3}+C$

Operating under the illusion that one cannot have enough of a good thing:

EXAMPLE 7.15

Evaluate:

(a) $\int\sqrt{x^2+2x+2}\,dx$ (b) $\int\frac{x^3+1}{(x^2+4)^2}dx$

SOLUTION: (a) Turning to the completing the square method of the previous section we have:

$$\int\sqrt{x^2+2x+2}\,dx = \int\sqrt{(x^2+2x+1)+(2-1)}\,dx = \int\sqrt{(x+1)^2+1}\,dx$$

Motivated by the identity $\tan^2\theta+1 = \sec^2\theta$:

$$\int\sqrt{(x+1)^2+1}\,dx \underset{x+1\,=\,\tan\theta,\ x\,=\,\tan\theta-1,\ dx\,=\,\sec^2\theta d\theta}{=} \int\sqrt{\tan^2\theta+1}\sec^2\theta d\theta = \int\sec^3\theta d\theta$$

Turning to the Integration by parts technique of Section 1:

$$\int\sec^3\theta d\theta = \int\sec\theta\sec^2\theta d\theta = \int u dv = uv-\int v du$$

$u = \sec\theta \qquad dv = \sec^2\theta d\theta$
$du = \sec\theta\tan\theta d\theta \qquad v = \tan\theta$

$$= \sec\theta\tan\theta-\int\sec\theta\tan^2\theta d\theta$$
$$= \sec\theta\tan\theta-\int\sec\theta(\sec^2\theta-1)d\theta$$
$$= \sec\theta\tan\theta+\int\sec\theta d\theta-\int\sec^3\theta d\theta$$

Focusing on the start and end of the above development we have:

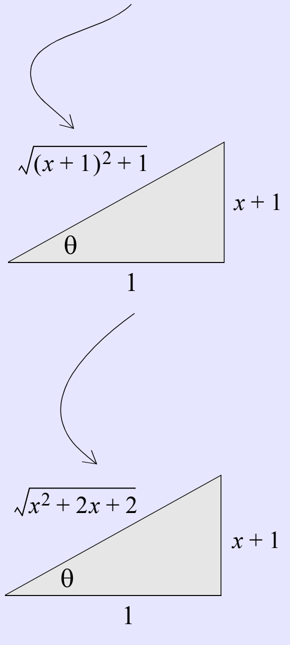
$$\int\sec^3\theta d\theta = \sec\theta\tan\theta+\int\sec\theta d\theta-\int\sec^3\theta d\theta$$

$$2\int\sec^3\theta d\theta = \sec\theta\tan\theta+\int\sec\theta d\theta = \sec\theta\tan\theta+\underset{\text{Theorem 6.4(c), page 227)}}{\ln}|\sec\theta+\tan\theta|+C$$

$$\int\sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)+C$$

see margin: $= \frac{1}{2}\left[(x+1)\sqrt{x^2+2x+2}+\ln\left|\sqrt{x^2+2x+2}+x+1\right|\right]+C$

$x+1 = \tan\theta$



$\sec\theta = \sqrt{x^2+2x+2}$

Conclusion:

$$\int\sqrt{x^2+2x+2}\,dx$$
$$= \frac{1}{2}\left[(x+1)\sqrt{x^2+2x+2}+\ln\left|\sqrt{x^2+2x+2}+x+1\right|\right]+C$$

(b) Turning to Figure 7.1(v) of page 273 we have:

$$\frac{x^3+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$x^3+1 = (Ax+B)(x^2+4)+Cx+D = Ax^3+Bx^2+(4A+C)x+(4B+D)$$

Equating coefficients of like powers of x:

$$\boxed{A = 1} \qquad \boxed{B = 0} \qquad 4A+C = 0 \qquad 4B+D = 1$$

$$4+C = 0 \Rightarrow \boxed{C = -4} \qquad \boxed{D = 1}$$

Bringing us to:

$$\int\frac{x^3+1}{(x^2+4)^2}dx = \int\frac{x}{x^2+4}dx + \int\frac{-4x+1}{(x^2+4)^2}dx$$

$$= \int\frac{\mathbf{x}}{\mathbf{x^2+4}}\mathbf{dx} - \int\frac{\mathbf{4x}}{\mathbf{(x^2+4)^2}}\mathbf{dx} + \int\frac{\mathbf{1}}{\mathbf{(x^2+4)^2}}\mathbf{dx}$$

Then:

$$\int\frac{\mathbf{x}}{\mathbf{x^2+4}}\mathbf{dx} = \frac{1}{2}\int\frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$\boxed{\begin{aligned} u &= x^2+4 \\ du &= 2xdx \end{aligned}}$$

$$= \frac{1}{2}\ln(x^2+4) + C$$

and

$$-\int\frac{\mathbf{4x}}{\mathbf{(x^2+4)^2}}dx = -2\int\frac{du}{u^2}$$

$$= \frac{2}{u} + C$$

$$= \frac{2}{x^2+4} + C$$

$$\int\frac{\mathbf{dx}}{\mathbf{(x^2+4)^2}} = \int\frac{2\sec^2\theta}{16\sec^4\theta}d\theta$$

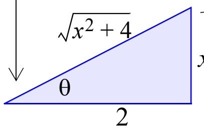
$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta d\theta \end{aligned}} \quad = \frac{1}{8}\int\cos^2\theta d\theta$$

Theorem 1.5(ix): (page 37) $= \frac{1}{8}\int\frac{1+\cos 2\theta}{2}d\theta$

$$= \frac{1}{16}\left(\int d\theta + \int\cos 2\theta d\theta\right)$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

Theorem 1.5(iv): (page 37) $= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C$



$$= \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}\right) + C$$

$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{8}\left(\frac{x}{x^2+4}\right) + C$$

A well-earned conclusion:

$$\int\frac{x^3+1}{(x^2+4)^2}dx = \frac{1}{2}\ln(x^2+4) + \frac{2}{x^2+4} + \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{8}\left(\frac{x}{x^2+4}\right) + C$$

CHECK YOUR UNDERSTANDING 7.16

Evaluate:

$$\int\frac{6x+11}{3x^2+2x+1}dx$$

Answer:

$$\ln(3x^2+2x+1) + \frac{9}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

-

EXERCISES

Exercise 1-35. Evaluate.

1. $\int \sin^2 x\, dx$

2. $\int \cos^2 x\, dx$

3. $\int \sin^3 3x dx$

4. $\int \frac{\cos^3 x}{\sin^2 x} dx$

5. $\int \cos^4 2x dx$

6. $\int (\sin^2 2x)(\cos^2 2x)\, dx$

7. $\int \sin^2 x \cos^4 x dx$

8. $\int \sin^4 x \cos^2 x\, dx$

9. $\int \tan^3 x dx$

10. $\int \sec^3 4x dx$

11. $\int \csc^4 2x dx$

12. $\int \tan^4 x dx$

13. $\int \frac{\sec^4 x}{\cot^3 x} dx$

14. $\int \tan 5x \sec^3 5x dx$

15. $\int \tan 5x \sec^4 5x dx$

16. $\int \cot 3x \csc^5 3x dx$

17. $\int x \sec x \tan x dx$

18. $\int \tan^6 x \sec^4 x dx$

19. $\int \frac{\tan^5 x}{\cos x} dx$

20. $\int \sqrt{\sec x} \tan x dx$

21. $\int \cot^3 x \csc^3 x dx$

22. $\int \sin^3 x \sqrt{\cos x} dx$

23. $\int \frac{1 - \tan^2 x}{\sec^2 x} dx$

24. $\int x \sin^3 x^2 \cos^2 x^2\, dx$

25. $\int \sqrt{\tan x} \sec^4 x\, dx$

26. $\int \sin x \cos \frac{x}{2} dx$

27. $\int_0^{\frac{\pi}{4}} \cos^3 x dx$

28. $\int_0^{\pi} \cos^2 2x dx$

29. $\int_0^{\frac{\pi}{3}} \sin^4 3x \cos^3 3x dx$

30. $\int_0^{\frac{\pi}{3}} \tan^5 x \sec^6 x dx$

31. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 x dx$

32. $\int_0^{\frac{\pi}{12}} \tan^2 3x dx$

33. $\int_0^{2\pi} \cos^4 x dx$

34. $\int_0^{\frac{\pi}{3}} x \sin^4 2x^2 \cos^3 2x^2 dx$

35. $\int_0^{\frac{\pi}{3}} 10 \sec^6 x dx$

Exercise 36-68. (Trigonometric Substitution) Evaluate.

36. $\int \frac{dx}{\sqrt{4 - x^2}}$

37. $\int \frac{x^2}{\sqrt{1 - x^2}} dx$

38. $\int \frac{dx}{\sqrt{x^2 + 16}}$

39. $\int \frac{x}{\sqrt{9-x^2}}dx$

40. $\int \frac{x^3 dx}{\sqrt{x^2+1}}$

41. $\int \frac{dx}{x^2\sqrt{25-x^2}}$

42. $\int \frac{\sqrt{x^2-1}}{x^2}dx$

43. $\int \frac{dx}{x\sqrt{4x^2+9}}$

44. $\int \frac{dx}{\sqrt{16-x^2}}$

45. $\int \frac{dx}{x^2\sqrt{4x^2-9}}$

46. $\int \frac{dx}{x^2\sqrt{x^2-9}}$

47. $\int \frac{dx}{(9x^2-1)^{3/2}}$

48. $\int \sqrt{1-4x^2}dx$

49. $\int \frac{x^3}{\sqrt{2-x^2}}dx$

50. $\int \frac{\sqrt{x^2-4}}{x}dx$

51. $\int \frac{\sqrt{x^2+16}}{x^4}dx$

52. $\int \frac{dx}{x\sqrt{5-x^2}}$

53. $\int \sqrt{1-4x^2}dx$

54. $\int \sqrt{4-(x+2)^2}dx$

55. $\int e^{2x}\sqrt{1-e^{2x}}dx$

56. $\int \frac{dx}{(x-2)\sqrt{(x-2)^2+9}}$

57. $\int \sqrt{21+4x-x^2}dx$

58. $\int \frac{dx}{\sqrt{x^2-6x+13}}$

59. $\int \frac{x}{\sqrt{x^2+x+1}}dx$

60. $\int \frac{x^2}{(x^2+1)(x^2+4)^2}dx$

61. $\int \frac{2x^3-8x^2+20x-5}{x^2-4x+8}dx$

62. $\int \frac{x^4+x^3+8x^2+15}{x(x^2+4)^2}dx$

63. $\int_{2\sqrt{2}}^{4} \frac{4}{\sqrt{x^2-4}}dx$

64. $\int_0^1 x\sqrt{1-x^2}dx$

65. $\int_0^1 \frac{3}{\sqrt{16+9x^2}}dx$

66. $\int_0^1 \frac{dx}{2x^2+3}$

67. $\int_1^{\frac{4}{3}} \frac{10}{(25x^2-16)^{3/2}}dx$

68. $\int_{\sqrt{5}}^{\sqrt{10}} \frac{2}{(x^2-1)^{3/2}}dx$

69. **(Area)** Find the area enclosed by the graph of the function $f(x) = \frac{\sqrt{9-x^2}}{3}$.

70. **(Area)** Find the area of the region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

71. **(Area)** Find the area of the region between $y = \sin x$ and $y = \sin^2 x$ for $0 \le x \le 2\pi$.

72. **(Volume)** Find the volume of the solid obtained by rotating the finite region bounded by the graph of the function $f(x) = x\sqrt{1-x^2}$ and the x-axis about the line $x = 1$.

73. **(Volume)** Find the volume of the solid obtained by rotating the region bounded by the graph of the function $f(x) = \frac{4}{x^2+4}$ over the interval $[0, 2]$ about the x-axis.

74. **(Volume)** Find the volume of the torus obtained by rotating the region bounded by the circle $x^2+(y-4)^2 = 4$ about the x-axis.

75. **(Volume)** Show that the volume of the torus obtained by rotating the region bounded by the circle $x^2+(y-R)^2 = r^2$ about the x-axis is given by $2\pi^2 Rr^2$, where $r \in R$.

Exercise 76-77. (Theory) Establish the given integral formula.

76. For m a positive even integer: $\int \tan^n x \sec^m x\,dx = \int (1+\tan^2 x)^{\frac{m-2}{2}} (\tan^n x)\sec^2 x\,dx$

(Note that since $(\tan x)' = \sec^2 x$, and since the above integral is of the form $\int p(\tan x)\sec^2 x\,dx$ for a polynomial p, that integral can always be evaluated.)

77. For n a positive odd integer and $m \geq 1$:

$$\int \tan^n x \sec^m x\,dx = \int \sec^{m-1} x(\sec^2 x - 1)^{\frac{n-1}{2}} \sec x \tan x\,dx$$

(Note that since $(\sec x)' = \sec x \tan x$, and since the above integral is of the form $\int p(\sec x)\sec x \tan x\,dx$ for a polynomial p, that integral can always be evaluated.)

§4. A HODGEPODGE OF INTEGRALS

While we offer a few additional integral examples in this section, the primary reason for its inclusion is the exercises. In each case we suggest that you take a good look at the integral and ask yourself:

Is it a straight forward integral, like $\int 5\sec x\tan x\,dx$?

Will a u-substitution crack the case, as with $\int \frac{5\sin x}{\sqrt{3\cos^5 x}}dx$?

How about integration by parts: $\int x^2e^{3x}dx$?

Completing the square: $\int \frac{dx}{x^2+6x+25}$?

Partial fractions: $\int \frac{x+2}{x^4+2x^3-3x^2}dx$?

Will a trig-identity help: $\int \sin^2 x\cos^4 x\,dx$?

How about a trig-substitution: $\int \sqrt{x^2-4}\,dx$?

And what if none of the above help?

Answer: ***GIVE IT YOUR BEST SHOT***!

EXAMPLE 7.16 Evaluate:

$$\int \frac{dx}{1+\sqrt{x}}$$

SOLUTION:

$$\begin{array}{r} 2 \\ u+1\overline{)\,2u} \\ \underline{2u+2} \\ -2 \end{array}$$

$$\int \frac{dx}{1+\sqrt{x}} \underset{\uparrow}{=} \int \frac{\mathbf{2u}}{1+u}\mathbf{du} \overset{\substack{\text{margin}\\ \downarrow}}{=} \int\left(2-\frac{2}{u+1}\right)du$$

$$\begin{aligned} u &= x^{1/2} \\ du = \frac{1}{2x^{1/2}}dx \Rightarrow dx &= 2x^{1/2}du \\ &= \mathbf{2udu} \end{aligned}$$

$$\begin{aligned} &= 2\left(\int du + \int\frac{du}{1+u}\right) \\ &= 2u + 2\ln|1+u| + C \\ &= 2\sqrt{x} + 2\ln(1+\sqrt{x}) + C \end{aligned}$$

EXAMPLE 7.17 Evaluate:

$$\int \frac{x^{1/2}}{x^{1/3}+4}dx$$

SOLUTION: Our initial substitution $x = u^6$ will take us to an expression involving only integer exponents:

Note that $x^{1/2}$ and $x^{1/3}$ are powers of $x^{1/6}$.

$x = u^6 \Rightarrow x^{1/2} = u^3$ and $x^{1/3} = u^2$

$dx = \mathbf{6u^5du}$

$$\int \frac{x^{1/2}}{x^{1/3}+4}dx = \mathbf{6}\int \frac{u^3\boldsymbol{u^5}}{u^2+4}\boldsymbol{du}$$

$$= 6\int \frac{u^8}{u^2+4}du$$

$$\begin{array}{r} u^6-4u^4+16u^2-64 \\ u^2+4\,\big)\,u^8 \qquad\qquad\qquad \\ \underline{u^8+4u^6} \qquad\qquad \\ -4u^6 \qquad\qquad \\ \underline{-4u^6-16u^4} \qquad \\ 16u^4 \qquad \\ \underline{16u^4+64u^2} \\ -64u^2 \\ \underline{-64u^2-256} \\ 256 \end{array}$$

see margin: $$= 6\int\left(u^6-4u^4+16u^2-64+\frac{256}{u^2+4}\right)du$$

$$= 6\left(\frac{u^7}{7}-4\frac{u^5}{5}+16\frac{u^3}{3}-64u\right)+\frac{6(256)}{4}\int\frac{du}{\left(\frac{u}{2}\right)^2+1}$$

$$= 6\left(\frac{u^7}{7}-4\frac{u^5}{5}+16\frac{u^3}{3}-64u\right)+384\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$= 6\left(\frac{x^{7/6}}{7}-\frac{4x^{5/6}}{5}+\frac{16x^{1/2}}{3}-64x^{1/6}\right)+384\tan^{-1}\left(\frac{x^{1/6}}{2}\right)+C$$

EXAMPLE 7.18

Evaluate:
$$\int\frac{x^2-4x+6}{x^2-4x+5}dx$$

SOLUTION: Try partial fractions:

$$\frac{x^2-4x+6}{x^2-4x+5} = 1+\frac{1}{x^2-4x+5} \quad \text{(see margin)}$$

$$\begin{array}{r} 1 \\ x^2-4x+5\,\big)\,x^2-4x+6 \\ \underline{x^2-4x+5} \\ 1 \end{array}$$

Or, more simply:
$$\frac{x^2-4x+6}{x^2-4x+5} = \frac{(x^2-4x+5)+1}{x^2-4x+5}$$

But that's as far as it goes, since x^2-4x+5 is irreducible.

If you try: $\frac{1}{x^2-4x+5} = \frac{Ax+B}{x^2-4x+5}$, you will just find that $A = 0$ and $B = 1$.

So: $$\int\frac{x^2-4x+6}{x^2-4x+5}dx = \int\left(1+\frac{1}{x^2-4x+5}\right)dx = x+\int\frac{dx}{x^2-4x+5}$$

And then complete the square:

$$\int\frac{dx}{x^2-4x+5} = \int\frac{dx}{(x^2-4x+4)+1} = \int\frac{dx}{(x-2)^2+1}$$

$u = x-2, du = dx$: $$= \int\frac{du}{u^2+1} = \tan^{-1}u+C$$

$$= \tan^{-1}(x-2)+C$$

Conclusion: $$\int\frac{x^2-4x+6}{x^2-4x+5}dx = x+\tan^{-1}(x-2)+C$$

EXAMPLE 7.19

Evaluate:

$$\int \frac{dx}{(x+3)\sqrt{x+1}}$$

SOLUTION:

$$\int \frac{dx}{(x+3)\sqrt{x+1}} \uparrow = 2\int \frac{u}{(\mathbf{u^2-1}+3)u}du = 2\int \frac{du}{u^2+2}$$

$u = \sqrt{x+1} \Rightarrow u^2 = x+1$

$\mathbf{x = u^2 - 1}$

$dx = 2udu$

$$= \frac{2}{2}\int \frac{du}{\left(\frac{u}{\sqrt{2}}\right)^2 + 1}$$

$w = \frac{u}{\sqrt{2}},\ dw = \frac{du}{\sqrt{2}}:$

$$= \sqrt{2}\int \frac{dw}{w^2+1}$$

$$= \sqrt{2}\tan^{-1} w + C$$

$$= \sqrt{2}\tan^{-1}\frac{u}{\sqrt{2}} + C$$

$$= \sqrt{2}\tan^{-1}\frac{\sqrt{x+1}}{\sqrt{2}} + C$$

EXAMPLE 7.20

Evaluate:

$$\int \sin 3x \sin 2x dx$$

SOLUTION: While one can use the integration-by-parts method, an easier procedure for this (and other integrals) is available; namely:

To evaluate integrals of type:

$\int \sin mx \sin nx dx \qquad \int \sin mx \cos nx dx \qquad \int \cos mx \cos nx dx$

use the identities (Exercise 63):

$$\begin{aligned} 2\sin A \sin B &= \cos(A-B) - \cos(A+B) \\ 2\sin A \cos B &= \sin(A+B) + \sin(A-B) \\ 2\cos A \cos B &= \cos(A-B) + \cos(A+B) \end{aligned}$$

In particular:

$$\int \sin 3x \sin 2x dx = \frac{1}{2}\int(\cos x - \cos 5x)dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$$

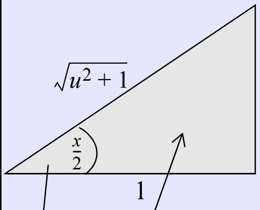
EXAMPLE 7.21

Evaluate:

$$\int \frac{dx}{\sin x + \cos x}$$

SOLUTION: This one also calls for a helping hand:

The integral of a rational function of $\sin x$ or $\cos x$ (or both) can be turned into a rational function of u via the substitution $u = \tan\frac{x}{2}$. How? Like this:



$$\sin\frac{x}{2} = \frac{u}{\sqrt{u^2+1}} \text{ and } \cos\frac{x}{2} = \frac{1}{\sqrt{u^2+1}}$$

From: $\tan\frac{x}{2} = \frac{u}{1}$

From Theorem 1.5(ii) and (iii), page 37:

$$\mathbf{\sin x} = \sin\left(\frac{x}{2}+\frac{x}{2}\right) = \sin\frac{x}{2}\cos\frac{x}{2} + \cos\frac{x}{2}\sin\frac{x}{2} = \mathbf{\frac{2u}{u^2+1}}$$

$$\mathbf{\cos x} = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos\frac{x}{2}\cos\frac{x}{2} - \sin\frac{x}{2}\sin\frac{x}{2} = \mathbf{\frac{1-u^2}{u^2+1}}$$

Also: $u = \tan\frac{x}{2} \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \Rightarrow \mathbf{dx} = \frac{2du}{\sec^2\left(\frac{x}{2}\right)} = \mathbf{\frac{2du}{u^2+1}}$

Summarizing:

If $\boxed{u = \tan\frac{x}{2}}$ then: $\sin x = \frac{2u}{u^2+1}$, $\cos x = \frac{1-u^2}{u^2+1}$, and $dx = \frac{2du}{u^2+1}$

In particular: $$\int\frac{dx}{\sin x + \cos x} \overset{u = \tan\frac{x}{2}}{=} \int\frac{\frac{2du}{u^2+1}}{\frac{2u}{u^2+1}+\frac{1-u^2}{u^2+1}} = -\underbrace{\int\frac{2}{u^2-2u-1}du}_{(*)}$$

If c is a zero of a polynomial $p(x)$, then $x-c$ is a factor.

We now set our sights on obtaining a partial fraction decomposition of the rational expression (*). Applying the quadratic formula to the polynomial u^2-2u-1 we see that it has two zeros: $1 \pm \sqrt{2}$. From Theorem 1.4, page 19 (see margin), we conclude that $u^2-2u-1 = (u-1-\sqrt{2})(u-1+\sqrt{2})$. An so we have:

$$\frac{2}{u^2-2u-1} = \frac{2}{(u-1-\sqrt{2})(u-1+\sqrt{2})} = \frac{A}{(u-1-\sqrt{2})} + \frac{B}{(u-1+\sqrt{2})}$$

Clearing denominators:

$$\mathbf{2 = A(u-1+\sqrt{2}) + B(u-1-\sqrt{2})}$$

letting $\boldsymbol{u} = \mathbf{1+\sqrt{2}}$: $2 = A(\mathbf{1+\sqrt{2}}-1+\sqrt{2})$

$$2 = A(2\sqrt{2}) \Rightarrow A = \frac{1}{\sqrt{2}}$$

letting $u = \mathbf{1-\sqrt{2}}$: $2 = B(\mathbf{1-\sqrt{2}}-1-\sqrt{2})$

$$2 = B(-2\sqrt{2}) \Rightarrow B = -\frac{1}{\sqrt{2}}$$

Putting it all together, we have:

$$\begin{aligned}\int\frac{dx}{\sin x+\cos x} &= -\int\frac{2}{u^2-2u-1}du\\ &= -\left[\int\frac{1/\sqrt{2}}{(u-1-\sqrt{2})}du-\int\frac{1/\sqrt{2}}{(u-1+\sqrt{2})}du\right]\\ &= \frac{1}{\sqrt{2}}\ln\left|u-1+\sqrt{2}\right|-\frac{1}{\sqrt{2}}\ln\left|u-1-\sqrt{2}\right|+C\\ &= \frac{1}{\sqrt{2}}\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|+C \underset{\substack{\uparrow\\ u\,=\,\tan\frac{x}{2}}}{=} \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1-\sqrt{2}}\right|+C\end{aligned}$$

SOME CLOSING REMARKS

1. Though $\int_a^b f(x)dx$ exists for every f that is continuous on $[a, b]$, an antiderivative of f need not exist.

A case in point:

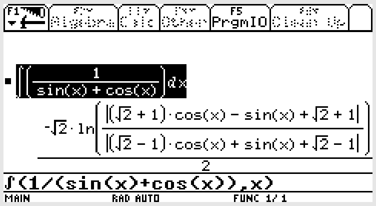
If $g(x) = \begin{cases}1 & \text{if } x\neq 1\\ 2 & \text{if } x = 1\end{cases}$, then $\int_0^2 g(x) = 2$ (Exercise 59. page 188). However, since $g(x)$ is not continuous, it has no antiderivative (Theorem 3.1, page 73).

2. While the Principal Theorem of Calculus (page 178) assures us that every continuous function f has an antiderivative, $\left(\int_a^x f(t)dt\right)' = f(x)$ it does not guarantee that the antiderivative has to be "nice."

A case in point:

Up to this point we have dealt exclusively with **elementary functions**: functions that can be expressed as a sum/difference, product/quotient, and composition of polynomials, rational, trigonometric, inverse trigonometric, exponential, and logarithmic functions (along with a few others). As it turns out, while the relatively nice continuous function $f(x) = e^{x^2}$ has an antiderivative, that antiderivative is not an elementary function. Later, we will see how its antiderivative can be represented by an infinite series.

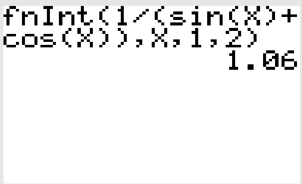
3. In the not-too distant past, every calculus book contained an extensive list of Integral Formulas (some still do). Before then, there were also trigonometric tables, and even square-root tables. Now they are gone. We've become digitalized, and rely more and more on graphing calculators or other instruments to perform many routine tasks. A case in point:



$$\int\left(\frac{1}{\sin(x)+\cos(x)}\right)dx \qquad \frac{-\sqrt{2}\cdot\ln\left(\frac{|(\sqrt{2}+1)\cdot\cos(x)-\sin(x)+\sqrt{2}+1|}{|(\sqrt{2}-1)\cdot\cos(x)+\sin(x)+\sqrt{2}-1|}\right)}{2}$$

```
∫(1/(sin(x)+cos(x)),x)
```

TI-89 through TI-voyage



```
fnInt(1/(sin(X)+
cos(X)),X,1,2)
                1.06
```

TI-83 through TI-84+

Note that the above TI-89 integral looks quite different than the one we arrived at in Example 7.21:

$$\frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1-\sqrt{2}}\right|+C \quad \text{versus} \quad -\sqrt{2}\ln\left(\frac{|(\sqrt{2}+1)\cdot\cos x-\sin(x)+\sqrt{2}+1|}{|(\sqrt{2}-1)\cdot\cos x-\sin(x)+\sqrt{2}-1|}\right)$$

Appearances aside, unless we or the calculator made a mistake (not as likely), the two expressions can, at most, differ by a constant (why?).

EXERCISES

Exercise 1-62. Evaluate by any method.

1. $\int \frac{x^2 - 3x + 2}{x^2} dx$
2. $\int \frac{x^2}{x^2 - 3x + 2} dx$
3. $\int \frac{x^2}{x^2 + 2x + 5} dx$
4. $\int \frac{dx}{x^4 + 1}$
5. $\int \frac{dx}{1 + 3x^2}$
6. $\int e^{2x} \cos 3x dx$
7. $\int x^5 e^{-x^3} dx$
8. $\int \frac{e^{2x}}{e^x + 1} dx$
9. $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$
10. $\int x^3 e^{-x} dx$
11. $\int \sin(\ln 2x) dx$
12. $\int x \tan^{-1} x dx$
13. $\int \frac{e^x}{e^{2x} + e^x - 2} dx$
14. $\int (2x^2 + 1) e^{x^2} dx$
15. $\int (x + 2) \sqrt{x - 5} dx$
16. $\int \frac{\sqrt{x}}{1 + x^{1/3}} dx$
17. $\int \frac{\sqrt{x}}{1 - x^{1/3}} dx$
18. $\int \frac{dx}{\sqrt{x} + \sqrt{x + 1}}$
19. $\int \frac{dx}{\sqrt{3x - x^2}}$
20. $\int \frac{\sqrt{x^2 - 1}}{x^2} dx$
21. $\int \frac{\sqrt{x^2 - 1}}{x} dx$
22. $\int \frac{dx}{\sqrt{x^2 - 2x + 10}}$
23. $\int \frac{x^2}{(x^2 + 9)^{3/2}} dx$
24. $\int \frac{dx}{(4 + 9x^2)^{5/2}}$
25. $\int \frac{dx}{(x^2 - 2x + 10)^{3/2}}$
26. $\int \frac{dx}{9x^2 - 36x + 52}$
27. $\int \frac{dx}{4x^2 + 8x + 29}$
28. $\int \frac{\cos x}{\sin^2 x + \sin x - 6} dx$
29. $\int \frac{\tan^{-1} x}{x^2} dx$
30. $\int \frac{\sin^2 x}{(1 + \cos x)^2} dx$
31. $\int \sec^3 x dx$
32. $\int \frac{\cos x}{(5 + 4\cos x)^2} dx$
33. $\int \frac{1 + \sin x}{1 - \sin x} dx$
34. $\int \frac{dx}{\sin 2x - \sin x}$
35. $\int \frac{\sin x \cos 2x}{\sin x + \sec x} dx$
36. $\int \frac{dx}{\sin^3 x}$
37. $\int \frac{\sec x}{2\tan x + \sec x - 1} dx$
38. $\int_0^{\pi} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx$
39. $\int_1^{e} \frac{dx}{x\sqrt{1 + (\ln x)^2}}$
40. $\int_1^{\sqrt{3}} \frac{3x^2 + x + 4}{x^3 + x} dx$
41. $\int_0^{1} \frac{dx}{x^3 + x^2 + x + 1}$
42. $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{\cos x}{\sin x \cos x + \sin x} dx$

43. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x}$

44. $\displaystyle\int_{-2}^{2} \ln(x + 3)\,dx$

45. $\displaystyle\int_0^2 xe^{2x}\,dx$

46. $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$

47. $\displaystyle\int_0^{\frac{\pi}{2}} x \sin 4x\,dx$

48. $\displaystyle\int_{-\pi}^{\pi} \cos^2 5x\,dx$

49. $\displaystyle\int_0^{\frac{\pi}{6}} \tan^2 2x\,dx$

50. $\displaystyle\int_0^{\frac{\pi}{6}} \sec^3 x \tan x\,dx$

51. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2 - 1}}$

52. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$

53. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{dx}{1 - \cos x}$

54. $\displaystyle\int_0^{\frac{\pi}{4}} \sin 5x \sin 3x\,dx$

55. $\displaystyle\int_{-\frac{\pi}{10}}^{0} \sin 2x \cos 3x\,dx$

56. $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} x \sec^2 x^2\,dx$

57. $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 3x\,dx$

58. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1 + 3\tan x}}\,dx$

59. $\displaystyle\int_{-1}^{0} x(x + 1)^{\frac{1}{3}}\,dx$

60. $\displaystyle\int_1^{16} \frac{\sqrt{x}}{\sqrt[4]{x} + 1}\,dx$

61. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \cos x}\,dx$

62. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3 + 2\cos x}$

63. Establish the given identity:

(a) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

(b) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

(c) $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$

CHAPTER SUMMARY

INTEGRATION BY PARTS	$\int u\,dv = uv - \int v\,du$ Basically, the integration by parts formula should be invoked when: **You can't perform:** $\int \underset{\textstyle\int u}{f(x)}\underset{dv}{g'(x)dx}$ **But you can this:** $\int \underset{\textstyle\int v}{g(x)}\underset{du}{f'(x)dx}$
THEOREM	$\int \ln x\,dx = x\ln x - x + C$
REDUCTION FORMULAS	$\int \sin^n x\,dx = -\frac{\cos x}{n}\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$ $\int \cos^n x\,dx = \frac{\sin x}{n}\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$
COMPLETING THE SQUARE	To turn: $x^2 + ax + \underset{\uparrow}{\boxed{?}}$ into a perfect square: this has to be the square of one-half the coefficient of x: $\left(\frac{a}{2}\right)^2$
PARTIAL FRACTIONS	To evaluate an integral of the form $\int \frac{p(x)}{q(x)}dx$ it may be necessary to represent the rational expression $\frac{p(x)}{q(x)}$ as a sum of a polynomial and rational expressions of the form: $\frac{A}{(ax+b)^n}$ or $\frac{Ax+B}{(ax^2+bx+c)^n}$ If the degree of the numerator, $p(x)$, is not less than that of the denominator, $q(x)$, then you should divide $q(x)$ into $p(x)$; after which you can express the remainder in terms of partial fractions of the form: $(ax+b)^n$ generates $\frac{A_1}{ax+b} + \ldots + \frac{A_n}{(ax+b)^n}$ $ax^2+bx+c)^n$ generates $\frac{A_1x+B_1}{ax^2+bx+c} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$ (See Figure 7.1, page 273, for details)

POWERS OF TRIGONOMETRIC FUNCTIONS

The Pythagorean identity $\sin^2 x + \cos^2 x = 1$ can be employed to evaluate integrals of the form $\int \sin^n x \cos^m x\,dx$ when at least one of the positive integer exponents, n and m, is **odd**.

In the event that both exponents are **even**, consider the identities:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For integrals of the form

$$\int \tan^n x \sec^m x\,dx \text{ and } \int \cot^n x \csc^m x\,dx$$

consider the identities:

$$\sec^2 x = 1 + \tan^2 x \quad \text{and} \quad \csc^2 x = 1 + \cot^2 x$$

To evaluate integrals of type:

$$\int \sin mx\ \sin nx\,dx, \int \sin mx\ \cos nx\,dx, \int (\cos mx\ \cos(nx))\,dx$$

use the identities:

$$\begin{aligned} 2\sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2\sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2\cos A \cos B &= \cos(A - B) + \cos(A + B) \end{aligned}$$

The integral of a rational function of $\sin x$ or $\cos x$ (or both) can be turned into a rational function of u via the substitution $u = \tan\frac{x}{2}$.

TRIGONOMETRIC SUBSTITUTION

The following table can be used to evaluate certain integrals that involve expressions of the form

$$\sqrt{a^2 - x^2},\ \sqrt{x^2 + a^2},\ \text{and } \sqrt{x^2 - a^2}$$

(1)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\cos^2\theta = 1 - \sin^2\theta$	$a^2 - x^2$ **with** $a^2\cos^2\theta$ Note that $\cos\theta$ is positive in the specified range.
(2)	$\sqrt{x^2 + a^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sec^2\theta = 1 + \tan^2\theta$	$x^2 + a^2$ **with** $a^2\sec^2\theta$ Note that $\sec\theta$ is positive in the specified range.
(3)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2} \\ \text{or} \\ \pi \le \theta < \frac{3\pi}{2} \end{cases}$	$\tan^2\theta = \sec^2\theta - 1$	$x^2 - a^2$ **with** $a^2\tan^2\theta$ Note that $\tan\theta$ is positive in the specified range.

CHAPTER 8

L'Hôpital's Rule and Improper Integrals

§1 L'HÔPITAL'S RULE

A limit such as $\lim\limits_{x \to 3} \frac{2x+5}{x+4}$ can be evaluated by direct substitution:

$$\lim_{x \to 3} \frac{2x+5}{x+4} = \lim_{x \to 3} \frac{2 \cdot 3 + 5}{3+4} = \frac{11}{7}$$

More interesting limits have previously been encountered and determined. For example:

$$\lim_{x \to 3} \frac{x^3 - 2x^2 - 3x}{x^2 + 2x - 15} = \frac{3}{2} \text{ (Example 2.1, page 44)}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ (Theorem 3.5, page 90)}$$

The above two limits are said to be limits of **indeterminate form of type "$\frac{0}{0}$."** The following method, established in Appendix B, page B-1, may be used to address such limits:

Guillaume Francois De l'Hôpital (1661-1704).

THEOREM 8.1

L'HÔPITAL'S RULE: "0/0" TYPE

Let c be a real number, or $\pm\infty$. Assume that, apart from c, f and g are differentiable on an open interval containing c with $g'(x) \neq 0$. If:

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

and if:

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$$

where L is a real number or $\pm\infty$, Then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L$$

For example:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos x}{1} = \lim_{x \to 0} \cos x = 1$$

(Compare with the proof of Theorem 3.5, page 90.)

Before turning to other examples, we want to emphasize that:

(1) When applying l'Hôpital's Rule to a limit, $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$, of indeterminate form, you differentiate f and g **separately**, you do **NOT** use the quotient rule.

(2) L'Hôpital's Rule only applies to limits of indeterminate form. In particular, $\lim\limits_{x \to 0} \frac{\sin x}{x+1} = \frac{0}{1} = 0$ — **IT IS NOT EQUAL TO** $\lim\limits_{x \to 0} \frac{(\sin x)'}{(x+1)'} = \lim\limits_{x \to 0} \frac{\cos x}{1} = 1$.

EXAMPLE 8.1

Use l'Hôpital's Rule to find:

(a) $\lim\limits_{x \to \frac{\pi}{3}} \frac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}}$ (b) $\lim\limits_{x \to 0} \frac{e^x - 1}{x^3}$

(c) $\lim\limits_{x \to \infty} \frac{x^{-4/3}}{\sin\left(\frac{1}{x}\right)}$

SOLUTION: Noting that, in each of the above we are dealing with an indeterminate form of type "$\frac{0}{0}$," we apply l'Hôpital's rule:

$$\text{(a)} \lim_{x \to \frac{\pi}{3}} \frac{\overset{\nearrow 0}{\cos x - \frac{1}{2}}}{\underset{\searrow 0}{x - \frac{\pi}{3}}} = \lim_{x \to \frac{\pi}{3}} \frac{\left(\cos x - \frac{1}{2}\right)'}{\left(x - \frac{\pi}{3}\right)'} = \lim_{x \to \frac{\pi}{3}} \frac{-\sin x}{1} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\text{(b)} \lim_{x \to 0} \frac{\overset{\nearrow 0}{e^x - 1}}{\underset{\searrow 0}{x^3}} = \lim_{x \to 0} \frac{(e^x - 1)'}{(x^3)'} = \lim_{x \to 0} \frac{e^x}{3x^2} \underset{\uparrow}{=} +\infty$$

since $\lim\limits_{x \to 0} e^x = 1$ and $\lim\limits_{x \to 0} 3x^2 = 0$ and $\frac{e^x}{3x^2} > 0$ for $x \neq 0$

$$\text{(c)} \lim_{x \to \infty} \frac{\overset{\nearrow 0}{x^{-4/3}}}{\underset{\searrow 0}{\sin\left(\frac{1}{x}\right)}} = \lim_{x \to \infty} \frac{(x^{-4/3})'}{\left[\sin\left(\frac{1}{x}\right)\right]'} = \lim_{x \to \infty} \frac{-\frac{4}{3}x^{-7/3}}{\cos\left(\frac{1}{x}\right) \cdot (-x^{-2})}$$

$$= \lim_{x \to \infty} \frac{\frac{4}{3}x^{-1/3}}{\cos\left(\frac{1}{x}\right)} = \frac{0}{1} = 0$$

CHECK YOUR UNDERSTANDING 8.1

Evaluate:

(a) $\lim\limits_{x \to 0} \frac{(8+x)^{\frac{1}{3}} - 2}{x}$ (b) $\lim\limits_{x \to 1} \frac{\tan(3x-3)}{\sin(2x-2)}$ (c) $\lim\limits_{x \to \infty} \frac{x^{-1/2}}{\tan(x^{-1/2})}$

Answers: (a) $\frac{1}{12}$ (b) $\frac{3}{2}$
(c) 1

L'Hôpital's rule may have to be employed more than once in an evaluation process. Consider the following example:

EXAMPLE 8.2 Evaluate:

$$\lim_{x \to 0} \frac{x - \sin x}{2 + 2x + x^2 - 2e^x}$$

SOLUTION: Observing that we are dealing with an indeterminate form of type "$\frac{0}{0}$" we have:

$$\lim_{x \to 0} \frac{x - \sin x}{2 + 2x + x^2 - 2e^x} = \lim_{x \to 0} \frac{(x - \sin x)'}{(2 + 2x + x^2 - 2e^x)'}$$

still indeterminate:

$$= \lim_{x \to 0} \frac{1 - \cos x}{2 + 2x - 2e^x}$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)'}{(2 + 2x - 2e^x)'}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 - 2e^x} = \lim_{x \to 0} \frac{(\sin x)'}{(2 - 2e^x)'}$$

still indeterminate:

$$= \lim_{x \to 0} \frac{\cos x}{-2e^x} = -\frac{1}{2}$$

CHECK YOUR UNDERSTANDING 8.2

Evaluate:

$$\lim_{x \to 0} \frac{1 - \cos 2x}{5x^2}$$

Answer: $\frac{2}{5}$

L'Hôpital's Rule also holds for one-sided limits. Consider the following example:

EXAMPLE 8.3 Evaluate:

$$\lim_{x \to (\pi/2)^-} \frac{\cos x}{\sqrt{\frac{\pi}{2} - x}}$$

SOLUTION: Observing that we are dealing with indeterminate forms of type "$\frac{0}{0}$" we have:

$$\lim_{x \to (\pi/2)^-} \frac{\cos x}{\sqrt{\frac{\pi}{2} - x}} = \lim_{x \to (\pi/2)^-} \frac{(\cos x)'}{\left(\sqrt{\frac{\pi}{2} - x}\right)'}$$

$$= \lim_{x \to (\pi/2)^-} \frac{-\sin x}{-\frac{1}{2\sqrt{\frac{\pi}{2} - x}}}$$

$$= \lim_{x \to (\pi/2)^-} 2(\sin x)\sqrt{\frac{\pi}{2} - x} = 2 \cdot 1 \cdot 0 = 0$$

CHECK YOUR UNDERSTANDING 8.3

Evaluate:

$$\lim_{x \to 0^-} \frac{\tan x}{x^2}$$

Answers: $-\infty$

We offer, without proof, the "$\frac{\infty}{\infty}$" variation of Theorem 8.1:

The theorem also holds for one-sided limits.

THEOREM 8.2

L'HÔPITAL'S RULE: "∞/∞" TYPE

Let c be a real number, or $\pm\infty$. Assume that, apart from c, f and g are differentiable on an open interval containing c with $g'(x) \neq 0$. If:

$$\lim_{x \to c} |f(x)| = \lim_{x \to c} |g(x)| = \infty$$

and if:

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$$

where L is a real number or $\pm\infty$, Then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L$$

EXAMPLE 8.4

Determine:

(a) $\displaystyle\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$ (b) $\displaystyle\lim_{x \to (\pi/2)^-} \frac{\sec x}{\ln \sec x}$

(c) $\displaystyle\lim_{x \to \infty} \frac{x^3}{e^x}$

SOLUTION: Observing that we are dealing with indeterminate forms of type "$\frac{\infty}{\infty}$" we have:

(a) $$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{(\ln x)'}{(x^{-1})'} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} \left(-\frac{x^2}{x}\right)$$
$$= \lim_{x \to 0^+} (-x) = 0$$

(b) $$\lim_{x \to (\pi/2)^-} \frac{\sec x}{\ln \sec x} = \lim_{x \to (\pi/2)^-} \frac{(\sec x)'}{(\ln \sec x)'} = \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\frac{\sec x \tan x}{\sec x}}$$
$$= \lim_{x \to (\pi/2)^-} \sec x = \infty$$

(c) $$\lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{3x^2}{e^x} = \lim_{x \to \infty} \frac{6x}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0$$

CHECK YOUR UNDERSTANDING 8.4

Determine:

(a) $\lim\limits_{x\to\infty}\dfrac{5x^2+1}{x^2-3}$ (b) $\lim\limits_{x\to 0^-}\dfrac{1/x}{\ln(-x)}$

Answers: (a) 5 (b) ∞

OTHER INDETERMINATE FORMS: “$0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞”

At times, these indeterminate forms can be evaluated by first rewriting them as indeterminate forms of type “$\frac{0}{0}$” or “$\frac{\infty}{\infty}$,” and then applying l’Hôpital’s Rule. Consider the following examples.

EXAMPLE 8.5 Determine:

(a) $\lim\limits_{x\to 0^+}\sin x\cdot\ln x$ (b) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

SOLUTION: (a) $\lim\limits_{x\to 0^+}\sin x\cdot\ln x$ of type “$0\cdot\infty$” can be converted into a “$\frac{\infty}{\infty}$” type:

$$\lim_{x\to 0^+}\sin x\cdot\ln x = \overset{\frac{\infty}{\infty}}{\lim_{x\to 0^+}\frac{\ln x}{\csc x}} \overset{\text{L'Hôpital's Rule}}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x}$$

invert and multiply:

$$= \lim_{x\to 0^+}\frac{-\sin x\tan x}{x}$$

$$= \left(\lim_{x\to 0^+}\frac{-\sin x}{x}\right)\left(\lim_{x\to 0^+}\tan x\right)$$

$$= (-1)(0) = 0$$

(b) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$ of type “$\infty-\infty$” can be converted into a “$\frac{0}{0}$” type:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \overset{\frac{0}{0}}{\lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right)} = \lim_{x\to 0}\left(\frac{\cos x - 1}{\boldsymbol{x\cos x}+\sin x}\right)$$

L’Hôpital’s Rule

$$= \lim_{x\to 0}\left(\frac{-\sin x}{\boldsymbol{-x\sin x+\cos x}+\cos x}\right)$$

$$= \frac{0}{2} = 0$$

CHECK YOUR UNDERSTANDING 8.5

Determine:

(a) $\lim\limits_{x\to -\frac{\pi}{4}}[(1+\tan x)\sec 2x]$ (b) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\tan x}\right)$

Answers: (a) 1 (b) 0

Limits of the form $\lim\limits_{x\to c} f(x)^{g(x)}$ and $\lim\limits_{x\to\infty} f(x)^{g(x)}$ may give rise to indeterminate forms of types $0^0, \infty^0$, and 1^∞, which can often be resolved by invoking the natural logarithmic function. Consider the following example.

EXAMPLE 8.6 Determine:

(a) $\lim\limits_{x\to\infty} x^{1/x}$ (b) $\lim\limits_{x\to\frac{\pi^-}{2}} (1+\cos x)^{\tan x}$

SOLUTION: (a) To evaluate $\lim\limits_{x\to\infty} x^{1/x}$, an indeterminate form of type ∞^0, we proceed as follows:

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln(x^{1/x})} \underset{\substack{\uparrow\\ \text{See margin}}}{=} e^{\lim\limits_{x\to\infty}\ln(x^{1/x})} = e^{\lim\limits_{x\to\infty}\frac{1}{x}\ln x}$$

$$= e^{\lim\limits_{x\to\infty}\frac{(\ln x)'}{x'}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^0 = 1$$

Exercise 41, page 62:
If f is continuous at b and if $\lim\limits_{x\to a} g(x) = b$, then:

$$\lim_{x\to a} f[g(x)] = f[\lim_{x\to a} g(x)]$$

(b) To evaluate $\lim\limits_{x\to\frac{\pi^-}{2}} (1+\cos x)^{\tan x}$ (an indeterminate form of type 1^∞) we let $y = (1+\cos x)^{\tan x}$ (see margin). Then:

$$\mathbf{\lim_{x\to\frac{\pi^-}{2}} \ln y} = \lim_{x\to\frac{\pi^-}{2}} \ln(1+\cos x)^{\tan x} = \lim_{x\to\frac{\pi^-}{2}} \tan x \ln(1+\cos x)$$

$$= \lim_{x\to\frac{\pi^-}{2}} \frac{\ln(1+\cos x)}{\cot x} \leftarrow \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\frac{\pi^-}{2}} \frac{[\ln(1+\cos x)]'}{(\cot x)'}$$

$$= \lim_{x\to\frac{\pi^-}{2}} \frac{\frac{-\sin x}{1+\cos x}}{-\csc^2 x}$$

$$= \lim_{x\to\frac{\pi^-}{2}} \frac{\sin^3 x}{1+\cos x} = \frac{1}{1} = \mathbf{1}$$

From $\displaystyle \boldsymbol{\lim_{x \to \frac{\pi}{2}^-} \ln y = 1}$ we have: $\displaystyle \lim_{x \to \frac{\pi}{2}^-} e^{\ln y} = e^1$

$$\lim_{x \to \frac{\pi}{2}^-} y = e$$

But $y = (1 + \cos x)^{\tan x}$. So: $\displaystyle \lim_{x \to \frac{\pi}{2}^-} (1 + \cos x)^{\tan x} = e$

CHECK YOUR UNDERSTANDING 8.6

Determine:

(a) $\displaystyle \lim_{x \to \infty} (e^x + 1)^{-\frac{2}{x}}$ (b) $\displaystyle \lim_{x \to 0} (\cos x)^{\frac{1}{x}}$

Answers: (a) $\frac{1}{e^2}$ (b) 1

EXERCISES

Exercise 1-56. Use L'Hôpital's rule to determine the given limit.

1. $\displaystyle\lim_{x\to 2}\frac{x^3-5x+2}{x^4+6x^2-40}$
2. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}$
3. $\displaystyle\lim_{x\to \infty}\frac{x^2-5x+2}{x^4+6x^2-40}$
4. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{\sin x}$
5. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x}$
6. $\displaystyle\lim_{x\to 0}\frac{\sin x}{e^x-e^{-x}}$
7. $\displaystyle\lim_{x\to \infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$
8. $\displaystyle\lim_{x\to 0}\frac{\sin x}{e^x-e^{-x}}$
9. $\displaystyle\lim_{x\to \infty}\frac{\ln x}{\sqrt{x}}$
10. $\displaystyle\lim_{x\to 1^-}\frac{\ln x}{\sqrt{1-x}}$
11. $\displaystyle\lim_{x\to 1}\frac{\frac{1}{x-1}}{\frac{1}{\ln x}}$
12. $\displaystyle\lim_{x\to \infty}\frac{\ln(\ln x)}{x}$
13. $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$
14. $\displaystyle\lim_{x\to -\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\sin\frac{1}{x}}$
15. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\sin 2x}{\pi-2x}$
16. $\displaystyle\lim_{x\to \frac{\pi}{2}^+}\frac{\ln\left(x-\frac{\pi}{2}\right)}{\tan x}$
17. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{1+\cos 2x}{1-\sin x}$
18. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$
19. $\displaystyle\lim_{x\to \infty}\frac{\sin\frac{3}{x}}{\sin\frac{9}{x}}$
20. $\displaystyle\lim_{x\to \pi}\frac{\ln\cos 2x}{(\pi-x)^2}$
21. $\displaystyle\lim_{x\to 0}\frac{x^2+2x-2e^x+2}{\sin x-x}$
22. $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$
23. $\displaystyle\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$
24. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\cos x-\sin x+1}{\cos x+\sin x-1}$
25. $\displaystyle\lim_{x\to 0}\frac{\sin x-x\cos x}{x-\sin x}$
26. $\displaystyle\lim_{x\to 0^+}\frac{\ln(\tan x)}{\ln(\tan 2x)}$
27. $\displaystyle\lim_{x\to 1^-}\frac{\ln(1-x)}{\cos\pi x}$
28. $\displaystyle\lim_{x\to 1}\frac{x^x-x}{1-x+\ln x}$
29. $\displaystyle\lim_{x\to 0}\frac{a^x-b^x}{x}$
30. $\displaystyle\lim_{x\to \frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}$

31. $\lim\limits_{x \to 0} \dfrac{x - \sin^{-1} x}{\sin^3 x}$

32. $\lim\limits_{x \to 0^+} \dfrac{\ln(\csc x)}{\ln(\cot x)}$

33. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{\sin x - x}$

34. $\lim\limits_{x \to 0^+} \dfrac{1 - \ln x}{e^{1/x}}$

35. $\lim\limits_{x \to \infty} \dfrac{\ln(\ln x)}{\ln x}$

36. $\lim\limits_{x \to \frac{1}{2}^-} \dfrac{\ln(1 - 2x)}{\tan \pi x}$

37. $\lim\limits_{x \to 0^+} \dfrac{\ln(x^2 + x)}{\ln x}$

38. $\lim\limits_{x \to 2^+} \dfrac{\cos x \ln(x - 2)}{\ln(e^x - e^2)}$

39. $\lim\limits_{x \to \infty} x^3 e^{-x^2}$

40. $\lim\limits_{x \to 0} \left(\dfrac{1 + x}{\sin x} - \dfrac{1}{x}\right)$

41. $\lim\limits_{x \to \frac{\pi}{2}} \tan x \cdot \ln(\sin x)$

42. $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right)$

43. $\lim\limits_{x \to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x - 1}\right)$

44. $\lim\limits_{x \to 1} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x}\right)$

45. $\lim\limits_{x \to \infty} [\ln 2x - \ln(x + 1)]$

46. $\lim\limits_{x \to \infty} (e^x + x)^{1/x}$

47. $\lim\limits_{x \to 0^+} (\sin x)^{\sin x}$

48. $\lim\limits_{x \to 0} (x + 2^x)^{1/x}$

49. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$

50. $\lim\limits_{x \to 0} (e^{3x} - 2x)^{-3/x}$

51. $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

52. $\lim\limits_{x \to 0} \dfrac{\int_0^x \sin t^2 dt}{x^2}$

53. $\lim\limits_{x \to \infty} \dfrac{\int_0^x e^{t^2} dt}{x}$

54. $\lim\limits_{x \to \infty} \dfrac{\int_0^x e^{t^2} dt}{e^x}$

55. $\lim\limits_{x \to \infty} \dfrac{e^x + x}{xe^x}$ and $\lim\limits_{x \to -\infty} \dfrac{e^x + x}{xe^x}$

56. $\lim\limits_{x \to \infty} \dfrac{\ln(x^2 + 1)}{\ln(x^3 + 1)}$ and $\lim\limits_{x \to -\infty} \dfrac{\ln(x^2 + 1)}{\ln(x^3 + 1)}$

Exercise 57-59. Use l'Hôpital's Rule to evaluate the limit of:

57. Example 2.1, page 44

58. Example 2.2, page 45

59. Example 2.3, page 45

Exercise 60-62. Use l'Hôpital's Rule to evaluate the limit $f'(c) = \lim\limits_{h \to 0} \dfrac{f(c + h) - f(c)}{h}$ of:

60. Example 3.2, page 67

61. Example 3.3, page 68

62. CYU 3.1, page 69

63. Find all values of a for which $\lim\limits_{x \to 0} \dfrac{\cos ax - 1}{x^2} = -8$.

64. Find all values a and b for which $\lim\limits_{x \to 0} \dfrac{\sin 2x + ax^3 + bx}{x^3} = 0$.

65. **(Theory)** Verify that for every positive integer n $\lim\limits_{x \to \infty} \dfrac{e^x}{x^n} = \infty$

§2 IMPROPER INTEGRALS

Up to this point we have only considered definite integrals of the form $\int_a^b f(x)dx$, with f continuous throughout the closed interval $[a, b]$.

What if the interval is not finite — as is the case with the expressions: $\int_0^\infty xe^x dx$, $\int_{-\infty}^0 xe^x dx$, and $\int_{-\infty}^\infty xe^x dx$?

What if f is discontinuous within the interval of integration — as is the case with the expression $\int_0^4 \frac{dx}{x-1}$?

Unencumbered by any sense of political correctness, such integrals continue to be called **improper integrals**. Let's "properize" them, starting with:

DEFINITION 8.1

IMPROPER INTEGRALS (INFINITE INTERVAL)

Let f be continuous on $[a, \infty)$. If $\lim\limits_{t \to \infty} \int_a^t f(x)dx$ exists, then we say that $\int_a^\infty f(x)dx$ **converges** to that (finite) limit.

Let f be continuous on $(-\infty, a]$. If $\lim\limits_{t \to -\infty} \int_t^a f(x)dx$ exists, then we say that $\int_{-\infty}^a f(x)dx$ **converges** to that (finite) limit.

Any number c can replace the "0" in $\int_t^0 f(x)dx$ and $\int_0^t f(x)dx$

Let f be continuous on $(-\infty, \infty)$. If both $\lim\limits_{t \to -\infty} \int_t^0 f(x)dx$ and $\lim\limits_{t \to \infty} \int_0^t f(x)dx$ exist, then we say that $\int_{-\infty}^\infty f(x)dx$ **converges** to $\lim\limits_{t \to -\infty} \int_t^0 f(x)dx + \lim\limits_{t \to \infty} \int_0^t f(x)dx$.

In the above situations, if the integral does not converge, then it is said to **diverge**.

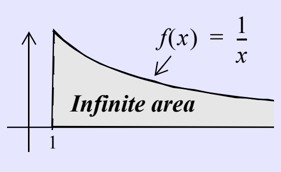
EXAMPLE 8.7 Determine if the given integral converges. If it does, find its value.

(a) $\int_1^{\infty}\frac{1}{x}dx$ (b) $\int_{-\infty}^{0} e^x dx$

(c) $\int_{-\infty}^{\infty} x^3 dx$ (d) $\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx$

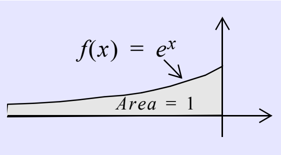
SOLUTION: (a) Since:

$$\int_1^{\infty}\frac{1}{x}dx = \lim_{t\to\infty}\int_1^{t}\frac{1}{x}dx = \lim_{t\to\infty}\ln x\Big|_1^t = \lim_{t\to\infty}(\ln t-\ln 1)$$
$$= \lim_{t\to\infty}\ln t = \infty$$



$\int_1^{\infty}\frac{1}{x}dx$ diverges, as the limit is not finite (see margin).

(b) Since: $\int_{-\infty}^{0} e^x dx = \lim_{t\to-\infty}\int_t^{0} e^x dx = \lim_{t\to-\infty} e^x\Big|_t^0 = 1 - \lim_{t\to-\infty} e^t = 1$



$\int_{-\infty}^{0} e^x dx$ converges to 1 (see margin).

(c) First: $\int_{-\infty}^{\infty} x^3 dx = \int_{-\infty}^{0} x^3 dx + \int_0^{\infty} x^3 dx$. Since $\int_{-\infty}^{0} x^3 dx$ diverges:

$$\int_{-\infty}^{0} x^3 dx = \lim_{t\to-\infty}\int_t^{0} x^3 dx = \lim_{t\to-\infty}\frac{x^4}{4}\Bigg|_t^0 = \frac{1}{4}\lim_{t\to-\infty}-t^4 = -\infty$$

so does $\int_{-\infty}^{\infty} x^3 dx$. (See margin.)

In Definition 8.1 $\int_{-\infty}^{\infty} f(x)dx$ is not defined to be $\lim_{t\to\infty}\int_{-t}^{t} x^3 dx$, which turns out to be 0 (verify). Rather, $\int_{-\infty}^{\infty} f(x)dx$ is the sum of two integrals, both of which need to converge in order for $\int_{-\infty}^{\infty} f(x)dx$ to converge.

(d) $$\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx = \int_{-\infty}^{0}\frac{1}{x^2+1}dx + \int_0^{\infty}\frac{1}{x^2+1}dx$$.

$$= \lim_{t\to-\infty}\int_t^{0}\frac{1}{x^2+1}dx + \lim_{t\to\infty}\int_0^{t}\frac{1}{x^2+1}dx$$

Since $f(x) = \frac{1}{x^2+1}$ is an even function: $$= 2\lim_{t\to\infty}\int_0^{t}\frac{1}{x^2+1}dx$$

Theorem 6.17(e): $$= 2\lim_{t\to\infty}\tan^{-1}x\Big|_0^t = 2\lim_{t\to\infty}(\tan^{-1}t-\tan^{-1}0)$$

Figure 6.6(b), page 251: $$= 2\left(\frac{\pi}{2}-0\right) = \pi$$

CHECK YOUR UNDERSTANDING 8.7

Does $\int_0^{\infty} xe^{-x}dx$ converge? If so, find its value.

Answer: Yes, 1

EXAMPLE 8.8 (a) Determine the area of the region bounded on the left by the line $y = x$, below by the x-axis, and above by the graph of the function $f(x) = \frac{1}{x^2}$.

(b) Find the volume obtained by revolving the above area about the x axis.

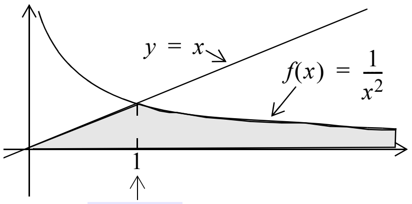
SOLUTION:

(a) $$A = \int_0^1 x\,dx + \int_1^{\infty} \frac{1}{x^2}dx$$

$$= \frac{x^2}{2}\bigg|_0^1 + \lim_{t \to \infty} \int_1^t x^{-2}\,dx$$

$$= \frac{1}{2} + \lim_{t \to \infty}\left(-\frac{1}{x}\bigg|_1^t\right)$$

$$= \frac{1}{2} + \lim_{t \to \infty}\left(-\frac{1}{t} + 1\right) = \frac{3}{2}$$



$$x = \frac{1}{x^2}$$
$$x^3 = 1$$
$$x = 1$$

(b) $$V = \pi\int_0^1 x^2dx + \pi\int_1^{\infty} \frac{1}{x^4}dx = \pi\frac{x^3}{3}\bigg|_0^1 + \pi \lim_{t \to \infty}\int_1^t x^{-4}\,dx$$

$$= \frac{\pi}{3} + \frac{\pi}{3}\lim_{t \to \infty}\left(-\frac{1}{t^3} + 1\right) = \frac{2\pi}{3}$$

CHECK YOUR UNDERSTANDING 8.8

While the area of the region lying to the right of $x = 1$ that is bounded below by the x-axis and above by the graph of the function $f(x) = \frac{1}{x}$ is infinite [see Example 8.7(a)], the volume obtained by revolving that region about the x-axis is finite. Find that volume.

Answer: π

We have seen that $\int_1^{\infty}\frac{1}{x}dx$ diverges [Example 8.7(a)], and that $\int_1^{\infty}\frac{1}{x^2}dx$ converges [see Solution of Example 8.8(a)]. In general:

THEOREM 8.3 The integral $\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \le 1$.

PROOF: We already know that the integral diverges if $p = 1$.

For $p \neq 1$: $\int_1^\infty \frac{1}{x^p}dx = \lim\limits_{t \to \infty}\int_1^t x^{-p}dx = \lim\limits_{t \to \infty}\frac{x^{-p+1}}{-p+1}\Bigg|_1^t$

$$= \frac{1}{-p+1}\lim_{t \to \infty}(t^{-p+1} - 1)$$

If $\boldsymbol{p > 1}$, then $\lim\limits_{t \to \infty}(t^{-p+1} - 1) = \lim\limits_{t \to \infty}\left(\underset{\searrow 0}{\boxed{\frac{1}{t^{p-1}}}} - 1\right) = -1$. So:

$\int_1^\infty \frac{1}{x^p}dx = \left(\frac{1}{-p+1}\right)(-1) = \frac{1}{p-1}$, and the integral converges.

If $\boldsymbol{p < 1}$, then $\lim\limits_{t \to \infty}(\underset{\searrow \infty}{\boxed{t^{-p+1}}} - 1) = \infty$ and $\int_1^\infty \frac{1}{x^p}dx$ diverges.

CHECK YOUR UNDERSTANDING 8.9

For what values of p and q does the indicated integral converge?

(a) $\int_1^\infty\left(\frac{1}{x^p} + \frac{1}{x^q}\right)dx$ (b) $\int_1^\infty\left(\frac{1}{x^p} \cdot \frac{1}{x^q}\right)dx$ (c) $\int_1^\infty \frac{1/x^p}{1/x^q}dx$

Answers:
(a) $p > 1$ and $q > 1$
(b) $p + q > 1$
(c) $p - q > 1$

Here is the $\int_a^b f(x)dx$-story when f is not continuous throughout $[a, b]$:

DEFINITION 8.2
IMPROPER INTEGRALS (DISCONTINUITY)

Let f be continuous on $[a, b)$ and discontinuous at b. If $\lim\limits_{t \to b^-}\int_a^t f(x)dx$ exists, then we say that $\int_a^b f(x)dx$ converges to that (finite) limit.

Let f be continuous on $(a, b]$ and discontinuous at a. If $\lim\limits_{t \to a^+}\int_t^b f(x)dx$ exists, then we say that $\int_a^b f(x)dx$ converges to that (finite) limit.

Let f be continuous at every point in $[a, b]$ other than $c \in (a, b)$. If both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge, then we say that $\int_a^b f(x)dx$ converges to $\int_a^c f(x)dx + \int_c^b f(x)dx$.

In the above situations, if the integral does not converge, then it is said to **diverge**.

EXAMPLE 8.9 Determine if the given integral converges. If if does, find its value.

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{-x+1}}$ (b) $\displaystyle\int_{-2}^2 \frac{dx}{x^4}$

(c) $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}}$

SOLUTION:

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{-x+1}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{-x+1}}$. Turning to $\displaystyle\int_0^t \frac{dx}{\sqrt{-x+1}}$ we have:

$$\int_0^t \frac{dx}{\sqrt{-x+1}} = -\int_1^{-t+1} u^{-1/2}du = -2u^{1/2}\Big|_1^{-t+1}$$
$$= -2(\sqrt{-t+1}-1)$$

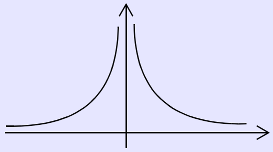
$u = -x+1$
$du = -dx$

And so: $\displaystyle\int_0^1 \frac{dx}{\sqrt{-x+1}} = \lim_{t\to 1^-} -2(\sqrt{-t+1}-1) = 2$ (converges)

If one does not spot the discontinuity at 0, then one might do this:

$$\int_{-2}^2 \frac{dx}{x^4} = -\frac{1}{x^3}\Big|_{-2}^2 = 0$$

WRONG — note graph:



(b) $$\int_{-2}^2 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^2 \frac{dx}{x^4} \underset{\uparrow}{=} 2\int_0^2 \frac{dx}{x^4} = 2\lim_{t\to 0^+}\int_t^2 x^{-4}dx$$

$f(x) = \dfrac{1}{x^4}$ is an even function: $f(-x) = f(x)$

$$= 2\lim_{t\to 0^+} -\frac{1}{3x^3}\Big|_t^2 = \infty$$

And so: $\displaystyle\int_{-2}^2 \frac{dx}{x^4}$ diverges. (See margin.)

(c) $$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$
$$= \lim_{t\to 2^-}\int_1^t \frac{dx}{(x-2)^{2/3}} + \lim_{t\to 2^+}\int_t^4 \frac{dx}{(x-2)^{2/3}} \quad (*)$$

Since $\displaystyle\int \frac{dx}{(x-2)^{2/3}} \underset{\uparrow}{=} \int u^{-\frac{2}{3}}du = 3u^{\frac{1}{3}} + C = 3(x-2)^{1/3} + C$:

$u = x-2$
$du = dx$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \lim_{t\to 2^-} 3(x-2)^{1/3}\Big|_1^t + \lim_{t\to 2^+} 3(x-2)^{1/3}\Big|_t^4$$
$$= 3\lim_{t\to 2^-}[(t-2)^{1/3}+1] + 3\lim_{t\to 2^+}[2^{1/3}-(t-2)^{1/3}]$$
$$= 3 + 3\cdot 2^{1/3} = 3(1+2^{1/3}) \qquad \text{(converges)}$$

Answers:
(a) Diverges.
(b) Converges:
$\frac{5}{4}(3^{4/5} - 2^{4/5})$

CHECK YOUR UNDERSTANDING 8.10

Determine if the given integral converges. If if does, find its value.

(a) $\displaystyle\int_1^3 \frac{dx}{(x-3)^2}$ (b) $\displaystyle\int_{-3}^2 \frac{dx}{(x+1)^{1/5}}$

EXERCISES

Exercise 1-42. Determine if the given integral converges. If if does, find its value.

1. $\int_1^\infty \frac{1}{x^2}dx$
2. $\int_1^\infty e^{-x}dx$
3. $\int_{-\infty}^0 xe^x dx$
4. $\int_1^\infty \frac{1}{\sqrt{x}}dx$
5. $\int_{-\infty}^0 \frac{1}{(2x-1)^3}dx$
6. $\int_0^\infty \sin x\,dx$
7. $\int_0^\infty \frac{1}{4+x^2}dx$
8. $\int_1^\infty \frac{\ln x}{x^2}dx$
9. $\int_0^\infty \cos \pi x\,dx$
10. $\int_{-\infty}^\infty \frac{x}{(x^2+4)^2}dx$
11. $\int_0^\infty \frac{x}{(1+x^2)^2}dx$
12. $\int_1^\infty xe^{-3x^2}dx$
13. $\int_{-\infty}^\infty xe^{-x^2}dx$
14. $\int_0^\infty \frac{e^x}{(e^x+1)^3}dx$
15. $\int_0^\infty \frac{6x^2+8}{(x^2+1)(x^2+2)}dx$
16. $\int_0^\infty e^{px}dx, \quad p>0$
17. $\int_e^\infty \frac{\ln x}{x}dx$
18. $\int_{-\infty}^\infty \frac{1}{4+x^2}dx$
19. $\int_0^8 x^{-1/3}dx$
20. $\int_1^2 \frac{1}{1-x}dx$
21. $\int_{-\infty}^\infty \frac{16}{9x^4+10x^2+1}dx$
22. $\int_0^3 \frac{1+x}{\sqrt{x}}dx$
23. $\int_0^{\frac{\pi}{2}} \tan x\,dx$
24. $\int_0^1 \ln x\,dx$
25. $\int_{-3}^{-1} 8(x+1)^{-1/5}dx$
26. $\int_0^{\frac{1}{2}} \frac{1}{x(1-x)^{1/3}}dx$
27. $\int_{-5}^{-1} 3(x+3)^{-2/5}dx$
28. $\int_1^4 \frac{1}{(x-2)^2}dx$
29. $\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}dx$
30. $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx$
31. $\int_0^1 \frac{1}{\sqrt{x}}dx$
32. $\int_0^3 \frac{1}{x-2}dx$
33. $\int_1^2 \frac{1}{\sqrt[3]{x-2}}dx$
34. $\int_0^1 \frac{1}{x}dx$
35. $\int_0^{\frac{\pi}{2}} \sec x\,dx$
36. $\int_0^\infty \frac{1}{x^2}dx$
37. $\int_0^\infty \frac{1}{\sqrt{x}(x+4)}dx$
38. $\int_0^8 \frac{1}{x^{2/3}}dx$
39. $\int_0^1 x\ln x\,dx$
40. $\int_{-1}^1 \frac{x}{(1+x^2)^{1/4}}dx$
41. $\int_1^{16} \frac{1}{\sqrt{x}}\left(\frac{1}{2-\sqrt{x}}\right)^{1/3}dx$
42. $\int_{-1}^4 \frac{1}{\sqrt{1+x}\sqrt{4-x}}dx$

Exercise 43-45. For what values of a does the given integral converge?

43. $\int_0^1 x^a dx$ 44. $\int_1^\infty x^a dx$ 45. $\int_0^\infty \left(\frac{x}{x^2+1} - \frac{a}{3x+1}\right) dx$

Exercise 46-48. For what values of n does the given integral converge?

46. $\int_0^1 x^n \ln x dx$ 47. $\int_0^1 x^n (\ln x)^2 dx$ 48. $\int_1^\infty \frac{\ln x}{x^n} dx$

49. Show that $\int_0^\infty \sin x dx$ and $\int_{-\infty}^0 \sin x dx$ diverge, and that $\lim_{t \to \infty} \int_{-t}^{t} \sin x dx = 0$.

50. Find the area of the region to the right of the origin that is bounded below by the x-axis, and above by the graph of the function $f(x) = \frac{1}{(1+x)^2}$.

51. Find the area of the region to the right of the origin that is bounded below by the x-axis, and above by the graph of the function $f(x) = e^{-x}$.

52. Find the area of the region bounded above by $xy = 1$, below by $y(x^2+1) = x$, and to the left by $x = 1$.

53. Find the area of the region above $(0, 1]$ and below the graph of the function $f(x) = x^{-1/4}$.

54. Find the volume obtained by rotating about the x-axis the region to the right of the origin that is bounded below by the x-axis, and above by the graph of the function $f(x) = e^{-x}$.

55. Find the volume obtained by rotating about the y-axis the region to the right of the origin that is bounded below by the x-axis, and above by the graph of the function $f(x) = e^{-x}$.

56. Find the volume obtained by rotating about the x-axis the region lying above $(0, 1]$ and below the graph of the function $f(x) = x^{-1/4}$.

57. Find the volume obtained by rotating about the y-axis the region lying above $(0, 1]$ and below the graph of the function $f(x) = x^{-1/4}$.

CHAPTER SUMMARY

L'HÔPITAL'S RULE	Let f and g be differentiable with $g'(c) \neq 0$ in an open interval containing c (except possibly at c). If: $\lim\limits_{x \to c} f(x) = 0$ and $\lim\limits_{x \to c} g(x) = 0$ OR $\lim\limits_{x \to c} f(x) = \pm\infty$ and $\lim\limits_{x \to c} g(x) = \pm\infty$ then: $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$ if the limit on the right exists (or is $\pm\infty$). The above also holds if "$x \to c$" is replaced by $x \to c^-$, $x \to c^+$, $x \to -\infty$, or $x \to \infty$.
IMPROPER INTEGRALS (INFINITE INTERVAL)	Let f be continuous on $[a, \infty)$. If $\lim\limits_{t \to \infty} \int_a^t f(x)dx$ exists, then we say that $\int_a^\infty f(x)dx$ **converges** to that (finite) limit. Let f be continuous on $(-\infty, a]$. If $\lim\limits_{t \to -\infty} \int_t^a f(x)dx$ exists, then we say that $\int_{-\infty}^a f(x)dx$ **converges** to that (finite) limit. Let f be continuous on $(-\infty, \infty)$. If both $\lim\limits_{t \to -\infty} \int_t^0 f(x)dx$ and $\lim\limits_{t \to \infty} \int_0^t f(x)dx$ exist, then we say that $\int_{-\infty}^\infty f(x)dx$ **converges** to $\lim\limits_{t \to -\infty} \int_t^0 f(x)dx + \lim\limits_{t \to \infty} \int_0^t f(x)dx$. In the above situations, if the integral does not converge, then it is said to **diverge**.
THEOREM	The integral $\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \leq 1$.

IMPROPER INTEGRALS (DISCONTINUITY)	Let f be continuous on $[a, b)$ and discontinuous at b. If $\lim\limits_{t \to b^-} \int_a^t f(x)dx$ exists, then we say that $\int_a^b f(x)dx$ converges to that (finite) limit. Let f be continuous on $(a, b]$ and discontinuous at a. If $\lim\limits_{t \to a^+} \int_t^b f(x)dx$ exists, then we say that $\int_a^b f(x)dx$ converges to that (finite) limit. Let f be continuous at every point in $[a, b]$ other than $c \in (a, b)$. If both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge, then we say that $\int_a^b f(x)dx$ converges to $\int_a^c f(x)dx + \int_c^b f(x)dx$. In the above situations, if the integral does not converge, then it is said to **diverge**.

CHAPTER 9

SEQUENCES AND SERIES

§1. SEQUENCES

Formally:

DEFINITION 9.1
SEQUENCE

A sequence of real numbers is a real-valued function with domain the set of positive integers: $f: Z^+ \to \Re$.

Formality aside, one seldom represents a sequence in function-form but rather as an infinite string of numbers, or terms:

$$(a_1, a_2, a_3, \dots) \text{ or } (a_n)_{n=1}^{\infty} \text{ or simply } (a_n)$$

with a_n representing the function value $f(n)$.

Consider the three sequences:

$$\text{(a) } \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \qquad \text{(b) } \left(\frac{n+1}{n}\right)_{n=1}^{\infty} \quad \text{and (c) } (1, 2, 1, 2, 1, 2, \dots)$$

While the sequence in (a) appears to be heading to 0 and that of (b) to 1, the one in (c) does not look to be going anywhere in particular, as its terms keep jumping back and forth between 1 and 2. Appearances are well and good, but mathematics demands precision, bringing us to:

We remind you that $|a-b|$ represents the distance on the number line between the numbers a and b. For example: $|2-7| = 5$ is the distance between 2 and 7, while $|3+4| = |3-(-4)| = 7$ is the distance between 3 and -4.

DEFINITION 9.2
CONVERGENT SEQUENCE

A sequence $(a_n)_{n=1}^{\infty}$ **converges** to the number L if for any given $\varepsilon > 0$ there exists a positive integer N (which depends on ε) such that:

$$n > N \Rightarrow |a_n - L| < \varepsilon$$

In the event that (a_n) converges to L we write $\lim\limits_{n \to \infty} a_n = L$, or $\lim a_n = L$, or $a_n \to L$, and call L the **limit** of the sequence. A sequence that converges is said to be a **convergent** sequence. A sequence that does not converge is said to **diverge**.

In spirit: $\lim\limits_{n \to \infty} a_n = L$ if the a_n's get arbitrarily close to L (within ε units of L), providing they are far enough in the sequence ($n > N$)

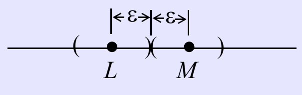
Compare with the spirit of Definition 2.2, page 53 $[\lim\limits_{x \to c} f(x) = L]$:

$f(x)$ gets arbitrarily close to L (within ε units of L), providing x is close enough to c: i.e. $0 < |x-c| < \delta$ for some $\delta > 0$.

Note that in the above definition we speak of **<u>the</u>** limit of a sequence, as opposed to "a limit." That is as it should be, for:.

THEOREM 9.1 If a sequence (a_n) converges, then it has a unique limit.

PROOF: Assume that $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} a_n = M$ with $L \neq M$ (we will arrive at a contradiction):



Let $\varepsilon = \frac{|L - M|}{2}$ (see margin). Since $\lim\limits_{n \to \infty} a_n = L$, there exists N_1 such that $n > N_1 \Rightarrow |a_n - L| < \varepsilon$. By the same token, since $\lim\limits_{n \to \infty} a_n = M$, there exists N_2 such that $n > N_2 \Rightarrow |a_n - M| < \varepsilon$. Choosing n_0 to be any integer greater than both N_1 and N_2, we are led to the conclusion that $|a_{n_0} - L| < \varepsilon$ and $|a_{n_0} - M| < \varepsilon$; but this cannot be, since no number lies both within ε units of L and ε units of M (see margin again).

CHECK YOUR UNDERSTANDING 9.1

Prove that for any constant c the sequence $(c, c, c, c, \ldots)$ converges to c.

Answer: See page A-48.

EXAMPLE 9.1

(a) Prove that $\lim\limits_{n \to \infty} \frac{n + 1}{n} = 1$

(b) Show that the sequence $(a_n) = (1, 0, 1, 0, 1, \ldots)$ diverges.

SOLUTION: (a) Let $\varepsilon > 0$ be given. We are to find N such that $n > N \Rightarrow \left|\frac{n + 1}{n} - 1\right| < \varepsilon$. Let's do it:

We want: $n > N \Rightarrow \left|\frac{\mathbf{n + 1}}{\mathbf{n}} - \mathbf{1}\right| < \varepsilon$

Let's rewrite our goal: $n > N \Rightarrow \left|\frac{n + 1 - n}{n}\right| < \varepsilon$

again: $n > N \Rightarrow \left|\frac{1}{n}\right| < \varepsilon$

and again: $n > N \Rightarrow \frac{1}{n} < \varepsilon$

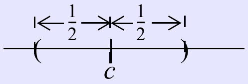
and finally: $n > N \Rightarrow \mathbf{n} > \frac{\mathbf{1}}{\varepsilon}$

So, to find an N such that $n > N \Rightarrow \left|\frac{\mathbf{n + 1}}{\mathbf{n}} - \mathbf{1}\right| < \varepsilon$ is to find an N such that $n > N \Rightarrow \mathbf{n} > \frac{\mathbf{1}}{\varepsilon}$. Easy: let N be the first integer greater than $\frac{1}{\varepsilon}$.

Note how N is dependent on ε — the smaller the given ε, the larger the N.

(b) We show that $(a_n) = (1, 0, 1, 0, 1, \ldots)$ diverges by demonstrating that no fixed number c can be the limit of the sequence:

> Let $\varepsilon = \frac{1}{2}$. For any N, both a_{N+1} and a_{N+2} cannot fall within ε units of c, for the simple reason that the distance between any two adjacent elements of $(1, 0, 1, 0, 1, \ldots)$ is 1, and any two numbers within $\frac{1}{2}$ unit of c are less than 1 unit apart (see margin). So, no N "works" for $\varepsilon = \frac{1}{2}$.



CHECK YOUR UNDERSTANDING 9.2

(a) Let $(a_n) = \left(7 - \frac{101}{n}\right)$.

- (i) Prove that $\lim\limits_{n \to \infty} a_n = 7$.
- (ii) Find the smallest positive integer N such that $n > N \Rightarrow |a_n - 7| < \frac{1}{10}$
- (iii) Find the smallest positive integer N such that $n > N \Rightarrow |a_n - 7| < \frac{1}{100}$.

(b) Prove that $\lim\limits_{n \to \infty} |a_n| = 0$ if and only if $\lim\limits_{n \to \infty} a_n = 0$.

(c) Find (a_n) for which $\lim\limits_{n \to \infty} |a_n| \neq \lim\limits_{n \to \infty} a_n$.

Answers: (a-i) See page A-48.
(a-ii) $N = 1010$
(a-iii) $N = 10{,}100$
(b) and (c) See page A-48.

THE ALGEBRA OF SEQUENCES

When it comes to sums, differences, products, and quotients, sequences behave nicely:

Compare with Theorem 2.3, page 55.

THEOREM 9.2 If $\lim a_n = A$ and $\lim b_n = B$, then:

(a) $\lim ca_n = cA$, for any $c \in \Re$.

(b) $\lim (a_n \pm b_n) = A \pm B$

(The limit of a sum (or difference) equals the sum (or difference) of the limits)

(c) $\lim (a_n b_n) = AB$

(The limit of a product equals the product of the limits)

(d) $\lim \frac{a_n}{b_n} = \frac{A}{B}$, providing no $b_n = 0$ (see margin) and $B \neq 0$.

(The limit of a quotient equals the quotient of the limits)

Actually, we need only require that "eventually" no $b_n = 0$; which is to say that for some integer N $b_n \neq 0$ for all $n > N$. After all, whether a sequence converges or not has nothing to do with the start of the sequence, but only on what happens as $n \to \infty$.

PROOF: We establish (a) and the (sum-part) of (b). Proofs of (c) and (d) appear in Appendix B, page B-2.

(a) Case 1. $c = 0$. If $c = 0$, then each entry of the sequence (ca_n) is 0. Consequently: $\lim\limits_{n \to \infty} ca_n = 0 = 0A = c \lim\limits_{n \to \infty} a_n$.

Case 2. $c \neq 0$. For given $\varepsilon > 0$ we will exhibit an N such that:

$$n > N \Rightarrow |ca_n - cA| < \varepsilon$$

$$\text{i.e:} \quad n > N \Rightarrow |c||a_n - A| < \varepsilon$$

$$\text{i.e:} \quad \boldsymbol{n > N \Rightarrow |a_n - A| < \frac{\varepsilon}{|c|}}$$

Since $\lim a_n = A$, we know that for **<u>any</u>** $\bar{\varepsilon} > 0$ there exists an N such that $n > N \Rightarrow |a_n - A| < \bar{\varepsilon}$. In particular, for $\bar{\varepsilon} = \frac{\varepsilon}{|c|}$ we can choose N such that $\boldsymbol{n > N \Rightarrow |a_n - A| < \frac{\varepsilon}{|c|}}$, and we are done.

(b) Let $\varepsilon > 0$ be given. We are to find N such that

$$n > N \Rightarrow |(a_n + b_n) - (A + B)| < \varepsilon \quad (*)$$

Note that:

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \underset{\uparrow\ \text{triangle inequality}}{\le} |(a_n - A)| + |(b_n - B)|$$

So, if we can arrange things so that both $|(a_n - A)|$ and $|(b_n - B)|$ are less than $\frac{\varepsilon}{2}$, then (*) will hold. Let's arrange things:

Since $a_n \to A$, there exists N_A such that:

$$n > N_A \Rightarrow |a_n - A| < \frac{\varepsilon}{2}$$

Since $b_n \to B$, there exists N_B such that: $n > N_B \Rightarrow |b_n - B| < \frac{\varepsilon}{2}$

Letting $N = \max\{N_A, N_B\}$ (the larger of N_A and N_B), we find that for $n > N$: $|(a_n + b_n) - (A + B)| \le |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

A similar argument can be used to show that $\lim (a_n - b_n) = A - B$.

The next result is reminiscent of the Pinching Theorem of page 89:

THEOREM 9.3
PINCHING THEOREM FOR SEQUENCES

If the sequences $(a_n), (c_n)$, and (b_n) are such that (eventually) $a_n \le c_n \le b_n$, and if $\lim a_n = \lim b_n = L$, then $\lim c_n = L$.

PROOF: Let $\varepsilon > 0$ be given. We are to find N such that $n > N \Rightarrow |c_n - L| < \varepsilon$, which is equivalent to finding N such that $n > N \Rightarrow -\varepsilon < c_n - L < \varepsilon$ (why?). Let's do it:

$$\text{Since } a_n \le c_n \le b_n\text{: } a_n - L \le c_n - L \le b_n - L \quad (*)$$

Since $\lim a_n = \lim b_n = L$, we can choose N such that $n > N$ implies that both $|a_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$, which is to say that both $-\varepsilon < a_n - L < \varepsilon$ and $-\varepsilon < b_n - L < \varepsilon$.

Returning to (*) we have:

$$n > N \Rightarrow -\varepsilon < a_n - L \le c_n - L \le b_n - L < \varepsilon$$

trimming the above we have: $n > N \Rightarrow -\varepsilon < c_n - L < \varepsilon$

CHECK YOUR UNDERSTANDING 9.3

(a) Let (a_n) and (b_n) be such that, eventually, $a_n \le b_n$. Show that if $a_n \to A$ and $b_n \to B$, then $A \le B$.

(b) Give an example of two sequences (a_n) and (b_n) with $a_n < b_n$ such that $\lim a_n = \lim b_n$.

Answers: See page A-48

The following result offers a link between continuity and sequence convergence.

THEOREM 9.4 Let $(a_n)_{n=1}^{\infty}$ be a sequence, and let the set $\{a_n\}_{n=1}^{\infty}$ be contained in the domain of a function *f*. If $\lim\limits_{n \to \infty} a_n = L$ and, if *f* is continuous at *L*, then $\lim\limits_{n \to \infty} f(a_n) = f(L)$.

To put it succinctly:
$\lim f(a_n) = f(\lim a_n)$

PROOF: Given $\varepsilon > 0$ we are to find N such that $n > N \Rightarrow |f(a_n) - f(L)| < \varepsilon$. Let's do it:

Since *f* is continuous at *L*, we can choose $\delta > 0$ such that:

$$|x - L| < \delta \Rightarrow |f(x) - f(L)| < \varepsilon \quad (*)$$

We are given that $\lim\limits_{n \to \infty} a_n = L$. Letting $\delta > 0$ play the role of ε in Definition 9.2, page 321, we choose *N* such that:

$$n > N \Rightarrow |a_n - L| < \delta \quad (**)$$

Putting (*) and (**) together we have:

$$n > N \underset{(**)}{\Rightarrow} |a_n - L| < \delta \underset{(*)}{\Rightarrow} |f(a_n) - f(L)| < \varepsilon$$

EXAMPLE 9.2 Show that $\lim\limits_{n \to \infty} \sin\left(\frac{\pi n^2 + 10n}{2n^2}\right) = 1$.

SOLUTION: Since the sine function is continuous, we set our sights on determining $\lim\left(\frac{\pi n^2 + 10n}{2n^2}\right)$. Taking advantage of Theorem 9.2, we take the easy way out:

$$\lim\left(\frac{\pi n^2+10n}{2n^2}\right) = \lim\frac{\pi n^2}{2n^2}+\lim\frac{10n}{2n^2}$$

$$= \lim\frac{\pi}{2}+\lim\frac{5}{n} = \frac{\pi}{2}+0 = \frac{\pi}{2}$$

Applying Theorem 9.4 we have:

$$\lim\left[\sin\left(\frac{\pi n^2+10n}{2n^2}\right)\right] = \sin\left[\lim\left(\frac{\pi n^2+10n}{2n^2}\right)\right] = \sin\frac{\pi}{2} = 1$$

CHECK YOUR UNDERSTANDING 9.4

Evaluate:

(a) $\lim\limits_{n\to\infty}\sqrt{\frac{n+1}{n}}$ (b) $\lim\limits_{n\to\infty}\ln\left(\frac{n+1}{n}\right)$

Answers: (a) 1 (b) 0

L'HÔPITAL'S RULE AND SEQUENCES.

L'Hôpital's rule can be a useful tool in determining the limit of certain sequences. Consider the following example.

EXAMPLE 9.3 Verify that

(a) $\lim\limits_{n\to\infty}\frac{\ln n}{n} = 0$ (b) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

SOLUTION: (a) L'Hôpital's rule deals with differentiable functions and not sequences. But once we verify that $\lim\limits_{x\to\infty}\frac{\ln x}{x} = 0$, we will be able to conclude that $\lim\limits_{n\to\infty}\frac{\ln n}{n} = 0$, for we can let x "walk to infinity by stepping only on integer values." Let's verify:

$$\lim_{x\to\infty}\frac{\ln x}{x} \underset{\substack{\uparrow\\ \text{Theorem 8.2, page 304}}}{=} \lim_{x\to\infty}\frac{(\ln x)'}{x'} = \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = \lim_{x\to\infty}\frac{1}{x} = 0$$

(b) Since $\left(1+\frac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x}$, we first show that

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right) = 1:$$

$$\lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\left[\ln\left(1+\frac{1}{x}\right)\right]'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty}\frac{\frac{-x^{-2}}{1+\frac{1}{x}}}{-x^{-2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

In the exercises you are invited to show that

$$\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

for every $a \in \Re$.

In particular: $\lim\limits_{n \to \infty} n \ln\left(1 + \frac{1}{n}\right) = 1$

Theorem 9.4: $\lim\limits_{n \to \infty} e^{n \ln\left(1 + \frac{1}{n}\right)} = e^1$

$$\lim_{n \to \infty} e^{\ln\left(1 + \frac{1}{n}\right)^n} = e$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \text{(see margin)}$$

CHECK YOUR UNDERSTANDING 9.5

Verify:

(a) $\lim\limits_{n \to \infty} n^{1/n} = 1$ (b) $\lim\limits_{n \to \infty} \left(\frac{n+1}{n-1}\right)^n = e^2$

Answers: See page A-50

MONOTONE SEQUENCES

DEFINITION 9.3 A sequence (a_n) is:

INCREASING **Increasing** if there exists an integer N such that

$$a_n \le a_{n+1} \text{ for all } n > N.$$

DECREASING **Decreasing** if there exists an integer N such that

$$a_n \ge a_{n+1} \text{ for all } n > N.$$

MONOTONE **Monotone** if it is either increasing or decreasing.

BOUNDED **Bounded** if there exists a number M, called a **bound** of (a_n), such that $|a_n| \le M$ for all n.

Note that an increasing sequence is bounded below by its first element while a decreasing sequence is bounded above by its first element.

EXAMPLE 9.4 Show that the given sequence is monotone and bounded.

(a) $\left(\frac{n}{n+1}\right)$ (b) $\left(\frac{n}{e^n}\right)$

SOLUTION: (a) To get a feeling for the sequence $\left(\frac{n}{n+1}\right)$, we look at a few of its initial terms: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$. It certainly appears that the sequence is (strictly) increasing. Let's prove it:

$$\frac{n}{n+1} < \frac{n+1}{n+2} \Leftrightarrow n(n+2) < (n+1)(n+1)$$

$$\Leftrightarrow n^2 + 2n \le n^2 + 2n + 1$$

$$\Leftrightarrow 2n < 2n + 1 \quad \text{Yes!}$$

Since $0 < \frac{n}{n+1} < 1$, $\left(\frac{n}{n+1}\right)$ is bounded.

Why the derivative? Because the sign of the derivative can shed light on whether the function is increasing or decreasing.

(b) To determine if $\left(\frac{n}{e^n}\right)$ is monotone, we turn to the derivative of the function $f(x) = \frac{x}{e^x}$: $f'(x) = \left(\frac{x}{e^x}\right)' = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x}$. Since $f'(x) < 0$ for $x > 1$, the positive sequence $\left(\frac{n}{e^n}\right)$ is (strictly) decreasing. Consequently $0 < \frac{n}{e^n} < \frac{1}{e^1}$, and the sequence is bounded.

CHECK YOUR UNDERSTANDING 9.6

(a) Show that a sequence (a_n) with each $a_n > 0$ is:

(i) Increasing if $\frac{a_{n+1}}{a_n} \geq 1$.

(ii) Strictly decreasing if $\frac{a_{n+1}}{a_n} < 1$.

(b) Use (a) to show that the sequence $\left(\frac{e^n}{n!}\right)$ is strictly decreasing for $n > 1$.

Answers: See page A-49.

In Example 9.4 we showed that the sequences $\left(\frac{n}{n+1}\right)$ and $\left(\frac{n}{e^n}\right)$ are monotone and bounded. As such, they must converge; for:

THEOREM 9.5 Every bounded monotone sequence converges.

A proof of the above result appears in Appendix B, page B-4.

CHECK YOUR UNDERSTANDING 9.7

Prove that if a sequence converges, then it is bounded.

Answers: See page A-49.

SUBSEQUENCES

Roughly speaking, to generate a subsequence of (a_n) simply pluck, in order of appearance, some of its elements. Formally:

In terms of Definition 9.1: A subsequence of the sequence $f: Z^+ \to \mathfrak{R}$ is a composite function $f \circ h: Z^+ \to \mathfrak{R}$ where $h: Z^+ \to Z^+$ is a strictly increasing function.

DEFINITION 9.4
SUBSEQUENCE

$(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$ is a **subsequence** of (a_n) if each a_{n_i} is a term of (a_n), and $n_1 < n_2 < n_3 < \cdots$.

For example, $(1, 3, 5, 7, 9, \ldots)$ is a subsequence of $(1, 2, 3, 4, \ldots)$.

THEOREM 9.6 If the sequence (a_n) converges to L, then every subsequence of (a_n) converges to L.

PROOF: Let (a_{n_k}) be a subsequence of (a_n). We are to show that for any $\varepsilon > 0$ there exists N such that $n_k > N \Rightarrow |a_{n_k} - L| < \varepsilon$. Let's do it:

Since (a_n) converges to L, we can choose N such that $n > N \Rightarrow |a_n - L| < \varepsilon$. It follows that $n_k > N \Rightarrow |a_{n_k} - L| < \varepsilon$.

CHECK YOUR UNDERSTANDING 9.8

Construct a sequence (a_n) with a subsequence converging to 0 and another subsequence converging to 1.

Answers: See page A-51

THEOREM 9.7 If $|r| < 1$, then $\lim\limits_{n \to \infty} r^n = 0$.

PROOF: Assume, first, that $0 < r < 1$. Since $r > r^2 > r^3 > \cdots > 0$, (r^n) is monotone and bounded. As such, (r^n) converges to some number L (Theorem 9.5), as must the subsequence (r^{2n}) (Theorem 9.6). We then have:

$$L = \lim_{n \to \infty} r^{2n} = \lim_{n \to \infty} (r^n r^n) = \lim_{n \to \infty} (r^n) \lim_{n \to \infty} (r^n) = L^2$$

$$\text{But if } L = L^2\text{, then: } L = 0 \text{ or } L = 1$$

We can eliminate the $L = 1$ possibility, since $1 > r > r^2 > r^3, \ldots$.

To see that $\lim\limits_{n \to \infty} r^n = 0$ for $-1 < r < 0$, use the above result and the fact that for any sequence (a_n):

$$\lim_{n \to \infty} a_n = 0 \text{ if and only if } \lim_{n \to \infty} |a_n| = 0 \quad \text{[CYU 9.2(b)].}$$

Finally, if $r = 0$, then surely $\lim\limits_{n \to \infty} r^n = 0$.

CHECK YOUR UNDERSTANDING 9.9

Show that (r^n) diverges if $|r| > 1$ or $r = -1$, and that it converges for $-1 < r \le 1$.

Answers: See page A-50

EXERCISES

Exercises 1-3. Find a formula for the general n^{th} term of the sequence $(a_n)_{n=1}^{\infty}$, assuming that the indicated pattern continues.

1. $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$
2. $\left(\frac{2}{3}, -\frac{3}{9}, \frac{4}{27}, -\frac{5}{81}, \ldots\right)$
3. $\left(\frac{1}{2}, -\frac{4}{5}, \frac{9}{8}, -\frac{16}{11}, \frac{25}{14}, -\frac{36}{17} \ldots\right)$

Exercises 4-8. (a) Determine the limit L of the given sequence (a_n).

(b) Find the smallest integer N for which $n > N \Rightarrow |a_n - L| < \frac{1}{10}$.

(c) Find the smallest integer N for which $n > N \Rightarrow |a_n - L| < \frac{1}{100}$.

(d) Find the smallest integer N for which $n > N \Rightarrow |a_n - L| < \varepsilon$, for $\varepsilon > 0$.

4. $\left(\frac{1}{n}\right)$
5. $\left(1 + \frac{1}{\sqrt{2n}}\right)$
6. $\left(\frac{100}{5n+3}\right)$
7. $\left(\frac{2n+1}{5n}\right)$
8. $\left(\frac{2n+5}{n-1}\right)$

Exercises 9-12. Show that the given sequence (a_n) diverges.

9. $a_n = \frac{n}{10^{100}}$
10. $a_n = \frac{n^2}{n+100}$
11. $a_n = \frac{n}{\sqrt{n+100}}$
12. $\left(1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots\right)$

Exercise 13-28. Establish whether or not the given sequence (a_n) converges. If it does, determine its limit.

13. $a_n = \frac{4}{n}$
14. $a_n = \frac{n}{4}$
15. $a_n = 5 - \frac{1}{n}$
16. $a_n = 1 + \frac{(-1)^n}{n}$
17. $a_n = 1 + (-1)^n$
18. $a_n = \frac{n}{n+1}$
19. $a_n = \frac{(-1)^n n}{n+1}$
20. $a_n = \frac{(-1)^n n}{n^2+1}$
21. $a_n = \frac{n}{n^2+1}$
22. $a_n = \frac{(n-1)!}{n}$
23. $a_n = \frac{(n+1)!}{n!}$
24. $a_n = \sin n\pi$
25. $a_n = \cos n\pi$
26. $a_n = \frac{\sin n}{\sqrt{n}}$
27. $a_n = \frac{\sin\frac{n\pi}{3}}{\cos\frac{n\pi}{3}}$
28. $a_n = \left(\frac{\sin\frac{n\pi}{3}}{\cos\frac{n\pi}{3}}\right)^2$

Exercises 29-35. Employ the Pinching Theorem to find the limit of the given sequence (a_n).

29. $a_n = 5 + \left(-\frac{1}{2}\right)^n$
30. $a_n = \frac{(-1)^n}{n}$
31. $a_n = \frac{n + |\cos n|}{n}$
32. $a_n = \frac{\sin n}{n}$
33. $a_n = \sqrt{4 + \left(\frac{1}{n}\right)^2}$
34. $a_n = 2 + \frac{(-1)^n}{n}$
35. $a_n = \frac{3n + \left(-\frac{1}{3}\right)^n}{5n}$

Exercises 36-43. Employ Theorem 9.4 to find the limit of the given sequence (a_n).

36. $a_n = \sin\frac{\pi}{n}$ 37. $a_n = \left|\sin\frac{5n\pi}{4n-1}\right|$ 38. $a_n = \ln\left(\frac{n+1}{n}\right)$ 39. $a_n = \ln\left(\frac{en+1}{n-1}\right)$

40. $a_n = e^{\frac{3n}{n+1}}$ 41. $a_n = \ln n^2 - \ln 5n^2$ 42. $a_n = \tan\sqrt{\frac{(n\pi)^2-8}{16n^2}}$ 43. $a_n = \sqrt{\frac{4n}{\sqrt{n^2+1}}}$

Exercises 44-51. Employ l'Hôspital's Rule to determine if the given sequence (a_n) converges.

44. $a_n = \frac{n^3}{e^n}$ 45. $a_n = \frac{e^n}{n^3}$ 46. $a_n = \left(1+\frac{2}{n}\right)^n$ 47. $a_n = \frac{(\ln n)^2}{n}$

48. $a_n = n\sin\frac{1}{n}$ 49. $a_n = \frac{n^2\sin\frac{1}{n}}{2n-1}$ 50. $a_n = n\left(1-\cos\frac{1}{n}\right)$ 51. $a_n = n-\sqrt{n^2-n}$

Exercises 52-59. Determine if the given sequence (a_n) is increasing, decreasing, or neither.

52. $a_n = \frac{5n}{100n+51}$ 53. $a_n = \frac{n}{2n+1}$ 54. $a_n = \frac{4^n}{n!}$ 55. $a_n = \frac{n}{2^n}$

56. $a_n = \frac{(n+2)!}{5^n}$ 57. $a_n = \frac{-n}{2n+1}$ 58. $a_n = \frac{e^n}{n^2}$ 59. $a_n = \frac{\ln(n+1)}{n+1}$

Exercises 60-62. Apply Theorem 9.5 to show that the given sequence (a_n) converges.

60. $a_n = \frac{2^n}{n!}$ 61. $a_n = \frac{\ln n}{n}$ 62. $a_n = \frac{n^2}{n-1} - \frac{n^2}{n+1}$

63. Prove that the constant sequence $-1, 1, -1, 1, \ldots$ diverges.

64. (a) Exhibit two convergent sequences (a_n) and (b_n) such that $\lim_{n\to\infty}(a_n+b_n) = 5$.

(b) Exhibit two divergent sequences (a_n) and (b_n) such that $\lim_{n\to\infty}(a_n+b_n) = 5$

65. Construct a sequence that contains two convergent subsequences with different limits.

66. Construct a sequence that contains infinitely many convergent subsequences no two of which converge to the same value.

67. Prove that if $r \neq 0$, then $\left(\frac{n}{rn+1}\right)$ converges to $\frac{1}{r}$.

68. Prove that if $\lim_{n\to\infty} a_n = 0$ and if (b_n) is bounded, then $\lim_{n\to\infty} a_nb_n = 0$.

69. Prove that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty}|a_n| = |L|$.

70. Prove that $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$ for any $a \in \Re$.

§2. SERIES

We can certainly add the first three (or 3000) numbers of a given sequence $(a_n)_{n=1}^{\infty}$:

$$a_1 + a_2 + a_3 \quad \text{and} \quad \sum_{n=1}^{3000} a_n = a_1 + a_2 + a_3 + \cdots + a_{3000}.$$

But what about $\sum_{n=1}^{\infty} a_n$? Can we perform an infinite sum? Sometimes, and in the following sense:

DEFINITION 9.5

INFINITE SERIES

An **infinite series** is an expression of the form:

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_k + \cdots$$

TERMS

The numbers $a_1, a_2, \ldots$ are called the **terms** of the series.

For any $n \in Z^+$ the number

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

PARTIAL SUMS

is called the $\boldsymbol{n}^{\textbf{th}}$ **partial sum** of the series.

CONVERGING SERIES

The series $\sum_{i=1}^{\infty} a_i$ is said to **converge** to the number *L*, written $\sum_{i=1}^{\infty} a_i = L$, if the sequence of its partial sums $(s_n)_{n=1}^{\infty}$ converges to *L*.

A series that converges is said to be a **convergent series**. A series that does not converge is said to **diverge** or to be a **divergent series**.

One need not, of course, choose the letter n as the indexer of a series, nor start the sum at $n = 1$; e.g:

$$\sum_{i=1}^{\infty} a_i \text{ and } \sum_{n=0}^{\infty} a_n$$

Note, also that:

$$\sum_{n=0}^{\infty} a_i = \sum_{n=1}^{\infty} a_{n-1}$$

In general:

A change in the start of the index from $n = i$ to $n = i + k$ requires a change from a_n to a_{n-k}:

$$\sum_{n=i}^{\infty} a_n \text{ and } \sum_{n=i+k}^{\infty} a_{n-k}$$

for any integer *k*.

As you can see, there is a small step that takes us from the concept of a convergent sequence to that of a convergent series; namely:

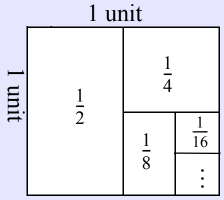
$$\sum_{i=1}^{\infty} a_i = L \quad \text{if and only if} \quad \lim_{n \to \infty} \underset{\substack{\uparrow \\ \text{partial sums}}}{s_n} = L$$

EXAMPLE 9.5

(a) Show that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

(b) Show that $\sum_{n=0}^{\infty} (-1)^n$ diverges.

A geometrical approach for (a). Start off with a square of length 1, and divide it into two equal pieces, each of area $\frac{1}{2}$:



Divide one of the two smaller regions again into two equal pieces, each of area $\frac{1}{4}$. Continuing this process indefinitely we arrive at a decomposition of the original square of area 1 into boxes of area $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, etc.; bringing us to: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1$.

SOLUTION: (a) Focusing on the sequence (s_n) of partial sums we have:

$$\begin{aligned}
s_1 &= \frac{1}{2}\\
s_2 &= \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4}\\
s_3 &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8}\\
&\vdots\\
s_n &= 1-\frac{1}{2^n} = 1-\left(\frac{1}{2}\right)^n
\end{aligned}$$

Since $\lim\limits_{n\to\infty}\left(1-\frac{1}{2^n}\right) \underset{\substack{\uparrow \\ \text{Theorem 9.7, page 329}}}{=} 1$, $\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$.

(b) A direct consequence of the fact that the sequence of partial sums of $\sum_{n=0}^{\infty}(-1)^n$, $s_0 = 1, s_1 = 1-1 = 0, s_2 = 1-1+1 = 1, \dots$; namely: 1, 0, 1, 0,..., diverges [Example 9.1(b), page 322].

You can also appeal to the following fact to conclude that $\sum_{n=0}^{\infty}(-1)^n$ diverges:

THEOREM 9.8
DIVERGENCE TEST

If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

PROOF: (By contradiction) If $\sum_{n=1}^{\infty} a_n$ converges to L then, since $a_n = s_n - s_{n-1}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) \underset{\substack{\uparrow \\ \text{Theorem 9.2(b), page 323}}}{=} \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0$$

The above theorem tells us that in order for a series to converge, it is necessary that its terms tend to zero. Necessary, yes; but not sufficient:

EXAMPLE 9.6 Show that the so-called **harmonic series**:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

diverges.

SOLUTION: Grouping the terms of the series as follows:

$$1+\frac{1}{2}+\underset{>2\cdot\frac{1}{4}=\frac{1}{2}}{\overset{2\text{ terms}}{\left(\frac{1}{3}+\frac{1}{4}\right)}}+\underset{>4\cdot\frac{1}{8}=\frac{1}{2}}{\overset{4\text{ terms}}{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}}+\underset{>8\cdot\frac{1}{16}=\frac{1}{2}}{\overset{8\text{ terms}}{\left(\frac{1}{9}+\ldots+\frac{1}{16}\right)}}+\underset{>16\cdot\frac{1}{32}=\frac{1}{2}}{\overset{16\text{ terms}}{\left(\frac{1}{17}+\ldots+\frac{1}{32}\right)}}+\ldots$$

we see that the sum of the entries in any of the above 2^n-blocks of terms exceeds $\frac{1}{2}$. As there are infinitely many such blocks, the series diverges [despite the fact that $a_n = \frac{1}{n}\to 0$].

CHECK YOUR UNDERSTANDING 9.10

(a) Find the fourth partial sum s_4 of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

(b) Find an expression for the n^{th} partial sum of the series.

(c) Does the series converge? If so, what is its sum.

Answers: (a) $\frac{4}{5}$ (b) $1-\frac{1}{n+1}$
(c) 1

GEOMETRIC SERIES

A **geometric series** is a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$, with $a \neq 0$.

Here is the whole geometric-series story:

THEOREM 9.9 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a+ar+ar^2+\ldots$$

is convergent if $|r|<1$, with sum:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

The geometric series diverges if $|r|\geq 1$.

As previously noted, we can rewrite $\sum_{n=1}^{\infty} ar^{n-1}$ as $\sum_{n=0}^{\infty} ar^{n}$.

PROOF: The Divergence Test tells us that $\sum\limits_{n=1}^{\infty} ar^{n-1}$ diverges if $|r| \geq 1$, because $ar^{n-1} \not\to 0$.

In the event that $|r| < 1$, we turn to the n^{th} partial sum:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

multiply by r: $rs_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

and subtract:
$$s_n - rs_n = a - ar^n$$
$$s_n(1-r) = a(1-r^n)$$
$$s_n = \frac{a(1-r^n)}{1-r} \quad (*)$$

Recalling that for any $-1 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$ (Theorem 9.7, page 329), we have:

$$\lim_{n\to\infty} s_n \underset{\substack{\uparrow \\ (*)}}{=} \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \underset{\substack{\uparrow \\ \text{Theorem 9.2, page 323}}}{=} \frac{a(1-\overset{= 0}{\overbrace{\lim\limits_{n\to\infty} r^n}})}{1-r} = \frac{a}{1-r}$$

EXAMPLE 9.7

Determine if the given series converges. If it does, find its sum.

(a) $\sum\limits_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ (b) $\sum\limits_{n=1}^{\infty} 5\left(\frac{3}{2}\right)^{n-1}$ (c) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$

SOLUTION: (a) Since $|r| = \frac{2}{3} < 1$, the series $\sum\limits_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$
$$\downarrow \quad \downarrow$$
$$\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1} = \frac{5}{1-\frac{2}{3}} = 15$$

(b) Since $|r| = \frac{3}{2} > 1$, the series $\sum\limits_{n=1}^{\infty} 5\left(\frac{3}{2}\right)^{n-1}$ diverges.

(c) While the series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ is not exactly in the form $\sum\limits_{n=1}^{\infty} ar^{n-1}$, it can be molded into that form: $\sum\limits_{n=1}^{\infty} \frac{1}{3^n} = \sum\limits_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$.

CHECK YOUR UNDERSTANDING 9.11

(a) Determine the sum of the series $\sum_{n=1}^{\infty}(-1)^n\frac{2}{3^n}$.

(b) Use the fact that $0.232323\ldots = 0.23 + 0.0023 + 0.000023 + \cdots$ is a convergent geometric series to show that $0.232323\ldots = \frac{23}{99}$.

Answers: (a) $-\frac{1}{2}$
(b) See page A-50

THEOREM 9.10

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then, for any $c \in \Re$:

$$\sum_{n=1}^{\infty}(a_n+b_n),\quad \sum_{n=1}^{\infty}(a_n-b_n),\text{ and } \sum_{n=1}^{\infty} ca_n$$

converge; moreover:

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n \text{ and } \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

PROOF: We establish the sum part of the theorem and relegate the remaining two parts to the exercises.

Let s_{n_a}, s_{n_b} and s_n denote the partial sums of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty}(a_n+b_n)$, respectively. Since

$$\begin{aligned} s_n &= (a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n) \\ &= (a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n) = s_{n_a}+s_{n_b} \end{aligned}:$$

$$\sum_{n=1}^{\infty}(a_n+b_n) = \lim_{n\to\infty}(s_n) \underset{\substack{\uparrow \\ \text{Theorem 9.2(b), page 322}}}{=} \lim_{n_a\to\infty}(s_{n_a}) + \lim_{n_b\to\infty}(s_{n_b}) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

EXAMPLE 9.8

Evaluate $\sum_{n=1}^{\infty}\left[\frac{9}{2^n} - 5\left(\frac{2}{3}\right)^{n-1}\right]$

SOLUTION: From Example 9.5(a) and Example 9.7(a):

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = 1 \text{ and } \sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1} = 15$$

Thus:

$$\sum_{n=1}^{\infty}\left[\frac{9}{2^n}-5\left(\frac{2}{3}\right)^{n-1}\right] = 9\sum_{n=1}^{\infty}\frac{1}{2^n}-\sum_{n=1}^{\infty}5\left(\frac{2}{3}\right)^{n-1} = 9-15 = -6$$

CHECK YOUR UNDERSTANDING 9.12

Evaluate: $\sum_{n=1}^{\infty}\left[\frac{2}{3^n}+\frac{3}{2^n}\right]$

Answers: 4

ALTERNATING SERIES

An **alternating series** is a series whose terms are alternately positive and negative; as is the case with the so-called **alternating harmonic series**:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

While the harmonic series of Example 9.6 diverges, its alternating cousin converges, by virtue of the following result:

THEOREM 9.11
ALTERNATING SERIES TEST

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = \underset{(\text{each } a_n>0)}{a_1-a_2+a_3-a_4+\cdots}$$

is such that:

$$a_{n+1}\le a_n \text{ for all } n, \text{ and } \lim_{n\to\infty}a_n = 0$$

then the series converges.

No mention of a specific limit appears in this theorem. That's okay, since in many applications one need only know whether or not a given series converges.

PROOF: We first consider partial sums with an even number of terms:

$$s_{2n} = (a_1-a_2)+(a_3-a_4)+(a_5-a_6)+\ldots+(a_{2n-1}-a_{2n})$$

The condition $a_{n+1}\le a_n$ assures us that the difference within each pair of parentheses is nonnegative. Consequently:

$$s_2\le s_4\le s_6\le s_8\le\cdots \qquad (*)$$

Pairing off the partial sums with an odd number of terms as follows:

$$s_{2n+1} = a_1-(a_2-a_3)-(a_4-a_5)-\ldots-(a_{2n}-a_{2n+1})$$

and noting that the difference within each pair of parentheses is nonnegative, we conclude that:

$$s_1\ge s_3\ge s_5\ge s_7\ge\ldots \qquad (**)$$

Moreover, since $s_{2n+1}-s_{2n} = a_{2n+1}>0$:

$$\mathbf{s_{2n}<s_{2n+1}} \qquad (***)$$

Combining (*), (**), and (***) brings us to:

$$\underset{\substack{\uparrow \\ \text{lower bound}}}{a_1 - a_2} = s_2 \le s_4 \le s_6 \le \cdots \le \boldsymbol{s_{2n}} < \boldsymbol{s_{2n+1}} \le s_{2n-1} \le \ldots \le s_5 \le s_3 \le s_1 = \underset{\substack{\uparrow \\ \text{upper bound}}}{a_1}$$

Since $(s_2, s_4, s_6, \ldots)$ and $(s_1, s_3, s_5, \ldots)$ are monotone and bounded, both sequences must converge (Theorem 9.5, page 328):

$$\lim_{n \to \infty} s_{2n+1} = L \quad \text{and} \quad \lim_{n \to \infty} s_{2n} = M$$

From: $L - M = \lim\limits_{n \to \infty} s_{2n+1} - \lim\limits_{n \to \infty} s_{2n}$

$$= \lim_{n \to \infty} (s_{2n+1} - s_{2n}) = \lim_{n \to \infty} a_{2n+1} = 0$$

we see that $L = M$. Consequently $\lim\limits_{n \to \infty} s_n = L$, and the alternating series converges.

EXAMPLE 9.9 Determine if the given alternating series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{2n-1} = 1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \cdots$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n-3}{n^2 - n - 19}$

(d) $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \frac{1}{5} - \frac{1}{32} + \ldots$

Solution: (a) Since $\dfrac{1}{(n+1)!} < \dfrac{1}{n!}$ and $\lim\limits_{n \to \infty} \dfrac{1}{n!} = 0$, the alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n!}$ converges by the Alternating Series Test.

(b) Since $\lim\limits_{n \to \infty} \dfrac{n}{2n-1} = \dfrac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{2n-1}$, regardless of its alternating nature, diverges by the Divergence Test.

(c) For $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}a_n = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{n-3}{n^2 - n - 19}$ does $a_n \to 0$? Is it true that $a_{n+1} \le a_n$? For both questions we turn to the function $f(x) = \dfrac{x-3}{x^2 - x - 19}$. Applying l'Hôpital's rule we find that:

$$\lim_{x \to \infty} \frac{x-3}{x^2-x-19} = \lim_{x \to \infty} \frac{(x-3)'}{(x^2-x-19)'} = \lim_{x \to \infty} \frac{1}{2x-1} = 0$$

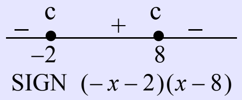
Consequently: $\lim_{n \to \infty} \frac{n-3}{n^2-n-19} = 0$.

While the initial terms of a series might affect its value, they have no affect on whether or not the series converges. Think about it!

To see if $a_{n+1} \le a_n$ (eventually -- see margin), we consider $f'(x)$:

$$\left(\frac{x-3}{x^2-x+19}\right)' = \frac{(x^2-x+19)(1)-(x-3)(2x-1)}{(x^2-x+19)^2}$$

$$= \frac{-x^2+6x+16}{(x^2-x+19)^2} = \frac{(-x-2)(x-8)}{(x^2-x+19)^2}$$



Noting that the denominator of $\frac{(-x-2)(x-8)}{(x^2-x+19)^2}$ is never negative and that its numerator is negative to the right of 8 (margin), we find that $f'(x) < 0$ for $x > 8$ and conclude that the terms $a_n = \frac{n-3}{n^2-n+19}$ decrease for all $n > 8$.

Conclusion: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-3}{n^2-n-19}$ converges by the Alternating Series Test.

(d) Although

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \frac{1}{5} - \frac{1}{32} + \dots$$

is an alternating series with $\lim_{n \to \infty} |a_n| = 0$ there is no assurance that it converges, as the condition $|a_{n+1}| < |a_n|$ of Theorem 9.11 does not hold for all n, rendering it useless for this series. Indeed, the series diverges:

Consider the positive and negative terms in the partial sum:

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \dots + \frac{1}{n} - \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \underset{\substack{\uparrow \\ \text{Theorem 9.9}}}{=} \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

The sum of its negative terms $-\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n}\right)$ are bounded below by -1 (see margin), while the sum of its positive terms $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ tend to ∞ (partial sums of the harmonic series of Example 9.6). It follows that $\lim_{n \to \infty} s_n = \infty$ and that the series therefore diverges.

CHECK YOUR UNDERSTANDING 9.13

Determine if the given alternating series converges.

(a) $\sum_{n=1}^{\infty}(-1)^n\frac{2}{3^n}$ (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+3}{n^2+n}$

Answers: (a) Converges.
(b) Converges.

APPROXIMATING THE SUM OF AN ALTERNATING SERIES

Here is a useful addition to Theorem 9.11:

THEOREM 9.12
ALTERNATING SERIES ERROR ESTIMATE

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

where each $a_n > 0$

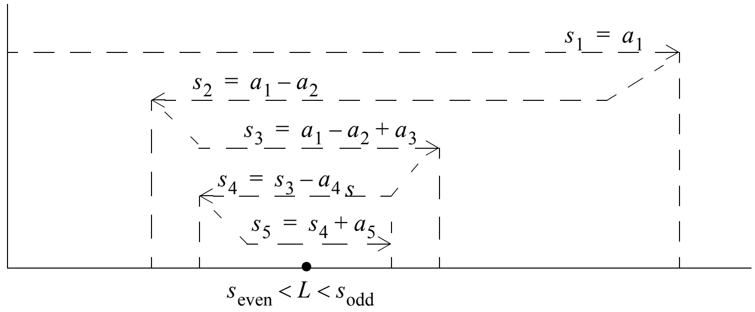
is such that:

$$a_{n+1} \le a_n \text{ for all } n\text{, and } \lim_{n\to\infty} a_n = 0$$

then the error E_N resulting by only summing the first N terms of the series is less than the $(N+1)^{\text{th}}$ term of the series:

$$E_N < a_{N+1}$$

PROOF: By the Alternating Series Test, the series converges. Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = L$. We observe that L lies between any two consecutive sums s_n and s_{n+1}:



Consequently: $E_N = |L - s_N| < |s_{N+1} - s_N| = a_{N+1}$

EXAMPLE 9.10 Consider the convergent alternating series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!} = 1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots = L$$

(a) Use Theorem 9.12 to find an upper bound for

$$\left|L-\left(1-\frac{1}{3!}+\frac{1}{5!}\right)\right|$$

(b) How many terms need to be added to insure that their sum falls within 0.00001 units of L?

SOLUTION: (a) Theorem 9.12 assures us that:

$$\left|L-\left(1-\frac{1}{3!}+\frac{1}{5!}\right)\right| < \frac{1}{7!} \approx 0.000198$$

(b) From the above we know that we will need to sum more that the first three terms of the series to be within 0.00001 units of L. Four terms, however, will certainly do the trick:

$$\left|L-\left(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}\right)\right| < \frac{1}{9!} \approx 0.000003$$

In a subsequent section we will show that:

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

As it turns out:

$$\left|(\sin 1)-\left(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}\right)\right| \approx 0.0000027$$

CHECK YOUR UNDERSTANDING 9.14

(a) Show that $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots$ converges.

(b) Approximate the sum $1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots$ to three decimal places.

Answers: (a) See page A51.
(b) 0.368

EXERCISES

Exercises 1-6. Express the given sum using the sigma notation: $\sum\limits_{n=}$. Do this in two ways — one with the index n starting at 1, and the other with n starting at 0.

1. $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$
2. $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}$
3. $1+10+100+1000+10000$
4. $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\dots$
5. $-\frac{5}{2}+\frac{10}{4}-\frac{15}{8}+\frac{20}{16}-\dots$
6. $\frac{x^2}{5}-\frac{x^3}{10}+\frac{x^4}{15}-\frac{x^5}{20}+\dots$

Exercises 7-8. Find the sum of the given series.

7. (a) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n-1}$ (b) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n}$ (c) $\sum\limits_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n}$

8. (a) $\sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$ (b) $\sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}$ (c) $\sum\limits_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n}$

Exercises 9-23. Determine if the series converges. If it does, find its sum.

9. $\sum\limits_{n=1}^{\infty}\left(\frac{5}{7}\right)^{n-1}$
10. $\sum\limits_{n=1}^{\infty}\frac{5}{7^n}$
11. $\sum\limits_{n=0}^{\infty}\frac{5}{100^n}$
12. $\sum\limits_{n=1}^{\infty}\frac{1}{2^{n-1}}$
13. $\sum\limits_{n=1}^{\infty}\frac{2}{3^{n-1}}$
14. $\sum\limits_{n=1}^{\infty}\frac{3^{n-1}}{2^n}$
15. $\sum\limits_{n=1}^{\infty}\frac{n^n}{n!}$
16. $\sum\limits_{n=1}^{\infty}\sin n\pi$
17. $\sum\limits_{n=0}^{\infty}\frac{1}{\sin\frac{n\pi}{4}}$
18. $\sum\limits_{n=0}^{\infty}\frac{2^n}{3^{n-1}}$
19. $\sum\limits_{n=1}^{\infty}\frac{3^{n-1}}{9^n}$
20. $\sum\limits_{n=1}^{\infty}\frac{3^{n-1}}{5^{2n+1}}$
21. $\sum\limits_{n=1}^{\infty}\left[\frac{3}{2^n}+\left(\frac{1}{4}\right)^n\right]$
22. $\sum\limits_{n=0}^{\infty}\frac{5}{2^n}+\left(-\frac{1}{4}\right)^n$
23. $\frac{1}{3}-\frac{3}{2}+\frac{1}{9}-\frac{3}{4}+\frac{1}{27}-\frac{3}{8}+\frac{1}{3^4}-\frac{3}{2^4}+\dots$

Exercises 24-32. Determine if the given alternating series converges.

24. $\sum\limits_{n=2}^{\infty}(-1)^{n-1}\frac{n^2}{n^2-1}$
25. $\sum\limits_{n=0}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$
26. $\sum\limits_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$

27. $\sum\limits_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$ 28. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$ 29. $\sum\limits_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$

30. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{e^n}{n^{10}}$ 31. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{\sin^2 n}$ 32. $\sum\limits_{n=0}^{\infty}(-1)^n\frac{e^{1/n}}{n}$

Exercises 33-38. Determine the number of terms that need to be added to insure that their sum falls within 0.0001 units of the value of the given convergent alternating series .

33. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{n}$ 34. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{2^n}$ 35. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{(2n)!}$

36. $\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{n2^n}$ 37. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ 38. $\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!}$

Exercises 39-44. Find the fourth partial sum s_4 of the given series, and an expression for its n^{th} partial sum. Determine the sum of the series.

39. $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ 40. $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$ 41. $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$

42. $\sum\limits_{n=1}^{\infty}\frac{1}{(4n-3)(4n+1)}$ 43. $\sum\limits_{n=1}^{\infty}\frac{3}{(2n-1)(2n+1)}$ 44. $\sum\limits_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$

Suggestion: Use partial fractions Suggestion: Use partial fractions Suggestion: Use partial fractions

45. Let $\sum\limits_{n=1}^{\infty}a_n$ and $\sum\limits_{n=1}^{\infty}b_n$ be convergent series. Prove that:

(a) The series $\sum\limits_{n=1}^{\infty}(a_n-b_n)$ converges, and $\sum\limits_{n=1}^{\infty}(a_n-b_n) = \sum\limits_{n=1}^{\infty}a_n-\sum\limits_{n=1}^{\infty}b_n$

(b) For any constant c, $\sum\limits_{n=1}^{\infty}ca_n$ converges, and $\sum\limits_{n=1}^{\infty}ca_n = c\sum\limits_{n=1}^{\infty}a_n$

46. (a) If $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, need both of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge? Justify your answer.

(b) If $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty} a_n$ converge, can $\sum_{n=1}^{\infty} b_n$ diverge? Justify your answer.

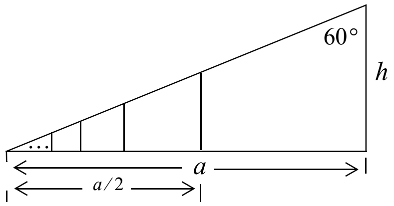
47. (a) Prove that if $\sum a_n$ converges and $\sum b_n$ diverges then $\sum(a_n + b_n)$ diverges.

(b) Show, by means of an example, that $\sum(a_n + b_n)$ may converge or may diverge when both $\sum a_n$ and $\sum b_n$ diverge.

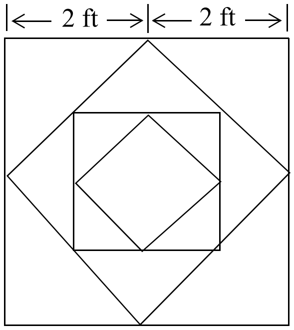
48. Show that both the series consisting of the positive terms and the series of the negative terms (in order) of the convergent series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ diverge.

49. A ball is dropped from a height of 60 feet. Each times it strikes the ground it bounces back two -thirds of the previous height. Determine the total vertical distance traveled by the ball before it comes to rest.

50. Find the sum of the bases, of the heights, and of the hypotenuses of the nested sequence of triangles depicted in the adjacent figure.



51. Find the sum of the areas of the nested sequence of squares depicted in the adjacent figure, wherein each square gives rise to the included square obtained by joining the midpoint of the sides of that square.



52. (Cantor Set) From the closed unit interval $[0, 1]$ remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ to arrive at $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Remove the middle third of each of those two resulting closed intervals to arrive at $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Remove the middle third of each of those four resulting intervals, and then the middle third of the resulting eight intervals, and continue the procedure indefinitely. Show that the sum of the lengths of all removed intervals equals 1, even though infinitely many numbers in $[0, 1]$ remain.

§3. SERIES OF POSITIVE TERMS

We will say that $\sum a_n$ is a **positive series** if there exists N such that $a_n > 0$ for every $n > N$.

> While the sum of the first N terms of a convergent series may effect its value, it will have no bearing whatsoever on whether or not the series converges. That being the case, when concerned solely on whether or not a series converges, we will let $\sum a_n$ represent the series, without indicating its starting point.

Consider a positive series $\sum a_n$. Since, eventually, each term is positive, the sequence of partial sums (s_n) is (eventually) increasing and bounded from below by its first term. That being the case we have:

THEOREM 9.13 A positive series converges if and only if its sequence (s_n) of partial sums is bounded from above.

PROOF: Clearly if (s_n) is not bounded, then the series diverges to ∞. On the other hand, if the monotonic sequence (s_n) is bounded from above, then it is bounded and must converge (Theorem 9.5, page 328).

Here is a particularly important consequence of the above theorem:

THEOREM 9.14
INTEGRAL TEST

Note: The "1" in $x \ge 1$ and throughout can be replaced by any positive integer c.

Let the continuous function f be such that:

(i) $f(x) > 0$ for all $x \ge 1$

(ii) $f(x) \ge f(y)$ if $1 \le x \le y$ (decreasing)

Let $a_n = f(n)$ for all $n \ge 1$. Then:

$\sum a_n$ converges if and only if $\int_1^\infty f(x)dx$ converges.

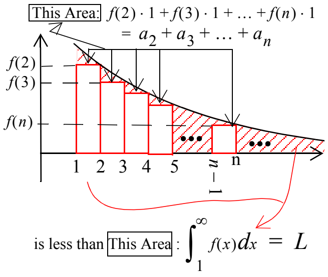
PROOF: Suppose $\int_1^\infty f(x)dx$ converges with $\int_1^\infty f(x)dx = L$. We show that the partial sums s_n of $\sum a_n$ are bounded from above:

Since, for $x \ge 1$, $f(x) > 0$: $\int_1^n f(x)dx < L$ for any n.

Since f is decreasing:

$$f(2) + f(3) + \cdots + f(n) \overset{\substack{\text{see margin} \\ \downarrow}}{\le} \int_1^n f(x)dx < L$$

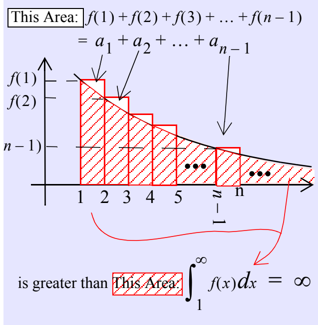
$$\text{Consequently:}\boldsymbol{f}(\mathbf{1}) + f(2) + f(3) + \cdots + f(n) \le \boldsymbol{f}(\mathbf{1}) + L \quad (*)$$



Since $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n)$ we have:

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &= f(1) + f(2) + f(3) + \ldots + f(n) \underset{\substack{\uparrow \\ (*)}}{\leq} f(1) + L = a_1 + L \end{aligned}$$

The convergence of $\sum a_n$ now follows from Theorem 9.13.



Now suppose that $\int_1^\infty f(x)dx = \infty$. A glance at the figure in the margin should convince you that $\sum_{n=1}^{\infty} a_n = \infty$.

EXAMPLE 9.11 Determine if the given series converges.

(a) $\sum \frac{n}{e^{n^2}}$ (b) $\sum \frac{\ln n}{n}$

SOLUTION: (a) The continuous function $f(x) = \frac{x}{e^{x^2}}$ is certainly positive for all $x \geq 1$. Moreover, since

$$f'(x) = \left(\frac{x}{e^{x^2}}\right)' = \frac{e^{x^2} - x(e^{x^2} \cdot 2x)}{(e^{x^2})^2} = \frac{1 - 2x^2}{e^{x^2}}$$

is negative for $x \geq 1$, f decreases over that interval.

Having observed that the hypotheses of the Integral Test are met, we turn to the improper integral $\int_1^\infty \frac{x}{e^{x^2}} dx$.

From $\int \frac{x}{e^{x^2}} dx \underset{\substack{\uparrow \\ u = x^2 \\ du = 2xdx}}{=} \frac{1}{2}\int e^{-u} du = -\frac{1}{2}e^{-u} + C = -\frac{1}{2e^{x^2}} + C$ we have:

$$\int_1^\infty \frac{x}{e^{x^2}} dx = \lim_{t \to \infty} -\frac{1}{2e^{x^2}}\Bigg|_1^t = -\frac{1}{2}\lim_{t \to \infty}\left(\frac{1}{e^{t^2}} - \frac{1}{e}\right) = \frac{1}{2e}$$

Since $\int_1^\infty \frac{x}{e^{x^2}} dx$ converges, so then does $\sum \frac{n}{e^{n^2}}$, by the Integral Test.

For $\sum \frac{n+5}{n(n+4)}$:

(b) We first note that the positive continuous function $f(x) = \frac{\ln x}{x}$ is decreasing for $x > e$:

$$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{x(\ln x)' - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0.$$

From $\int \frac{\ln x}{x} dx \underset{\substack{\uparrow \\ u = \ln x,\ du = \frac{1}{x}dx}}{=} \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$ we have:

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t \to \infty} \frac{(\ln x)^2}{2}\bigg|_1^t = \frac{1}{2}\lim_{t \to \infty} (\ln t)^2 = \infty$$

Since $\int_1^{\infty} \frac{\ln x}{x} dx$ diverges, so does $\sum \frac{\ln n}{n}$, by the Integral Test.

CHECK YOUR UNDERSTANDING 9.15

Use the Integral Test to show that the harmonic series $\sum \frac{1}{n}$ diverges.

(Compare your solution with that of Example 9.6, page 334.)

Answers: See page A-52

P-SERIES

Series of the form $\sum \frac{1}{n^p}$ are called *p*-series, and here is their story:

THEOREM 9.15

CONVERGENCE OF *P*-SERIES

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

converges if $p > 1$ and diverges if $p \le 1$.

PROOF: A direct consequence of Theorem 8.3, page 313, and Theorem 9.14.

The following result is a direct consequence of Theorem 9.13:

THEOREM 9.16
COMPARISON TEST

If the positive series $\sum a_n$ converges and if $\sum b_n$ is such that (eventually) $0 \le b_n \le a_n$, then $\sum b_n$ converges.

If the positive series $\sum a_n$ diverges, and if $a_n < b_n$, then $\sum b_n$ diverges.

EXAMPLE 9.12

(a) Show that $\sum \frac{\sin^2 n}{2^n + n}$ converges.

(b) Show that $\sum \frac{n+5}{n(n+4)}$ diverges.

SOLUTION: (a) Since $0 \le \sin^2 x \le 1$ for all x:

$$0 \le \frac{\sin^2 n}{2^n + n} \le \frac{1}{2^n + n} < \frac{1}{2^n}$$

Since the geometric series $\sum\left(\frac{1}{2}\right)^n$ converges (Theorem 9.9, page 334), so must $\sum\frac{\sin^2 n}{2^n + n}$, by the Comparison Test.

For $\sum\frac{n+5}{n(n+4)}$:

(b) Since $\frac{n+5}{n(n+4)} = \left(\frac{n+5}{n+4}\right)\frac{1}{n} > \frac{1}{n}$ and since the harmonic series $\sum\frac{1}{n}$ diverges, so must $\sum\frac{n+5}{n(n+4)}$, by the Comparison Test.

CHECK YOUR UNDERSTANDING 9.16

Determine if the given series converges.

(a) $\sum\frac{1}{n^2+n}$ (b) $\sum\frac{\sqrt{n}}{n-1}$

Answers: (a) Converges.
(b) Diverges.

Since the terms of the series $\sum\frac{1}{n^2+n}$ are less than the corresponding terms of the convergent series $\sum\frac{1}{n^2}$, $\sum\frac{1}{n^2+n}$ must also converge, by the Comparison Test. But what about the series $\sum\frac{1}{n^2-n}$ whose terms $\frac{1}{n^2-n}$ are greater than $\frac{1}{n^2}$? In a sense, "close enough is good enough:"

In the **EXERCISES** you are asked to verify that:

If $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

If $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

THEOREM 9.17
LIMIT COMPARISON TEST

If $\sum a_n$ and $\sum b_n$ are positive series and if

$$\lim_{n\to\infty}\frac{a_n}{b_n} = L > 0$$

then both series converge or both series diverge.

PROOF: If $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = L > 0$, then there exists N such that:

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2} \text{ for all } n > N.$$

Leading us to: $\frac{L}{2}b_n < a_n < \frac{3L}{2}b_n$ for all $n > N$ (*).

If $\sum a_n$ converges then so does $\sum \frac{L}{2} b_n$ by the Comparison Test (Theorem 9.16), as well as $\frac{2}{L}\sum \frac{L}{2} b_n = \sum b_n$ (see Theorem 9.10, page 336).

Similarly, if $\sum b_n$ converges, then so does $\sum \frac{3L}{2} b_n$, as well as $\sum a_n$, by the Comparison Test.

It follows, from the above argument, that if one of the series diverges, then so must the other (think about it).

EXAMPLE 9.13 Determine if the given series converges.

(a) $\sum \frac{n^2 - 2n + 1}{3n^4 + 5n}$ (b) $\sum \frac{2n^2 - 500}{n^3 + 50}$

SOLUTION: (a) Recalling that as $x \to \pm\infty$, the graph of the rational function $f(x) = \frac{\boldsymbol{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{\boldsymbol{b_m x^m} + b_{m-1}x^{m-1} + \cdots + b_0}$ resembles, in shape, that of $g(x) = \frac{\boldsymbol{a_n x^n}}{\boldsymbol{b_m x^m}}$ (see page 138), we might very well suspect that $\sum \frac{n^2 - 2n + 1}{3n^4 + 5n}$ will behave like the convergent p-series $\sum \frac{1}{n^2}$. Invoking the Limit Comparison Test, we find that it does:

$$\lim_{n\to\infty} \frac{\frac{n^2 - 2n + 1}{3n^4 + 5n}}{\frac{1}{n^2}} \underset{\substack{\uparrow \\ \text{invert and multiply}}}{=} \lim_{n\to\infty} \frac{n^4 - 2n^3 + n^2}{3n^4 + 5n} \underset{\substack{\uparrow \\ \text{divide numerator and} \\ \text{denominator by } n^4}}{=} \lim_{n\to\infty} \frac{1 - \frac{2}{n} + \frac{1}{n^2}}{3 + \frac{5}{n^3}} = \frac{1}{3} > 0$$

For $\sum \frac{2n^2 - 500}{n^3 + 50}$:

(b) We compare $\sum \frac{2n^2 - 500}{n^3 + 50}$ and $\sum \frac{n^2}{n^3} = \sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{2n^2 - 500}{n^3 + 50}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^3 - 500n}{n^3 + 50} = \lim_{n\to\infty} \frac{2 - \frac{500}{n^2}}{1 + \frac{50}{n^3}} = 2 > 0$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, so does $\sum \frac{2n^2 - 500}{n^3 + 50}$, by the Limit Comparison Test.

CHECK YOUR UNDERSTANDING 9.17

Determine if the given series converges.

(a) $\sum \frac{1}{3^n - 100}$ (b) $\sum \frac{5\sqrt{n} + 100}{\sqrt{n^3 - 3n + 1}}$

Answers: (a) Converges
(b) Diverges

THE RATIO TEST

Comparing a positive series with a geometric series leads us to the following important result:

THEOREM 9.18

RATIO TEST
(FOR POSITIVE SERIES)

Let $\sum a_n$ be a positive series with

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$$

$$\text{If } \begin{cases} L < 1, \text{ then the series converges} \\ L > 1 \text{ or } L = \infty, \text{ then the series diverges} \\ L = 1, \text{ then the test is inconclusive} \end{cases}$$

PROOF: Assume that $L < 1$, and let $\varepsilon > 0$ be small enough so that:

$$L + \varepsilon < 1$$

Since $\frac{a_{n+1}}{a_n} \to L$, we can choose N such that:

$$\frac{a_{n+1}}{a_n} < L + \varepsilon \text{ if } n \geq N$$

We then have:

$$\begin{aligned} a_{N+1} &< a_N(L+\varepsilon) \\ a_{N+2} &< a_{N+1}(L+\varepsilon) < a_N(L+\varepsilon)(L+\varepsilon) = \mathbf{a_N(L+\varepsilon)^2} \\ a_{N+3} &< a_{N+2}(L+\varepsilon) < \mathbf{a_N(L+\varepsilon)^2}(L+\varepsilon) = a_N(L+\varepsilon)^3 \\ &\vdots \\ a_{N+k} &< a_N(L+\varepsilon)^k \end{aligned}$$

Since $L + \varepsilon < 1$, the geometric series $\sum a_N(L+\varepsilon)^k$ converges. By the Comparison Test, so must $\sum a_n$ converge, since eventually $a_n < a_N(L+\varepsilon)^k$.

As for the rest of the proof:

CHECK YOUR UNDERSTANDING 9.18

Referring to Theorem 9.18:

(a) Verify that the series $\sum a_n$ diverges if $L > 1$ or $L = \infty$.

(b) Show that $L = 1$ for both the divergent series $\sum \frac{1}{n}$ and the convergent series $\sum \frac{1}{n^2}$.

Answers: See page A-53

EXAMPLE 9.14 Determine if the given series converges or diverges.

(a) $\sum \frac{1}{n!}$ (b) $\sum \frac{2^n}{n^{10}}$

SOLUTION: (a) Since:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \to \infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n \to \infty} \frac{n!}{n!(n+1)} = \lim_{n \to \infty} \frac{1}{n+1} = 0 < 1$$

$\sum \frac{1}{n!}$ converges, by the Ratio Test.

For $\sum \frac{2^n}{n^{10}}$:

(b) Since:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{2^{n+1}}{(n+1)^{10}}}{\frac{2^n}{n^{10}}} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)^{10}} \cdot \frac{n^{10}}{2^n}$$

$$= \lim_{n \to \infty} 2\left(\frac{n}{n+1}\right)^{10} = 2 \cdot 1^{10} = 2 > 1$$

the series $\sum \frac{2^n}{n^{10}}$ diverges, by the Ratio Test.

CHECK YOUR UNDERSTANDING 9.19

Determine if the given series converges.

(a) $\sum \frac{n^3}{5^n}$ (b) $\sum \frac{(2n)!}{(n!)^2}$

Answers: (a) Converges
(b) Diverges

THE ROOT TEST

Here is another powerful convergence test:

THEOREM 9.19
ROOT TEST

Let $\sum a_n$ be a positive series with

$$\lim_{n \to \infty} \sqrt[n]{a_n} = L$$

$$\text{If } \begin{cases} L < 1, \text{ then the series converges} \\ L > 1 \text{ or } L = \infty, \text{ then the series diverges} \\ L = 1, \text{ then the test is inconclusive} \end{cases}$$

PROOF: Assume that $L < 1$, and let $\varepsilon > 0$ be small enough so that:

$$L + \varepsilon < 1$$

Since $\sqrt[n]{a_n} \to L$, we can choose N such that:

$$\sqrt[n]{a_n} < L + \varepsilon \text{ or } a_n < (L+\varepsilon)^n \text{ for } n > N.$$

Since $L + \varepsilon < 1$, the geometric series $\sum\limits_{n=N+1} (L+\varepsilon)^n$ converges. By the Comparison Test, so must $\sum a_n$, since eventually $a_n < (L+\varepsilon)^n$.

As for the rest of the proof:

CHECK YOUR UNDERSTANDING 9.20

(a) Referring to Theorem 9.19, verify that the series $\sum a_n$ diverges if $L > 1$ or $L = \infty$.

(b) Show that $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = 1$ for any $p > 0$.

(c) Establish the claim that the Root Test is inconclusive if $L = 1$.

Answers: See page A-54

EXAMPLE 9.15 Determine if the given series converges.

(a) $\sum \frac{n}{(n^2+6)^n}$ (b) $\sum \frac{2^n}{n^3}$

SOLUTION: (a) Applying the Root Test we find that $\sum \frac{n}{(n^2+6)^n}$ converges:

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{(n^2+6)^n}} = \lim_{n\to\infty} \frac{\overset{\text{1 (CYU 9.5(a), page 327}}{n^{1/n}\nearrow}}{\underset{\infty}{n^2+6\searrow}} = 0 < 1$$

(b) Applying the Root Test we find that $\sum \frac{2^n}{n^3}$ diverges:

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^3}} = \lim_{n\to\infty} \frac{2}{(n^{1/n})^3} = \frac{2}{[\lim\limits_{n\to\infty} n^{1/n}]^3} \overset{\substack{\text{1 (CYU 9.5(a), page 327} \\ \downarrow}}{=} \frac{2}{1^3} = 2 > 1$$

CHECK YOUR UNDERSTANDING 9.21

Determine if the given series converges.

(a) $\sum \left(\frac{3n+2}{2n+1}\right)^n$ (b) $\sum \frac{1}{(\ln n)^n}$

Answers: (a) Diverges
(b) Converges

We've presented several methods which may enable you to determine if a series converges. Which should you use? Well, one that works is certainly a priority. That said, we hasten to point out that more than one of the methods might do the trick. If you addressed the series in (a) of the above CYU, chances are that you probably attacked it using the Root Test as the "*n*-exponent" stands out in the expression $\left(\frac{3n+2}{2n+1}\right)^n$. That's fine, for the Root Test will certainly do the job. But you could have simply observed that $a_n = \left(\frac{3n+2}{2n+1}\right)^n$ rather dramatically does not approach 0 as $n \to \infty$, and be done with it, by the Divergence Test.

EXERCISES

Exercises 1-3. Use the Integral Test to determine if the given series converges.

1. $\sum \frac{n}{n^2+1}$
2. $\sum \frac{1}{50n\ln n}$
3. $\sum \frac{n}{e^n}$

Exercises 4-6. Use the Comparison Test to determine if the given series converges.

4. $\sum \frac{n+1}{n^2}$
5. $\sum \frac{6}{6^n+3}$
6. $\sum \frac{1}{(n+3)^{5/4}}$

Exercises 7-9. Use the Limit Comparison Test to determine if the given series converges.

7. $\sum \frac{1}{n^2+5n}$
8. $\sum \frac{\sqrt{n}}{n-3}$
9. $\sum \frac{5\sqrt{n}+9}{2n^2+3}$

Exercises 10-12. Use the Ratio Test to determine if the given series converges.

10. $\sum \frac{n}{3^n}$
11. $\sum \frac{2^n}{n!}$
12. $\sum \frac{n^n}{n!}$

Exercises 13-15. Use the Root Test to determine if the given series converges.

13. $\sum \frac{(3n+5)^n}{(2n-1)^n}$
14. $\sum \frac{n}{5^n}$
15. $\sum \frac{3^n}{n^{10}}$

Exercises 16-57. Determine if the given series converges.

16. $\sum \frac{1}{\sqrt{n^2+1}}$
17. $\sum \frac{1}{\sqrt{n(n-1)}}$
18. $\sum \frac{\sqrt{n}+5}{n^2}$
19. $\sum \frac{1}{n\sqrt{n^2-1}}$
20. $\sum \sin\frac{1}{n}$
21. $\sum \frac{1}{n\ln n}$
22. $\sum_{n=1}^{\infty} \frac{3n+7}{n2^n}$
23. $\sum \frac{1}{n\ln(n+1)}$
24. $\sum \frac{\ln n}{n+1}$
25. $\sum \frac{2^n}{n^4}$
26. $\sum \frac{n^2}{5^n}$
27. $\sum \frac{5^n}{n!}$
28. $\sum \frac{e^n}{(1+e^n)^2}$
29. $\sum ne^{-2n}$
30. $\sum n\left(\frac{2}{3}\right)^n$
31. $\sum \frac{(n!)^2}{(2n)!}$
32. $\sum \frac{n^2}{2^n}$
33. $\sum \frac{\ln n}{n^2}$
34. $\sum \frac{1}{1+\sqrt{n}}$
35. $\sum \frac{\ln n}{e^n}$
36. $\sum \frac{(k+3)!}{3^k k!}$
37. $\sum \ln\left(\frac{n^2+1}{n^2}\right)$
38. $\sum \frac{1}{3^{n-1}+2}$
39. $\sum \frac{n+1}{(n+2)2^n}$

40. $\sum \frac{n^{50}}{e^n}$

41. $\sum \frac{n!}{n2^n}$

42. $\sum \frac{2^n\sqrt{n}}{4^n}$

43. $\sum \frac{2n-1}{2^n}$

44. $\sum \left(1+\frac{1}{n}\right)^n$

45. $\sum \left(\frac{n^2+10n}{2n^2+5}\right)^n$

46. $\sum \frac{(2n)!}{(3n)!}$

47. $\sum \frac{(2n)!}{n!2^n}$

48. $\sum \frac{2^n n!}{n^n}$

49. $\sum \frac{2^n}{1+(\ln n)^2}$

50. $\sum \frac{\tan^{-1} n}{1+n^2}$

51. $\sum \frac{1}{(n+1)[\ln(n+1)]^2}$

52. $\sum \left(\frac{1}{n^2}+\frac{n^2}{n^3+1}\right)$

53. $\sum \left(\frac{1}{n(\ln n)^2}-\frac{1}{n^2}\right)$

54. $\sum \left(\frac{n^3}{3n^4+1}+\frac{1}{n^{3/2}}\right)$

55. $\sum \left(\frac{1}{n^{3/2}}+\frac{e^n}{(1+e^n)^2}\right)$

56. $\sum \left(\frac{1}{n}-\frac{1}{n+1}\right)$

57. $\sum \left(\frac{\sqrt{n}+5}{n^2}+\frac{3n+1}{n2^n}\right)$

58. Prove that (a) If $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(b) If $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

59. Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

60. For what values of p does the series $\sum \left(\frac{n}{n^2+1}\right)^p$ converge?

61. Let $f(x) = \sin^2 \pi x + \frac{1}{x^2}$. Note that f is positive and continuous for $x \ge 1$. Show that $\int_1^{\infty} f(x)dx$ diverges while $\sum_{n=1}^{\infty} f(n)$ converges. Does this violate the Integral Test?

Exercise 62-67. Indicate True or False.

62. If $a_n > 0$ and $\sum a_n$ converges then $\sum \ln(1+a_n)$ converges.

63. If $a_n > 0$ and $\sum \ln(1+a_n)$ converges then $\sum a_n$ converges.

64. If $a_n > 0$ and $\sum a_n$ converges then $\sum \sin a_n$ converges.

65. If $a_n > 0$ and $\sum a_n$ converges then $\sum a_n^2$ converges.

66. If $\sum a_n$ converges then so does $\sum p(n)a_n$ for any polynomial p.

§4. ABSOLUTE AND CONDITIONAL CONVERGENCE

As it turns out, it is "easier" for a series $\sum a_n$ to converge than it is for $\sum |a_n|$ to converge:

THEOREM 9.20 If $\sum |a_n|$ converges, then $\sum a_n$ converges.

PROOF: Adding $|a_n|$ across the inequalities $-|a_n| \le a_n \le |a_n|$ we have:

$$0 \le a_n + |a_n| \le 2|a_n| \quad (*)$$

Since $\sum |a_n|$ converges, so does $\sum 2|a_n|$. Employing the Comparison Test (page 348) we find that $\sum a_n + |a_n|$ also converges [see (*)]. Noting that:

$$a_n = (a_n + |a_n|) - |a_n|$$

we conclude that $\sum a_n$ converges [see Theorem 9.10, page 336].

Does the converse of Theorem 9.20 hold? No:

The alternating series $\sum (-1)^n \frac{1}{n}$ converges, while the series $\sum \left|(-1)^n \frac{1}{n}\right| = \sum \frac{1}{n}$ does not (it is the harmonic series).

Bringing us to:

DEFINITION 9.6

ABSOLUTELY AND CONDITIONALLY CONVERGENT SERIES

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.

A convergent series $\sum a_n$ is **conditionally convergent** if $\sum |a_n|$ diverges.

Note that if $\sum |a_n|$ diverges, no conclusion can be drawn about the convergence or divergence of $\sum a_n$. For example:

$$\sum \left|(-1)^n \frac{1}{n}\right| \text{ diverges while } \sum (-1)^n \frac{1}{n} \text{ converges, and}$$

$$\sum \left|-\frac{1}{n}\right| \text{ diverges while } \sum -\frac{1}{n} = -\sum \frac{1}{n} \text{ diverges.}$$

EXAMPLE 9.16 Does the given series converge absolutely? If not, does it converge conditionally?

(a) $\sum \frac{\sin n}{n^2}$ (b) $\sum (-1)^{n-1} \frac{n-3}{n^2-n-19}$

SOLUTION: (a) Since $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$ and since the p-series $\sum \frac{1}{n^2}$ converges, the positive series $\sum \left|\frac{\sin n}{n^2}\right|$ converges by the Comparison Theorem. Thus, $\sum \frac{\sin n}{n^2}$ converges absolutely.

(b) The convergence of the alternating series $\sum (-1)^{n-1} \frac{n-3}{n^2-n-19}$ has already been established [see Example 9.9(c), page 338]. Does it converge absolutely? No:

Comparing the positive series $\sum \left|(-1)^{n-1} \frac{n-3}{n^2-n-19}\right| = \sum \frac{n-3}{n^2-n-19}$ with the divergent series $\sum \frac{1}{n}$ we have:

$$\lim_{n \to \infty} \frac{\frac{n-3}{n^2-n-19}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2-3n}{n^2-n-19} = \lim_{n \to \infty} \frac{1-\frac{3}{n}}{1-\frac{1}{n}-\frac{19}{n^2}} = 1 > 0$$

Since $\sum \left|(-1)^n \frac{n-3}{n^2-n-19}\right|$ diverges by the Limit Comparison Test, $\sum (-1)^n \frac{n-3}{n^2-n-19}$ is conditionally convergent.

CHECK YOUR UNDERSTANDING 9.22

Does the given series converge absolutely? If not, does it converge conditionally?

(a) $\sum \frac{n^2 \cos n}{3^n}$ (b) $\sum (-1)^n \frac{n^n}{n!}$

Answers:
(a) Converges Absolutely
(b) Diverges

THEOREM 9.21 If $\sum a_n$ is such that the series of its positive terms and the series of its negative terms both converge, then $\sum a_n$ converges absolutely.

PROOF: Plucking the positive elements (in order of appearance) of $\sum a_n$ we arrive at the positive series $\sum p_n$. Let $\sum p_n = P$.

Let $\sum q_n$ denote the series of the remaining (negative) elements of $\sum a_n$. Clearly, if $\sum q_n = Q$, then $\sum |q_n| = -Q$.

Since the sequence of partial sums of the series $\sum |a_n|$ is increasing and bounded above by $P - Q$, $\sum |a_n|$ converges (Theorem 9.13, page 345).

Why not use the Alternating Series Test of page 337? Because $|a_{n+1}| \le |a_n|$ does not hold here.

EXAMPLE 9.17 Does the alternating series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \dots + \frac{1}{2^n} - \frac{1}{3^n} + \dots$$

converge?

SOLUTION: Since both p-series $\sum \frac{1}{2^n}$ and $\sum \frac{1}{3^n}$ converge, the given series converges absolutely and therefore converges.

CHECK YOUR UNDERSTANDING 9.23

Does the given series converge?

(a) $\frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{16} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} - \frac{1}{2^{n+3}} + \dots$

(b) $\frac{1}{3} - 1 + \frac{1}{3^2} - \frac{1}{2} + \frac{1}{3^3} - \frac{1}{3} + \dots + \frac{1}{3^n} - \frac{1}{n} + \dots$

Answers:
(a) Converges (absolutely)
(b) Diverges

The convergence theorems of the previous section can be used to test a series $\sum a_n$ for absolute convergence — just apply the test to the positive series $\sum |a_n|$. In particular, we have:

THEOREM 9.22
RATIO TEST

For a given series $\sum a_n$ (not necessarily positive), with $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

$$\text{If } \begin{cases} L < 1: \sum a_n \text{ converges absolutely.} \\ L > 1 \text{ or } L = \infty: \sum a_n \text{ diverges.} \\ L = 1, \text{ the test is inconclusive} \end{cases}$$

PROOF: For $L < 1$: Apply the Ratio Test of page 350, to the positive series $\sum |a_n|$.

For $L > 1$ or $L = \infty$: Assume that $\left|\frac{a_{n+1}}{a_n}\right| \to L > 1$ or $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$.

Let N be such that $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for $n > N$. Since $|a_{n+1}| > |a_n| \ge 0$ for

$n > N$, $\lim\limits_{n \to \infty} |a_n| \neq 0$ which implies that $\lim\limits_{n \to \infty} a_n \neq 0$. That being the case, $\sum a_n$ diverges by the Divergence Test.

As for the inconclusive part of the Ratio Test, you can easily show that $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$ for both the convergent series $\sum a_n = \sum (-1)^n \dfrac{1}{n}$ and the divergent series $\sum a_n = \sum \dfrac{1}{n}$.

EXAMPLE 9.18 Use The Ratio Test to determine if the given series converges.

(a) $\sum \dfrac{(-100)^n}{n!}$ (b) $\sum \left(-\dfrac{1}{n}\right)^n \dfrac{(2n)!}{n!}$

SOLUTION: (a) Since, for $\sum a_n = \sum \dfrac{(-100)^n}{n!}$:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\dfrac{(100)^{n+1}}{(n+1)!}}{\dfrac{(100)^n}{n!}} = \lim_{n \to \infty} \frac{(100)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n}$$

$$= \lim_{n \to \infty} \frac{100}{n+1} = 0 < 1$$

the series $\sum \dfrac{(-100)^n}{n!}$ converges absolutely and therefore converges.

(b) For $\sum a_n = \sum \left(-\dfrac{1}{n}\right)^n \dfrac{(2n)!}{n!}$:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\left(\dfrac{1}{n+1}\right)^{n+1} \dfrac{[2(n+1)]!}{(n+1)!}}{\left(\dfrac{1}{n}\right)^n \dfrac{(2n)!}{n!}}$$

$$= \lim_{n \to \infty} \frac{(2n+2)!}{(n+1)^{n+1}(n+1)!} \cdot \frac{n^n(n)!}{(2n)!}$$

$$= \lim_{n \to \infty} \frac{(2n+2)!n!}{(2n)!(n+1)!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n \to \infty} \frac{(2n+1)(2n+2)}{(n+1)} \cdot \frac{\boldsymbol{n^n}}{(n+1)\boldsymbol{(n+1)^n}}$$

$$\text{(see margin):} \quad = \lim_{n \to \infty} \frac{4n^2}{n^2} \left(\frac{\boldsymbol{n}}{\boldsymbol{n+1}}\right)^{\boldsymbol{n}} = \frac{4}{e} > 1$$

Conclusion: $\sum \left(-\dfrac{1}{n}\right)^n \dfrac{(2n)!}{n!}$ diverges.

As $n \to \infty$:

$$\frac{(2n+1)(2n+2)}{(n+1)(n+1)} \to \frac{4n^2}{n^2} \to 4$$

As $n \to \infty$:

$$\left(\frac{\boldsymbol{n}}{\boldsymbol{n+1}}\right)^{\boldsymbol{n}} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \to \frac{1}{e}$$

Example 9.3(b), page 326

CHECK YOUR UNDERSTANDING 9.24

Determine if the given series converges.

(a) $\sum(-1)^n \frac{2^{2n-50}}{2n-100}$ (b) $\sum(-1)^n \frac{n^2}{2^n}$

Answers:
(a) Diverges
(b) Converges (absolutely)

REARRANGING THE TERMS OF A SERIES

Surely $2+9+7 = 9+7+2$, and so it is with any finite sum. But how about rearranging the terms of an infinite sum $\sum a_n$? Here is the surprising answer [at least for part (b)]:

THEOREM 9.23 (a) If $\sum a_n$ converges absolutely to L, then any series $\sum b_n$ obtained by rearranging the terms of $\sum a_n$ also converges to L.

(b) If $\sum a_n$ converges conditionally then, for **any** given L, the terms of the series can be rearranged so that the resulting series converges to L. The terms can also be rearranged so that the resulting series diverges.

You are invited to consider proofs of the above claims in Appendix B, page B-5. Here, we will content ourselves by showing that:

(a) If $\sum a_n$ converges absolutely then any series $\sum b_n$ obtained by rearranging the terms of $\sum a_n$ also converges absolutely.

(b) Let $\sum a_n$ be the conditionally convergent series $\sum(-1)^n \frac{1}{n}$. For **any** given L, the terms of the series can be rearranged so that the resulting series converges to L. The terms can also be rearranged so that the resulting series diverges.

For (a): If $\sum|a_n| = L$, then the partial sums of the series $\sum|b_n|$ are bounded above by L, and the series therefore converges (Theorem 9.13, page 345), as must $\sum b_n$ (Theorem 9.20).

(The argument does not establish the fact that $\sum a_n = \sum b_n$.)

For (b): We consider the two series consisting of the positive terms and of the negative terms of the alternating harmonic series:

(i) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$ (ii) $-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\ldots$

Since (i) diverges to ∞ (Exercise 48, page 344), starting at any point in that series, we can add enough subsequent terms to surpass any given positive number. Similarly, starting at any point in (ii), we can add enough subsequent terms to arrive at a number smaller than any given negative number. That being the case:

> For given L, we add enough of (i)'s elements to just get us to the right of L on the number line (none of them if $L \le 0$). We then add enough of (ii)'s elements to just get us to the left of L.
>
> Starting with the unused elements of (i) we again add enough of them to just get us again to the right of L, and then pick up enough of the unused elements of (ii) to just get us back to the left of L.
>
> The above process can be continued indefinitely, with each element of the original series appearing in the rearrangement. Since both the elements of (i) and (ii) approach 0 as $n \to \infty$, the amount by which the partial sums of the rearranged series differ from L must also approach 0.

CHECK YOUR UNDERSTANDING 9.25

Find a rearrangement of $\sum(-1)^n\frac{1}{n}$ which diverges to ∞.

Answers: See page A-66

EXERCISES

Exercises 1-33. Determine whether the series is absolutely convergent, conditionally convergent, or divergent

1. $\sum(-1)^n\frac{1}{\sqrt{n}}$
2. $\sum(-1)^n\frac{n}{n^3+1}$
3. $\sum(-1)^n\frac{1}{\ln n}$
4. $\sum(-1)^n\frac{(1.1)^n}{n^4}$
5. $\sum(-1)^n\frac{n^{50}}{n!}$
6. $\sum(-1)^n\frac{1}{n^{1/3}}$
7. $\sum(-1)^n\frac{1+n}{n^2}$
8. $\sum(-1)^n\frac{1}{2n}$
9. $\sum(-1)^n\frac{1}{n(\ln n)^2}$
10. $\sum(-1)^n\frac{1}{n\sqrt{n+1}}$
11. $\sum(-1)^n\frac{\sin(1/n)}{n}$
12. $\sum(-1)^n\frac{3+n}{4+n}$
13. $\sum(-1)^n\left(\frac{n^2+1}{2n^2+1}\right)^n$
14. $\sum\frac{(-500)^n}{n!}$
15. $\sum(-1)^n\frac{e^n}{n}$
16. $\sum\frac{10^n}{n3^{2n+1}}$
17. $\sum(-1)^n\frac{\tan^{-1}n}{n^2}$
18. $\sum(-1)^n\frac{1}{n(n+1)}$
19. $\sum(-1)^n 3^{1/n}$
20. $\sum(-1)^n\frac{\sin n}{n^2+1}$
21. $\sum(-1)^n\frac{1}{n\ln n}$
22. $\sum(-1)^n\frac{\ln n}{n}$
23. $\sum(-1)^n\frac{\ln n}{n}$
24. $\sum\left(\frac{n^2+1}{2n^2+1}\right)^n$
25. $\sum\frac{\cos(n\pi/4)}{n!}$
26. $\sum(-1)^n\frac{\sqrt{n}}{n^2+1}$
27. $\sum(-1)^n\frac{1}{n\sqrt{\ln n}}$
28. $\sum(-1)^n\frac{(2n)!}{n2^n n!}$
29. $\sum(-1)^n\frac{n+\ln n}{n^{3/2}}$
30. $\sum(-1)^n\left(1-\frac{1}{n}\right)^n$
31. $\sum(-1)^n\frac{(2n)!}{n2^n n!}$
32. $\sum(-1)^n(\sqrt{n+1}-\sqrt{n})$
33. $\sum(-1)^n(\sqrt{n+\sqrt{n}}-\sqrt{n})$

34. Show that the series $\sum\frac{x^n}{n!}$ converges for every real number x.

35. Prove that if $\sum a_n$ diverges, then $\sum|a_n|$ diverges.

36. (a) Prove that if $\sum a_n$ converges absolutely, then so does $\sum a_n^2$.

(b) Does the convergence of $\sum a_n^2$ imply the convergence of $\sum a_n$?

37. (a) Prove that if $\sum a_n$ and $\sum b_n$ converge absolutely, then so does $\sum (a_n + b_n)$.

    (b) Does the absolute convergence of $\sum (a_n + b_n)$ imply the absolute convergence of both $\sum a_n$ and $\sum b_n$?

    (c) Does the absolute convergence of $\sum (a_n + b_n)$ imply that either $\sum a_n$ or $\sum b_n$ converges?

    (d) Does the absolute convergence of $\sum a_n$ and $\sum b_n$ imply the absolute convergence of $\sum a_n b_n$?

    (e) Does the absolute convergence of $\sum a_n b_n$ imply the absolute convergence of both $\sum a_n$ and $\sum b_n$?

    (f) Does the absolute convergence of $\sum a_n b_n$ imply that either $\sum a_n$ or $\sum b_n$ converges?

38. (a) Prove that for any $|x| < 1$ the series $\sum x^n \sin n$ converges absolutely.

    (b) Prove that for any $|x| < 1$ and any y the series $\sum x^n \cos ny$ converges absolutely.

39. (a) Prove that if $\sum a_n$ converges absolutely and the sequence (y_n) is bounded, then $\sum a_n y_n$ converges absolutely.

    (b) Give an example of a convergent series $\sum a_n$ and a bounded sequence (y_n) for which $\sum a_n y_n$ does not converge.

§5. POWER SERIES

A **power series centered at 0**, is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

A **power series centered at a**, is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Here is a particularly important result concerning power series:

In the event that $a = 0$:

If $\sum c_n x^n$ converges at $x_0 \neq 0$, then it converges absolutely for all x such that $|x| < |x_0|$.

If $\sum c_n x^n$ diverges at x_0, then it diverges for all x such that $|x| > |x_0|$.

THEOREM 9.24 If $\sum c_n (x-a)^n$ converges at $x_0 \neq a$, then it converges absolutely for all x such that $|x-a| < |x_0 - a|$.

If $\sum c_n (x-a)^n$ diverges at $x = x_0$, then it diverges for all x such that $|x-a| > |x_0 - a|$.

PROOF: If $\sum c_n (x_0 - a)^n$ converges, then $c_n (x_0 - a)^n \to 0$ (Theorem 9.8, page 333). In particular we can find N such that:

$$|c_n (x_0 - a)^n| < 1 \Rightarrow |c_n| < \frac{1}{|x_0 - a|^n} \text{ for } n > N.$$

So, for any x and $n > N$:

$$|c_n||x-a|^n < \left|\frac{x-a}{x_0-a}\right|^n$$

It follows that the series $\sum c_n (x-a)^n$ converges absolutely for any x such that $|x-a| < |x_0 - a|$, as the terms of $\sum |c_n (x-a)^n|$ are (eventually) smaller than those of the convergent geometric series $\sum \left|\frac{x-a}{x_0-a}\right|^n$ (note that $\left|\frac{x-a}{x_0-a}\right| < 1$).

Now suppose that $\sum c_n (x_0 - a)^n$ diverges. Can $\sum c_n (x-a)^n$ converge at some x such that $|x-a| > |x_0 - a|$? No, for by the previous argument the convergence of $\sum c_n (x-a)^n$ would imply convergence of $\sum c_n (x_0 - a)^n$.

One additional step is required to take us from the above theorem to the one below — a proof of which appears in Appendix B, page B-6.

THEOREM 9.25

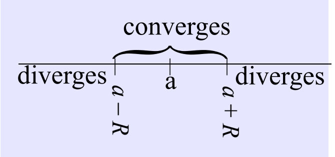
CONVERGENCE THEOREM FOR POWER SERIES

For a given power series $\sum c_n(x-a)^n$ there are only three possibilities:

(i) The series converges absolutely for all x.

(ii) The series converges only at $x = a$.

(iii) There exists $R > 0$ such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.



The number R in (iii) is called the **radius of convergence** of $\sum c_n(x-a)^n$. Moreover, in (i) and (ii) we write $R = \infty$ and $R = 0$, respectively.

The **interval of convergence** of a power series consists of those x for which the series converges:

In (i): $(-\infty, \infty)$ In (ii): $\{a\}$

In (iii):

$(a-R, a+R)$ with the possible addition of one or both endpoints.

As is illustrated in the following examples, one generally employs the Ratio Test (or the Root Test) to find the radius of convergence of a power series.

EXAMPLE 9.19 Find the radius of convergence and the interval of convergence of the given power series.

(a) $\sum \frac{x^n}{n!}$ (b) $\sum n!(x-2)^n$ (c) $\sum \frac{(x-4)^n}{n}$

The Ratio Test for $\sum a_n$:

If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ then:

$L < 1$: $\sum a_n$ converges (abs.)

$L > 1$: $\sum a_n$ diverges

$L = 1$: Test is inconclusive.

SOLUTION: (a) Applying the Ratio Test to $\sum \frac{x^n}{n!}$ we consider:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \frac{|x|^{n+1}}{(n+1)!}\frac{n!}{|x|^n} = \frac{|x|}{n+1}$$

Since, for any x, $\lim\limits_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$, the series $\sum \frac{x^n}{n!}$ converges (absolutely) for all x.

Conclusion: $\sum \frac{x^n}{n!}$ has radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

(b) Turning to $\sum n!(x-2)^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|(n+1)!(x-2)^{n+1}|}{|n!(x-2)^n|} = (n+1)|x-2|$$

In general, $\sum c_n(x-a)^n$ will certainly converge if $x = a$, as all of the terms are 0, for $n \geq 1$.

Since, for any $x \neq 2$, $(n+1)|x-2| \to \infty$ as $n \to \infty$, the series $\sum n!(x-2)^n$ converges only at $x = 2$ (see margin).

Conclusion: $\sum n!(x-2)^n$ has radius of convergence $R = 0$ and interval of convergence $\{2\}$.

(c) For $\sum \frac{(x-4)^n}{n}$ we turn to:

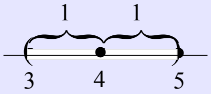
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{(x-4)^{n+1}}{(n+1)}\right|}{\left|\frac{(x-4)^n}{n}\right|} = \frac{|x-4|^{n+1}}{(n+1)} \cdot \frac{n}{|x-4|^n} = \frac{|x-4|n}{n+1}$$

Bringing us to: $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}|x-4|\frac{n}{n+1} = |x-4|$

The Ratio Test assures us that $\sum \frac{(x-4)^n}{n}$ converges (absolutely) when $|x-4| < 1$ and diverges when $|x-4| > 1$. It follows that the series $\sum\left|\frac{(x-4)^n}{n}\right|$ has radius of convergence $R = 1$.

We remind you that $|x-4|$ denotes the distance between x and 4 on the number line.

So:



Since the Ratio Test is inconclusive when $|x-4| = 1$, we need to consider the situation at $x = 3$ and at $x = 5$ separately. Let's do it:

At $x = 3$: $\sum \frac{(x-4)^n}{n} = \sum (-1)^n \frac{1}{n} \longleftarrow$ Converges (alternating harmonic series)

At $x = 5$: $\sum \frac{(x-4)^n}{n} = \sum \frac{1}{n} \longleftarrow$ Diverges (harmonic series)

It follows that the power series $\sum \frac{(x-4)^n}{n}$ has interval of convergence $[3, 5)$.

CHECK YOUR UNDERSTANDING 9.26

Find the radius of convergence and the interval of convergence of:

(a) $\sum \frac{x^n}{n}$ (b) $\sum n!x^n$ (c) $\sum \frac{(x+3)^n}{n-2}$

Answers: (a) $R = 1$, $[-1, 1)$
(b) $R = 0$, $\{0\}$
(c) $R = 1$, $[-4, -2)$

POWER SERIES FUNCTIONS

Just as the algebraic expression $\sqrt{x-5}$ serves to define the function $f(x) = \sqrt{x-5}$ with domain $[5, \infty)$, so then does the power series $\sum c_n(x-a)^n$ lead us to a function $f(x) = \sum c_n(x-a)^n$ with its interval of convergence as its domain.

Power series functions $f(x) = \sum c_n(x-a)^n$ behaves nicely when it comes to differentiation and integration. Specifically (proof omitted):

THEOREM 9.26 If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then:

(i) $f(x) = \sum c_n(x-a)^n$ is differentiable (and therefore continuous) on $(a-R, a+R)$, with:

$$\begin{aligned} f'(x) &= \left(\sum [c_n(x-a)^n]\right)' \\ &= \sum [c_n(x-a)^n]' = \sum nc_n(x-a)^{n-1} \end{aligned}$$

(ii)
$$\begin{aligned} \int f(x)dx &= \int\left(\sum c_n(x-a)^n\right)dx \\ &= \sum \int c_n(x-a)^n dx \\ &= \sum c_n \frac{(x-a)^{n+1}}{n+1} + C \end{aligned}$$

(iii) The power series in (i) and (ii) also have radius of convergence R.

It follows, from (i) and (iii), that the function

$$f(x) = \sum c_n(x-a)^n$$

has **derivatives of all orders** on the interval $(a-R, a+R)$.

EXAMPLE 9.20

(a) Verify that $\frac{1}{1-(x-a)} = \sum_{n=0}^{\infty} (x-a)^n$ for $|x-a| < 1$.

(b) Find a power series representation of $f(x) = \frac{1}{x+3}$, centered at 0, for $|x| < 1$; and centered at 1, for $|x-1| < 4$.

(c) Use Theorem 9.26(i) and (a) to find a power series representation of $f(x) = \frac{1}{(1-x)^2}$, centered at $x = 0$ for $|x| < 1$.

(d) Use Theorem 9.26(ii) and (a) to find a power series representation of $f(x) = \ln(1-x)$, centered at $x = 0$ for $|x| < 1$.

SOLUTION: (a) Employing Theorem 9.9, page 334 (margin):

$$\text{For } |x|<1:\ \frac{1}{1-\boxed{x}}=\sum_{n=0}^{\infty}x^n \qquad (*)$$

Replacing x with $x-a$ yields the desired result:

$$\text{For } |x-a|<1:\ \frac{1}{1-\boxed{(x-a)}}=\sum_{n=0}^{\infty}(x-a)^n \qquad (**)$$

For $|r|<1$:

$$\frac{a}{1-r}=\sum_{n=0}^{\infty}ar^n$$

In a more general form:

For $\square<1$:

$$\frac{a}{1-\square}=\sum_{n=0}^{\infty}a\square^n$$

b) [Centered at 0] The trick is to mold $\frac{1}{x+3}$ into the form $\frac{1}{1-\boxed{x}}$, and then take advantage of (*) in (a):

$$\frac{1}{x+3}=\frac{1/3}{1-\left(-\frac{x}{3}\right)}=\frac{1}{3}\cdot\frac{1}{1-\boxed{\left(-\frac{x}{3}\right)}}=\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n$$

$$=\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n+1}x^n$$

The above holds for $\left|-\frac{x}{3}\right|<1$, i.e: $|x|<3$.

[Centered at 1] We now mold $\frac{1}{x+3}$ into the form $\frac{1}{1-\boxed{x-1}}$ and then take advantage of (**) in (a):

$$f(x)=\frac{1}{x+3}=\frac{1}{(x-1)+4}=\frac{1}{4}\cdot\frac{1}{1-\boxed{\left(-\frac{x-1}{4}\right)}}$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x-1}{4}\right)^n=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^{n+1}(x-1)^n$$

The above holds for $\left|\left(-\frac{x-1}{4}\right)\right|<1$, i.e: $|x-1|<4$

(c) On the one hand, for $f(x)=\frac{1}{1-x}$ we have:

$$f'(x)=\left(\frac{1}{1-x}\right)'=[(1-x)^{-1}]'=\frac{1}{(1-x)^2}$$

On the other hand, from (a) and Theorem 9.26 we have:

$$f'(x)=\sum_{n=0}^{\infty}(x^n)'=\sum_{n=1}^{\infty}nx^{n-1}=0+1+2x+3x^2+\cdots$$

Consequently, for $|x|<1$:

$$f'(x)=\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}=1+2x+3x^2+4x^3+\cdots$$

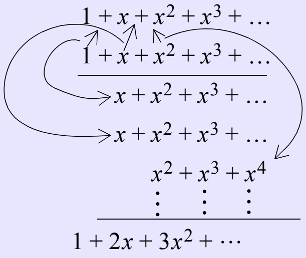
A point of interest:

If you "physically multiply" the infinite polynomial

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots$$

with itself you will also get

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots:$$



This is no fluke, for the above "product of two series" result does hold in general.

(d) For $|x| < 1$:

$$\ln(1-x) = -\int \frac{dx}{1-x}$$

From (a): $$= -\int\left(\sum_{n=0}^{\infty} x^n\right)dx$$

Theorem 9.26(ii): $$= -\sum_{n=0}^{\infty}\int x^n dx = \left(-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right) + C = \left(-\sum_{n=1}^{\infty}\frac{x^n}{n}\right) + C$$

Evaluating $\ln(1-x) = \left(-\sum_{n=1}^{\infty}\frac{x^n}{n}\right) + C$ at $x = 0$ we find that $C = 0$.

Consequently, for $|x| < 1$: $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$.

Note that if you differentiate the above power series representation of $\ln(1-x)$, term by term, you end up with the negative of the power series representation of $\frac{1}{1-x}$ in (a). Not surprising, as $[\ln(1-x)]' = -\frac{1}{1-x}$.

CHECK YOUR UNDERSTANDING 9.27

Find the second derivative of $f(x) = \frac{1}{1-x}$ along with a power series representation centered at $x = 0$ for $|x| < 1$.

Answer:

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

EXAMPLE 9.21 Find a power series representation for $f(x) = \tan^{-1}x$, for $|x| < 1$.

SOLUTION: Recalling that $(\tan^{-1}x)' = \frac{1}{1+x^2}$, we set our sights on finding a power series representation for $\frac{1}{1+x^2}$, and will then integrate it to arrive at a power series representation for $f(x) = \tan^{-1}x$:

Replacing x with $-x^2$ in Example 9.20(a), we conclude that, for $|-x^2| < 1$ (or $|x| < 1$):

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

So, for $|x| < 1$:

$$\tan^{-1}x = \int\frac{dx}{1+x^2} = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\int x^{2n}dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + C$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots\right) + C$$

Evaluating $\tan^{-1}x = \left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\right) + C$ at $x = 0$ we find that $C = 0$ (recall that $\tan^{-1}0 = 0$).

Consequently, for $|x| < 1$:

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

CHECK YOUR UNDERSTANDING 9.28

(a) Represent $f(x) = \frac{1}{x}$ as a power series centered at 1, for $|x-1| < 1$. Suggestion: Consider Example 9.20(a).

(b) Represent $f(x) = \frac{1}{x^2}$ as a power series centered at 1, for $|x-1| < 1$.

Answers:

(a) $\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$

(b) $\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$

EXERCISES

Exercises 1-15. Determine the radius of convergence and interval of convergence of the given power series.

1. $\sum \frac{x^n}{n+1}$
2. $\sum \frac{nx^n}{n+1}$
3. $\sum \frac{nx^n}{2^n}$
4. $\sum \frac{(-1)^n x^n}{n2^n}$
5. $\sum \frac{n(n+1)x^n}{5^n}$
6. $\sum n^2(x-2)^n$
7. $\sum \frac{(x+4)^n}{2^n}$
8. $\sum (-1)^n \frac{x^{2n}}{(2n)!}$
9. $\sum \frac{n2^n(x-1)^n}{n+1}$
10. $\sum \frac{(x+1)^n}{n^n}$
11. $\sum \frac{(x+4)^n}{n(n+1)}$
12. $\sum \frac{(x+4)^n}{2^n n^2}$
13. $\sum (-1)^{n-1} \frac{x^n}{\sqrt{n}}$
14. $\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
15. $\sum \frac{x^n}{1+n^2}$

Exercises 16-24. Express the given function as a power series centered at 0 and denote both its radius and interval of convergence.

16. $f(x) = \frac{1}{1-2x}$
17. $f(x) = \frac{1}{1+x}$
18. $f(x) = \frac{1}{2-x}$
19. $f(x) = \frac{2}{3-x}$
20. $f(x) = \frac{x}{1+x}$
21. $f(x) = \frac{x}{9+x^2}$
22. $f(x) = \frac{x^2}{9+x^2}$
23. $f(x) = \ln(5-x)$
24. $f(x) = \tan^{-1}\frac{x}{2}$

Exercises 25-26. Represent the given function in partial fractions form, and then express it as a sum of power series centered at 0. Denote both its radius and interval of convergence.

25. $f(x) = \frac{3}{x^2-x-2}$
26. $f(x) = \frac{x}{x^2+3x+2}$

Exercises 27-29. Use Theorem 9.26 to obtain a power series centered at 0 and denote both its radius and interval of convergence. [See Example 9.20]

27. $f(x) = \ln(1-2x)$
28. $f(x) = \frac{1}{(2x+3)^2}$
29. $f(x) = \ln(1-x^2)$

Exercises 30-32. Represent the given function as a power series centered at 0 and denote both its radius and interval of convergence.

30. $f(x) = \tan^{-1} 2x$

31. $f(x) = \tan^{-1} x^2$

32. $f(x) = \int_0^x \tan^{-1} t \, dt$

33. Show that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a solution of the differentiable equation $f''(x) - f(x) = 0$.

34. Show that $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ is a solution of the differentiable equation $f''(x) + f(x) = 0$.

35. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(n+s)!}{n!(n+t)!} x^n$, where s and t are positive integers.

36. Prove that if the power series $\sum_{n=0}^{\infty} c_n x^n$ has a finite radius of convergence R, then $\sum_{n=0}^{\infty} c_n x^{2n}$ has radius of convergence $\sqrt{R}$.

37. Show that if $\lim_{n \to \infty} |c_n|^{1/n} = L \neq 0$, then the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $\frac{1}{L}$.

38. Use the Root Test to find the interval of convergence of $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$.

§6. TAYLOR SERIES

We know that if a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function $f(x) = \sum c_n(x-a)^n$ has derivatives of all orders on $(a-R, a+R)$ (Theorem 9.26, page 367). As it turns out, those derivatives can be used to find the coefficients c_n of $\sum c_n(x-a)^n$:

THEOREM 9.27

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$, then:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Where, for any positive integer n, $f^{(n)}$ denotes the n^{th} derivative of f, and where $f^{(0)}$ is used to represent the function f.

You are invited to establish the above result in the exercises. For now:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \Rightarrow \boldsymbol{c_0} = f(a) = \frac{\boldsymbol{f(a)}}{\mathbf{0!}}$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \Rightarrow \boldsymbol{c_1} = f'(a) = \frac{\boldsymbol{f'(a)}}{\mathbf{1!}}$$

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + 4\cdot 5c_5(x-a)^3 + \cdots \Rightarrow \boldsymbol{c_2} = \frac{f''(a)}{2} = \frac{\boldsymbol{f''(a)}}{\mathbf{2!}}$$

$$f^{(3)}(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + 4\cdot 5\cdot 6c_6(x-a)^3 + \ldots \Rightarrow \boldsymbol{c_3} = \frac{\boldsymbol{f^{(3)}(a)}}{\mathbf{3!}}$$

$$f^{(4)}(x) = 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5(x-a) + 3\cdot 4\cdot 5\cdot 6c_6(x-a)^2 + 4\cdot 5\cdot 6\cdot 7c_8(x-a)^3 + \ldots \Rightarrow \boldsymbol{c_4} = \frac{\boldsymbol{f^{(4)}(a)}}{\mathbf{4!}}$$

We know, from the previous section, that $f(x) = \frac{1}{x^2}$ has a power series representation over the interval $(0, 2)$ [CYU 9.28(b), page 370], and that $f(x) = \frac{1}{(1-x)^2}$ has a power series representation over $(-1, 1)$ [Example 9.20(c), page 368]. That being the case, we should be able to arrive at those power series via Theorem 9.27; and so we shall:

EXAMPLE 9.22

(a) Express the function $f(x) = \frac{1}{x^2}$ as a power series, centered at 1, for $|x-1| < 1$.

(b) Express the function $f(x) = \frac{1}{(1-x)^2}$ as a power series, centered at 0, for $|x| < 1$.

SOLUTION:

(a) We are to find c_n such that $f(x) = \frac{1}{x^2} = \sum_{n=0}^{\infty} c_n(x-1)^n$ for $|x-1| < 1$. Theorem 9.27 tells us that $c_n = \frac{f^{(n)}(1)}{n!}$. Grinding away, with the hope of spotting a pattern, we find that:

$$f(x) = x^{-2} \qquad c_0 = \frac{f(1)}{0!} = \frac{1}{0!} = 1$$

$$f'(x) = -2x^{-3} \qquad c_1 = \frac{f'(1)}{1!} = \frac{-2}{1!} = -2$$

$$f''(x) = 2\cdot 3x^{-4} \qquad c_2 = \frac{f''(1)}{2!} = \frac{2\cdot 3}{2!} = 3$$

$$f^{(3)}(x) = -2\cdot 3\cdot 4x^{-5} \qquad c_3 = \frac{f^{(3)}(1)}{3!} = \frac{-2\cdot 3\cdot 4}{3!} = -4$$

$$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5x^{-6} \qquad c_4 = \frac{f^{(4)}(1)}{4!} = \frac{2\cdot 3\cdot 4\cdot 5}{4!} = 5$$

Pattern: $c_n = (-1)^n(n+1)$. Thus:

We know from CYU 9.28 that this power series has radius of convergence 1, a fact that could also be addressed at this point.

$$f(x) = \frac{1}{x^2} = \sum_{n=0}^{\infty} (-1)^n(n+1)(x-1)^n$$

(b) We are to find c_n such that $f(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} c_n x^n$. Let's do it:

$$f(x) = (1-x)^{-2} \qquad c_0 = \frac{f(0)}{0!} = 1$$

$$f'(x) = 2(1-x)^{-3} \qquad c_1 = \frac{f'(0)}{1!} = \frac{2}{1!} = 2$$

$$f''(x) = 2\cdot 3(1-x)^{-4} \qquad c_2 = \frac{f''(0)}{2!} = \frac{2\cdot 3}{2!} = 3$$

$$f^{(3)}(x) = 2\cdot 3\cdot 4(1-x)^{-5} \qquad c_3 = \frac{f^{(3)}(0)}{3!} = \frac{2\cdot 3\cdot 4}{3!} = 4$$

$$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5(1-x)^{-6} \qquad c_4 = \frac{f^{(4)}(0)}{4!} = 5$$

Pattern: $c_n = n+1$. Thus:

We know from Example 9.20, that this power series has radius of convergence 1, a fact that could also be addressed at this point.

$$f(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=1}^{\infty} nx^{n-1}$$

CHECK YOUR UNDERSTANDING 9.29

Use Theorem 9.27 to find a power series representation for $f(x) = \ln(1-x)$ over $(-1, 1)$. [See Example 9.20(d), page 367.]

Answer:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

At this point we know that if f has derivatives of all orders at a, and **IF it can be represented by a power series** centered at a with radius of convergence R (as we knew to be the case with the functions of Example 9.22) then:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < R.$$

Turning things around we ask the following question:

> If f has derivatives of all orders at a, and if $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ has radius of convergence R, need that power series converge back to $f(x)$ for $|x-a| < R$? The answer is, ***NOT ALWAYS*** — but first:

The bad news is that the Taylor series of f need not represent f. The good news is that it will for all "reasonable" functions.

DEFINITION 9.7 TAYLOR AND MACLAURIN SERIES

If f has derivatives of all orders at a, then the **Taylor series for *f* about *a*** is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor series for f about 0 has a special name — it is called the **Maclaurin series for *f***.

To illustrate, let's find the Maclaurin series for $f(x) = e^x$, as well as its Taylor series centered at 2.

Since $f(x) = f'(x) = f''(x) = f^{(3)}(x) = \cdots = e^x$, we have:

$f^{(n)}(0) = e^0 = 1$ for all n	$f^{(n)}(2) = e^2 =$ for all n
Maclaurin series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$	Taylor series about 2: $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

Using the Ratio Test, we can easily show that both of the above power series have radius of convergence $R = \infty$:

$\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \left\lvert\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right\rvert = \frac{\lvert x\rvert}{n+1\underset{\text{for all } x}{\uparrow}} \to 0 < 1$	$\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \left\lvert\frac{\frac{e^2(x-2)^{n+1}}{(n+1)!}}{\frac{e^2(x-2)^n}{n!}}\right\rvert = \frac{\lvert x-2\rvert}{n+1\underset{\text{for all } x}{\uparrow}} \to 0 < 1$

Yes, both the Maclaurin and Taylor series of $f(x) = e^x$ converge throughout $(-\infty, \infty)$, but the question remains as to whether or not they converge to $f(x) = e^x$. Generalizing our concern, we turn to the following question:

For a given function f with derivatives of all orders, when will

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n ?$$

There is an easy answer:

THEOREM 9.28 If the Taylor series of f has a radius of convergence R, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < R$$

if and only if

$$\lim_{N\to\infty}\left(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right) = 0$$

PROOF: For any x in $|x-a| < R$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \overset{\substack{\text{sequence of partial sums converge to } f(x) \text{ (see Definition 9.5, page 332)}\\ \downarrow}}{\Leftrightarrow} f(x) = \lim_{N\to\infty}\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\Leftrightarrow \lim_{N\to\infty}\left(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right) = 0$$

Unfortunately, the above limit is often difficult to evaluate. Fortunately, help is on the way:

With reference to the Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the partial sum

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the **Taylor polynomial of *f* of degree *N*.**

Note that the difference $E_N(x) = f(x) - p_N(x)$ is a measure of how well the Taylor polynomial of degree N of f approximates the function value at x. Focusing on that error, or remainder expression we have:

Joseph Louis Lagrange (1736, 1813).

THEOREM 9.29

LAGRANGE'S REMAINDER THEOREM

If f has derivatives of all orders in an open interval I containing a, then for each positive integer N and for each $x \in I$ there exists $\boldsymbol{c}$ between a and x such that

$$E_N(x) = \frac{f^{(N+1)}(\boldsymbol{c})}{(N+1)!}(x-a)^{N+1}$$

PROOF: Offered in Appendix B, page B-7.

When choosing the center a for the Taylor series of f one should take into account that the magnitude of $(x-a)^{N+1}$ increases as one moves away from a.

Observe that the above expression for $E_N(x)$:

$$\frac{f^{(N+1)}(\boldsymbol{c})}{(N+1)!}(x-a)^{N+1}$$

looks like the term preceding it in the Taylor series for f:

$$\frac{f^{(N)}(a)}{N!}(x-a)^N$$

with one notable exception:

> The a in the derivative is being replaced by some number c that lies somewhere between a and x.

That being the case, we often have to be content with finding a worst case scenario for $E_N(x)$; specifically:

THEOREM 9.30

TAYLOR'S INEQUALITY

If f has derivatives of all orders in an open interval I containing a, and if $|f^{(N+1)}(c)| \le M$ for every c between x and a, then for each positive integer N and for each $x \in I$:

$$|E_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

PROOF:
$$|E_N(x)| \underset{\substack{\uparrow \\ \text{Theorem 9.29}}}{=} \left|\frac{f^{(N+1)}(\boldsymbol{c})}{(N+1)!}(x-a)^{N+1}\right| = \frac{|f^{(N+1)}(c)|}{(N+1)!}|x-a|^{N+1} \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

Putting this all together we come to:

THEOREM 9.31

TAYLOR'S CONVERGENCE THEOREM

If $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ has a radius of convergence R, and if, for every $0 < d < R$, there exists M (which depends on d) such that $|f^{(n)}(x)| \le M$ for all n, and x in $|x-a| < d$ then:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < R.$$

PROOF: Theorem 9.30 and the given conditions assure us that for $|x-a| < d < R$:

$$|E_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

Since $\sum \frac{x^n}{n!}$ converges absolutely for all x [Example 9.19(a), page 365], $\sum \frac{M|x-a|^n}{n!}$ must also converge for all x. It follows, from the Divergence Theorem (page 333), that $\frac{M|x-a|^n}{n!} \to 0$ as $n \to \infty$ and that therefore $|E_N(x)| \to 0$ as $n \to \infty$.

At this point we know that the Taylor series of f converges to $f(x)$ on $(a-d, a+d)$ for any $d < R$. To see that it converges on $(a-R, a+R)$ simply note that for any $x \in (a-R, a+R)$ there exists $d < R$ such that $x \in (a-d, a+d)$.

THEOREM 9.32

THREE IMPORTANT MACLAURIN SERIES

For all x:

(i) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(ii) $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(iii) $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

PROOF: (i) We already know, from page 375, that $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$ is the Maclaurin series for $f(x) = e^x$ and that it converges everywhere.

Since $f^{(n)}(x) = e^x$ for every n, and since $f(x) = e^x$ is an increasing function, for any $d > 0$:

$$|f^{(n)}(x)| < \underset{\substack{\uparrow \\ \text{the } M \text{ in Theorem 9.31}}}{e^d} \text{ for } |x| < d$$

$$\text{Conclusion: } e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \text{ for all } x.$$

(ii) For $f(x) = \sin x$ we have:

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin(0) = \mathbf{0} \\ f'(x) &= \cos x & f'(0) &= \cos(0) = \mathbf{1} \\ f''(x) &= -\sin x & f''(0) &= -\sin(0) = \mathbf{0} \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos(0) = \mathbf{-1} \end{aligned}$$

(Note that $|f^{(n)}(0)| \le 1$ for all n)

Since $f^{(4)}(x) = \sin x = f(x)$, the above value-pattern of **0**, **1**, **0**, **–1** will keep repeating:

$$f^{(4)}(0) = \mathbf{0}, f^{(5)}(0) = \mathbf{1}, f^{(6)}(0) = \mathbf{0}, f^{(7)}(0) = \mathbf{-1}, f^{(8)}(0) = \mathbf{0}, \ldots$$

Bringing us to the Maclaurin series of the sine function:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \frac{\mathbf{0}}{0!}x^0 + \frac{\mathbf{1}}{1!}x^1 + \frac{\mathbf{0}}{2!}x^2 + \frac{\mathbf{-1}}{3!}x^3 + \frac{\mathbf{0}}{4!}x^4 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Which is seen to converge (absolutely) for all x:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{x^{2(n+1)+1}}{[2(n+1)+1]!}}{\dfrac{x^{2n+1}}{(2n+1)!}}\right| = \frac{(2n+1)!}{(2n+3)!}\left|\frac{x^{2n+3}}{x^{2n+1}}\right|$$

$$= \frac{1}{(2n+2)(2n+3)}|x^2| \to 0 < 1 \text{ as } n \to \infty$$

$$\text{(for all } x\text{)}$$

Since, for all x and n $|f^{(n)}(x)| \le \underset{\substack{\uparrow \\ \text{the } M \text{ in Theorem 9.31}}}{1}$:

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

As for (iii) (and beyond):

CHECK YOUR UNDERSTANDING 9.30

(a) As in the proof of Theorem 9.32(ii), show that, for all x:

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

(b) Find the Taylor series representation of $\sin x$, centered at $\frac{\pi}{2}$.

(c) Show that the term-by-term differentiation of the sine series yields the cosine series.

Answer: (a) and (c):
See Page A-59

(b) $\sum_{n=0}^{\infty}(-1)^n\frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}$

EXAMPLE 9.23 Determine the Maclaurin series of:

$$f(x) = x^3\sin\left(\frac{x}{2}\right)$$

SOLUTION: From $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$, we have:

$$\sin\frac{x}{2} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)!}$$

Consequently (see margin):

$$x^3\sin\frac{x}{2} = \sum_{n=0}^{\infty}(-1)^n\frac{x^3(x^{2n+1})}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{2^{2n+1}(2n+1)!}$$

In the exercises you are invited to verify that if $f(x) = \sum_{n=0}^{\infty}c_nx^n$, then, for any positive integer m:

$$x^mf(x) = \sum_{n=0}^{\infty}c_nx^{n+m}$$

CHECK YOUR UNDERSTANDING 9.31

Determine the Mclaurin series of:

$$f(x) = e^x+2e^{2x}$$

Answer: $\sum_{n=0}^{\infty}\frac{(2^{n+1}+1)x^n}{n!}$

THEOREM 9.33

BINOMIAL SERIES

For any real number r, and any $|x|<1$:

$$f(x) = (1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k$$

where: $\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$

I.e: $(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k = 1+rx+\frac{r(r-1)}{2!}x^2+\frac{r(r-1)(r-2)}{3!}x^3+\cdots$

PROOF: For $f(x) = (1+x)^r$ we have:

$$\begin{aligned} f(x) &= (1+x)^r & f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} & f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} & f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} & f'''(0) &= r(r-1)(r-2) \\ &\vdots & &\vdots \\ f^{(k)}(x) &= r(r-1)\ldots(r-k+1)(1+x)^{r-k} & f^{(n)}(0) &= r(r-1)\ldots(r-k+1) \end{aligned}$$

The Mclaurin series of $f(x) = (1+x)^r$ is therefore:

$$\sum_{k=0}^{\infty} \frac{r(r-1)\ldots(r-k+1)}{k!} x^n = \sum_{n=0}^{\infty} \binom{r}{k} x^k$$

In the exercises you are invited to show that the above power series does indeed converges to f for $|x| < 1$.

EXAMPLE 9.24

Find the Mclaurin series of $f(x) = \frac{1}{\sqrt{9-x}}$ and its radius of convergence.

SOLUTION: The first step is to express $\frac{1}{\sqrt{9-x}}$ in a form that displays the "$(1+x)^r$" appearing in the Theorem 9.33; namely:

$$\frac{1}{\sqrt{9-x}} = \frac{1}{3\sqrt{1-\frac{x}{9}}} = \frac{1}{3}\overset{(1+x)^r}{\left(1+\left(-\frac{x}{9}\right)\right)^{-1/2}}$$

Applying the theorem with $r = -\frac{1}{2}$ and with x replaced by $-\frac{x}{9}$ we have:

$$\frac{1}{\sqrt{9-x}} = \frac{1}{3}\left(1+\left(-\frac{x}{9}\right)\right)^{-1/2} = \frac{1}{3}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{9}\right)^n$$

$$= \frac{1}{3}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{9}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{9}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{9}\right)^3\right.$$

$$\left. + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{9}\right)^n + \cdots\right]$$

$$= \frac{1}{3}\left[1 + \frac{x}{18} + \frac{(1\cdot 3)/2^2}{2!9^2}x^2 + \frac{(3\cdot 5)/2^3}{3!9^3}x^3 + \cdots + \frac{[1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)]/2^n}{n!9^n}x^n + \cdots\right]$$

Returning to Theorem 3.33 we find that the radius of convergence of the above series is 9, for $\left|-\frac{x}{9}\right| < 1$ when $|x| < 9$.

CHECK YOUR UNDERSTANDING 9.32

Let n be a positive integer. Use Theorem 9.33 to show that for any a and b distinct from zero:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Answer: See page A-61

APPROXIMATING FUNCTION VALUES

If $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ over an interval I, then we know that for any $x \in I$ we can get as close as we want to $f(x)$ by summing enough terms of the given power series (Definition 9.5, page 332). Our concern here is to see how many terms need to be added in order to accommodate all elements of I simultaneously. Consider the following example.

EXAMPLE 9.25 Find the minimum number of terms in the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

that can be used to approximate e^x on the interval $0 \le x \le 4$ with an error no greater than 0.0001

SOLUTION: Noting that for $0 \le x \le 4 : 0 \le e^x \le e^4$ and that $|x-0| \le 4$, we invoke Taylor's Inequality (margin) and set our sights on finding the smallest N for which:

$$|E_N(x)| \le \frac{\overset{e^4}{\overset{\downarrow}{M}}}{(N+1)!}\underset{\underset{|x-0| \le 4}{\downarrow}}{|x-a|}^{N+1}$$

$$|E_N(x)| \le \frac{e^4}{(N+1)!}4^{N+1} \le 0.0001$$

Being faced with a somewhat unmanageable inequality, we turned to a calculator to evaluate $\frac{e^4}{(N+1)!}4^{N+1}$ for increasing values of N and found that while $\frac{e^4}{(N+1)!}4^{N+1} > 0.0001$ for $N \le 18$; at $N = 19$:

$$\frac{e^4}{(19+1)!}4^{19+1} \approx 0.00002 < 0.0001$$

Conclusion: Twenty terms are needed. Then:

$$\left|e^x - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{19}}{19!}\right)\right| < 0.0001 \text{ for } 0 \le x \le 4\,.$$

CHECK YOUR UNDERSTANDING 9.33

Find the minimum number of terms in the Taylor series of e^x centered at 2 that can be used to approximate e^x on the interval $0 \le x \le 4$ with an error no greater than 0.0001.

Answer: 13

EPILOGUE

Yes, **if f has a power series representation**, then f has derivatives of all orders within its radius of convergence, and, moreover:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Truth be told:
MATHEMATICS THRIVES ON ANOMALIES.

Though somewhat of an anomaly, there do exist functions f for which the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ converges to some function other than f. Here is an example of such a function:

$$\text{Let } f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Accepting the fact that $f^{(n)}(0) = 0$ for all n
(a fact that is typically established in an Analysis course)

we see that the Maclaurin series of f converges to 0 everywhere:

$$f(0) + f'(0)x + f''(0)x^2 + f^{(3)}(0)x^3 + \ldots = 0 + 0 + 0 + 0 + \ldots = 0$$

Conclusion: The Maclaurin series of f converges to $f(x)$ only at $x = 0$.

EXERCISES

Exercises 1-10. Use the definition of a Maclaurin series to find the Maclaurin series of f and its radius of convergence. [Do not verify that $E_N(x) \to 0$.]

1. $f(x) = \ln(1+x)$
2. $f(x) = e^{-x}$
3. $f(x) = \dfrac{1}{1+x}$
4. $f(x) = \dfrac{1}{1-x}$
5. $f(x) = \dfrac{1}{1-3x}$
6. $f(x) = e^{-4x}$
7. $f(x) = \dfrac{x}{e^x}$
8. $f(x) = x^2e^x$
9. $f(x) = 100\cos\pi x$
10. $f(x) = \sqrt{x+4}$

Exercises 11-20. Use the definition of a Taylor series to find the Taylor series of f and its radius of convergence. [Do not verify that $E_N(x) \to 0$.]

11. $f(x) = e^x$, $a = 1$
12. $f(x) = x^3+2x-1$, $a = 2$
13. $f(x) = \cos x$, $a = \dfrac{\pi}{6}$
14. $f(x) = \dfrac{3}{x}$, $a = 3$
15. $f(x) = \ln x$, $a = 1$
16. $f(x) = \cos x$, $a = \pi$
17. $f(x) = \sin\pi x$, $a = \dfrac{1}{2}$
18. $f(x) = x^{3/2}$, $a = 1$
19. $f(x) = \dfrac{1}{\sqrt{x}}$, $a = 9$
20. $f(x) = \dfrac{1}{b+x}$, $a \neq -b$

Exercises 21-35. Find the Taylor or Maclaurin series of f and the radius of convergence, using the Maclaurin series of e^x, $\sin x$, $\cos x$, and $\dfrac{1}{1-x}$.

21. $f(x) = e^x$, $a = 1$
22. $f(x) = \dfrac{3}{x}$, $a = 3$
23. $f(x) = \dfrac{1}{x^2+1}$, $a = 0$
24. $f(x) = \sin\pi x$, $a = \dfrac{1}{2}$
25. $f(x) = \tan^{-1}x$, $a = 0$
26. $f(x) = \ln x$, $a = 1$

27. $f(x) = \frac{1}{b+x}$, $a \neq -b$

28. $f(x) = \ln(1+x)$, $a = 0$

29. $f(x) = \frac{1}{1+x}$, $a = 0$

30. $f(x) = \frac{1}{1-3x}$, $a = 0$

31. $f(x) = \frac{x}{e^x}$, $a = 0$

32. $f(x) = x^2 e^x$, $a = 0$

33. $f(x) = \cos x$, $a = \pi$

34. $f(x) = x^2 \cos x$, $a = \pi$

35. $f(x) = x^2 \sin x$, $a = 0$

Exercises 36-38. Use the binomial series to expand f as a power series. State the radius of convergence.

36. $f(x) = \frac{1}{\sqrt{4-x}}$

37. $f(x) = \frac{1}{(2+x)^3}$

38. $f(x) = (1-x)^{2/3}$

Exercises 39-42. Find the minimum number of terms in the Taylor series of f centered at a that can be used to approximate f on the interval I with an error no greater than 0.0001.

39. $f(x) = \sin x$, $a = 0$, $I = [0, \pi]$

40. $f(x) = \cos x$, $a = 0$, $I = [0, \pi]$

41. $f(x) = \frac{1}{x^2+1}$, $a = 0$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$

42. $f(x) = \frac{1}{x^2+1}$, $a = 2$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$

Exercises 43-46. **[GC]**. Instruct your graphing calculator to sketch, on the same screen, the graph of f over the interval I along with the first N terms of its Maclaurin series for $N = 1, 2, 3$, and 4.

43. $f(x) = \sin x$, $I = [0, \pi]$

44. $f(x) = \cos x$, $I = [0, \pi]$

45. $f(x) = e^x$, $I = [-1, 2]$

46. $f(x) = \ln x$, $I = [1, e]$

47. (a) Find the Maclaurin series for $f(x) = \sin x^2$.

(b) Express $\int \sin x^2 dx$ as a power series.

(c) Use (b) to estimate $\int_0^1 \sin x^2 dx$ with an error of less that 0.001.

(d) Find a polynomial that will approximate $\int_0^t \sin x^2 dx$ with an error of less that 0.001, for any $t \in [0, 1]$.

48. (a) Find the Maclaurin series for $f(x) = e^{-x^2}$.

(b) Express $\int e^{-x^2}dx$ as a power series.

(c) Use (b) to estimate $\int_0^1 e^{-x^2}dx$ with an error of less that 0.001.

(d) Find a polynomial that will approximate $\int_0^t e^{-x^2}dx$ with an error of less that 0.001, for any $t \in [0, 1]$.

49. Prove Theorem 9.27.

Suggestion: First show that:

$$\frac{d^k}{dx^k}(x-a)^n = \left\{\left[\begin{array}{cc} 0 & \text{if } n < k \\ k! & \text{if } n = k \\ n(n-1)\cdots(n-k+1)(x-a)^{n-k} & \text{if } n > k \end{array}\right]\right.$$

And then consider the decomposition:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{k-1} c_n(x-a)^n + c_k(x-a)^k + \sum_{n=k+1}^{\infty} c_n(x-a)^n$$

50. Prove that if $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with radius of convergence R, then $x^m f(x) = \sum_{n=0}^{\infty} c_n x^{n+m}$ with radius of convergence R.

51. Verify that the power series $\sum_{k=0}^{\infty} \frac{r(r-1)\ldots(r-k+1)}{k!}x^k = \sum_{k=0}^{\infty} \binom{r}{k} x^k$ converges to $f(x) = (1+x)^r$ for $|x| < 1$.

CHAPTER SUMMARY

DEFINITION	A sequence $(a_n)_{n=1}^{\infty}$ **converges** to the number L if for any given $\varepsilon > 0$ there exists a positive integer N (which depends on ε) such that: $n > N \Rightarrow \lvert a_n - L \rvert < \varepsilon$ In the event that (a_n) converges to L we write $\lim\limits_{n \to \infty} a_n = L$, or $\lim a_n = L$, or $a_n \to L$, and call L the **limit** of the sequence.
ALGEBRA OF SEQUENCES	If $\lim a_n = A$ and $\lim b_n = B$, then: (a) $\lim ca_n = cA$, for any $c \in \Re$. (b) $\lim (a_n \pm b_n) = A \pm B$. (c) $\lim (a_n b_n) = AB$. (d) $\lim \dfrac{a_n}{b_n} = \dfrac{A}{B}$, providing no $b_n = 0$ and $B \neq 0$.
PINCHING THEOREM	If the sequences (a_n), (c_n), and (b_n) are such that (eventually) $a_n \le c_n \le b_n$, and if $\lim a_n = \lim b_n = L$, then $\lim c_n = L$.
SEQUENCES AND CONTINUOUS FUNCTIONS	Let $(a_n)_{n=1}^{\infty}$ be a sequence, and let the set $\{a_n\}_{n=1}^{\infty}$ be contained in the domain of a function *f*. If $\lim\limits_{n \to \infty} a_n = L$ and, if *f* is continuous at *L*, then $\lim\limits_{n \to \infty} f(a_n) = f(L)$.

SERIES	
DEFINITION	The series $\sum\limits_{i=1}^{\infty} a_i$ is said to **converge** to the number *L*, written $\sum\limits_{i=1}^{\infty} a_i = L$, if the sequence of its partial sums $(s_n)_{n=1}^{\infty}$ (where $s_n = \sum\limits_{i=1}^{n} a_i$) converges to *L*.

DIVERGENCE TEST	If $\lim\limits_{n \to \infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.
GEOMETRIC SERIES	The geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots = \sum\limits_{n=0}^{\infty} ar^n$ is convergent if $\lvert r \rvert < 1$, with sum: $\sum\limits_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ The geometric series diverges if $\lvert r \rvert \geq 1$.
ALGEBRA OF SERIES	If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge, then, for any $c \in \Re$: $\sum\limits_{n=1}^{\infty} (a_n \pm b_n) = \sum\limits_{n=1}^{\infty} a_n \pm \sum\limits_{n=1}^{\infty} b_n$ and $\sum\limits_{n=1}^{\infty} ca_n = c \sum\limits_{n=1}^{\infty} a_n$
ALTERNATING SERIES TEST	If the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n = \underset{(\text{each } a_n > 0)}{a_1 - a_2 + a_3 - a_4 + \cdots}$ is such that: $a_{n+1} \leq a_n$ for all n, and $\lim\limits_{n \to \infty} a_n = 0$ then the series converges.
SERIES OF POSITIVE TERMS	
CONVERGENCE THEOREM	A positive series converges if and only if its sequence (s_n) of partial sums is bounded from above.
INTEGRAL TEST	Let the continuous function f be such that: (i) $f(x) > 0$ for all $x \geq 1$ (ii) $f(x) \geq f(y)$ if $1 \leq x \leq y$ Let $a_n = f(n)$ for all $n \geq 1$. Then: $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

***P*-SERIES**	$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$ converges if $p > 1$ and diverges if $p \le 1$.
COMPARISON TEST	If the positive series $\sum a_n$ converges and if $\sum b_n$ is such that $0 \le b_n \le a_n$, then $\sum b_n$ converges. If the positive series $\sum a_n$ diverges, and if $a_n < b_n$, then $\sum b_n$ diverges.
LIMIT COMPARISON TEST	If $\sum a_n$ and $\sum b_n$ are positive series and if $\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$ then both series converge or both series diverge.
RATIO TEST **(FOR POSITIVE SERIES)**	Let $\sum a_n$ be a positive series with $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$ $\text{If} \begin{cases} L < 1, \text{ then the series converges} \\ L > 1 \text{ or } L = \infty, \text{ then the series diverges} \\ L = 1, \text{then the test is inconclusive} \end{cases}$
ROOT TEST	Let $\sum a_n$ be a positive series with $\lim_{n \to \infty} \sqrt[n]{a_n} = L$ $\text{If} \begin{cases} L < 1, \text{ then the series converges} \\ L > 1 \text{ or } L = \infty, \text{ then the series diverges} \\ L = 1, \text{then the test is inconclusive} \end{cases}$
ABSOLUTE AND CONDITIONAL CONVERGENCE	
DEFINITION	A series $\sum a_n$ is **absolutely convergent** if $\sum \lvert a_n \rvert$ converges. A convergent series $\sum a_n$ is **conditionally convergent** if $\sum \lvert a_n \rvert$ diverges.

RATIO TEST	For a given series $\sum a_n$ (not necessarily positive), with $\lim\limits_{n \to \infty}\left\|\frac{a_{n+1}}{a_n}\right\| = L$. If $\begin{cases} L < 1: \sum a_n \text{ converges absolutely.} \\ L > 1 \text{ or } L = \infty: \sum a_n \text{ diverges.} \\ L = 1, \text{ the test is inconclusive} \end{cases}$
POWER SERIES AND TAYLOR SERIES	
DEFINITION	A **power series centered at a**, is a series of the form $\sum\limits_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$
CONVERGENCE THEOREM	For a given power series $\sum c_n(x-a)^n$ there are only three possibilities: (i) The series converges absolutely for all x. (ii) The series converges only at $x = a$. (iii)There exists $R > 0$ (called the **radius of convergence**) such that the series converges absolutely if $\|x-a\| < R$ and diverges if $\|x-a\| > R$.
DERIVATIVE AND INTEGRAL THEOREM	If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then: (i) $f(x) = \sum c_n(x-a)^n$ is differentiable (and therefore continuous) on $(a-R, a+R)$, with: $f'(x) = (\sum [c_n(x-a)^n])'$ $= \sum [c_n(x-a)^n]' = \sum nc_n(x-a)^{n-1}$ (ii) $\int f(x)dx = \int(\sum c_n(x-a)^n)dx$ $= \sum \int c_n(x-a)^n dx$ $= \sum c_n \frac{(x-a)^{n+1}}{n+1} + C$ (iii) The power series in (i) and (ii) also have radius of convergence R.

THEOREM	If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on $(a-R, a+R)$, then: $$c_n = \frac{f^{(n)}(a)}{n!}$$
DEFINITION	If f has derivatives of all orders at a, then the **Taylor series for f about a** is the power series: $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ A Taylor series for f about 0 has a special name — it is called the **Maclaurin series for f**.
LAGRANGE'S REMAINDER THEOREM	If f has derivatives of all orders in an open interval I containing a, then for each positive integer N and for each $x \in I$ there exists $\boldsymbol{c}$ between a and x such that: $$E_N(x) = \frac{f^{(N+1)}(\boldsymbol{c})}{(N+1)!}(x-a)^{N+1}$$
TAYLOR'S INEQUALITY	If f has derivatives of all orders in an open interval I containing a, and if $\lvert f^{(N+1)}(c) \rvert \le M$ for every c between x and a, then for each positive integer N and for each $x \in I$: $$\lvert E_N(x) \rvert \le \frac{M}{(N+1)!}\lvert x-a \rvert^{N+1}$$
TAYLOR'S CONVERGENCE THEOREM	If $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ has a radius of convergence R, and if, for every $0 < d < R$, there exist M (which depends on d) such that $\lvert f^{(n)}(x) \rvert \le M$ for $\lvert x-a \rvert < d$ then: $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } \lvert x-a \rvert < R.$$

CHAPTER 10
PARAMETRIZATION OF CURVES AND POLAR COORDINATES

§1. PARAMETRIZATION OF CURVES

As you know, no vertical line can intersect the graph of a function $y = f(x)$ in more than one point. Some curves that fail the above "vertical-line test" may be described by means of a pair of functions, $x(t)$ and $y(t)$, where the variable t assumes values in some specified interval I. Basically, the idea is to choose I, $x(t)$, and $y(t)$, in such a way that as t traverses the interval I, the points $(x(t), y(t))$ trace out the curve of interest. To illustrate:

One way of tracing out the points (x, y) on the unit circle C in Figure 10.1, is to let $x = x(t) = \cos t$ and $y = y(t) = \sin t$, where $0 \le t \le 2\pi$. How so? Like so:

To say that $(x, y) \in C$ is to say that (x, y) is one unit from the origin, which is to say that $x^2 + y^2 = 1$. Since:

$$[x(t)]^2 + [y(t)]^2 = \cos^2 t + \sin^2 t \underset{\uparrow}{=} 1$$

Theorem 1.5(i), page 37

each point $(x(t), y(t))$ lies on C. Indeed, Definition 1.8, page 32, should convince you of the fact that as t runs from 0 to 2π the corresponding points $(x(t), y(t))$ start at $(1, 0)$ (when $t = 0$) and move along the unit circle in a counterclockwise direction ending up, once again, at the point $(1, 0)$ when $t = 2\pi$.

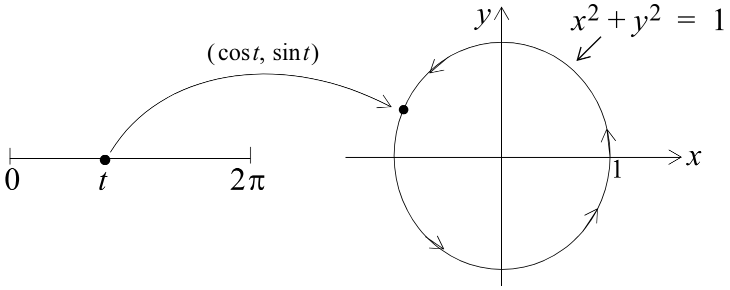


Figure 10.1

In general:

Let $x(t)$ and $y(t)$ be a pair of functions defined on an interval I. To each t in I we associate the point $(x(t), y(t))$ in the plane. As t ranges over I, the point $(x(t), y(t))$ traces out a path (curve) in the plane. Such a curve is said to be a **parametrized curve**, and the variable t is said to be a **parameter**.

If $I = [a, b]$, then $(x(a), y(a))$ and $(x(b), y(b))$ are said to be the **initial point** and **terminal point** of the curve, respectively,

Curves in three-dimensional space can also be accommodated with the introduction of a third parameter: $z(t)$.

An equation such as $x^2 + y^2 = 1$ is said to be in rectangular form. As is illustrated in the following example, one may be able to go from a parametric representation of a curve to its rectangular form by eliminating the parameter t:

EXAMPLE 10.1 Find the rectangular equation of the curve defined by the parametric equations:

$$x = t^2 - 2t \text{ and } y = t + 1 \text{ for } -\infty < t < \infty.$$

Sketch the curve, utilizing arrows to indicate the direction, or **orientation**, of the curve for increasing values of the parameter t.

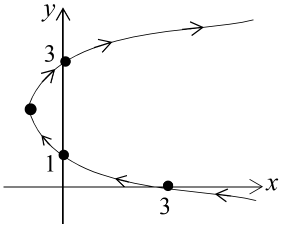
SOLUTION: From $y = t + 1$: $t = y - 1$. Substituting $t = y - 1$ in $x = t^2 - 2t$ we come to the rectangular equation:

$$\begin{aligned} x &= (y-1)^2 - 2(y-1) \\ &= y^2 - 4y + 3 = (y-3)(y-1) \end{aligned}$$

The above equation represents a parabola, opening to the right, with y-intercepts at 3 and at 1, and vertex at:

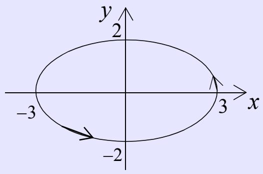
$$y = -\frac{b}{2a} = -\frac{-4}{2} = 2$$

$$x = 2^2 - 4 \cdot 2 + 3 = -1$$



Note that as t increases from $-\infty$ to ∞ the y-values, $y = t + 1$ will also increase — in harmony with the above displayed orientation.

Answer: $\frac{x^2}{9} + \frac{y^2}{4} = 1$



CHECK YOUR UNDERSTANDING 10.1

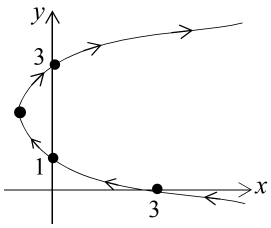
Find the rectangular equation of the curve defined by the parametric equations:

$$x = 3\cos t,\ y = 2\sin t, \quad 0 \le t \le 2\pi$$

Sketch the curve, indicating the orientation.

DERIVATIVES OF PARAMETRIZED CURVES

Returning to the curve of the previous example we see that the slope of the tangent line at $(0, 3)$ is positive, and that it is negative at $(3, 0)$. We could proceed as we did with the unit circle on page 103 to find those slopes, but choose, instead, to turn directly to the fact that the curve is parametrized by the equations $x = t^2 - 2t$ and $y = t + 1$:



Leibniz form of the chain rule (Theorem 3.8, page 94):

If $y = f(x)$ and $x = g(t)$

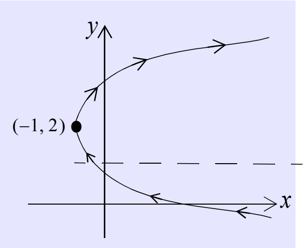
then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Applying the chain rule (see margin), we have:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2t-2} \quad (*)$$

To find the slope at $(0, 3)$, we consider the equation $y = t + 1$ and observe that $y = 3$ only at $t = 2$. Turning to (*) we can calculate the slope of the tangent line to the curve at $(0, 3)$: $\frac{dy}{dx} = \frac{1}{2(2)-2} = \frac{1}{2}$.

Setting y to 0 in $y = t + 1$, we find that the point $(3, 0)$ is encountered at $t = -1$. Turning to (*), we then have: $\frac{dy}{dx} = \frac{1}{2(-1)-2} = -\frac{1}{4}$.



The lower part of the curve in question (see margin) appears to be concave up, and its upper part: concave down. To formally address the concavity issue we turn to the second derivative $\frac{d^2y}{dx^2}$, which, being the derivative of the first derivative, brings us to:

Note that $\frac{d^2y}{dx^2}$ is **NOT** $\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

It is:

$$\boldsymbol{\frac{d^2y}{dx^2}} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \boldsymbol{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{d}{dt}(t^2-2t)} &= \frac{\frac{d}{dt}\left(\frac{1}{2t-2}\right)}{2t-2}\\ &= \frac{[(2t-2)^{-1}]'}{2t-2}\\ &= \frac{-2(2t-2)^{-2}}{2t-2}\\ &= -\frac{2}{(2t-2)^3} \qquad \text{SIGN: } \overset{+}{\longrightarrow}\underset{1}{\circ}\overset{-}{\longrightarrow} t\end{aligned}$$

From the above, we see that the curve is concave up for $-\infty < t < 1$ (the bottom part of the curve) and concave down for $t > 1$ (the top part of the curve), with the curve reaching the point $(-1, 2)$ at $t = 1$.

Let's underline two of the above observations:.

For a given point (x, y) on a parametrized curve with $x = x(t)$ and $y = y(t)$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$$

Providing, of course, that the indicated expressions are defined.

EXAMPLE 10.2 Determine the slope of the tangent line to the curve with parametric equations

$$x(t) = t^2,\ y(t) = t^3 - 3t,\ \ -\infty < t < \infty$$

at the point $(3, 0)$.

SOLUTION: In order for the curve to pass through the point $(3, 0)$, $x(t)$ must be 3. Solving for t we have:

$$x(t) = t^2 = 3 = 0 \Rightarrow t = \pm\sqrt{3}$$

As it turns out, $y(t) = t^3 - 3t$ is 0 for both values of t:

$$(3^{1/2})^3 - 3 \cdot 3^{1/2} = 0 \text{ and } (-3^{1/2})^3 - 3(-3^{1/2}) = 0$$

Consequently, the curve crosses the point $(3, 0)$ twice: once when $t = -\sqrt{3}$ and again when $t = \sqrt{3}$. Turning to:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

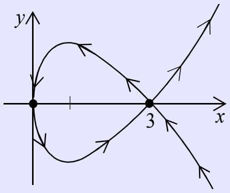
we see that when the curve crosses the point $(3, 0)$ at $t = -\sqrt{3}$ the tangent line has slope:

$$\frac{dy}{dx} = \frac{3(-\sqrt{3})^2 - 3}{2(-\sqrt{3})} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$

When it crosses again, at $t = \sqrt{3}$, the tangent line has slope

$$\frac{dy}{dx} = \frac{3(\sqrt{3})^2 - 3}{2(\sqrt{3})} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

In CYU 10.3 you are asked to sketch the curve of this example. Here is the end product:



Note that the tangent line at $(3, 0)$ when the point is crossed for the first time from right to left [at $t = -\sqrt{3}$] has negative slope, while its slope is positive when the point is crossed for the second time from left to right [at $t = \sqrt{3}$].

CHECK YOUR UNDERSTANDING 10.2

Referring to Example 10.2, find the points on the curve where horizontal or vertical tangent lines occur.

Verify that the curve is concave down for $-\infty < t < 0$ and concave up for $t > 0$.

Answer: Horizontal tangent line at $(1, \pm 2)$. Vertical tangent line at $(0, 0)$.

See page A-61 for the concavity issue.

EXAMPLE 10.3 Sketch the curve with parametrization:

$$x = t^2 - 4,\ y = t^2 + t \text{ for } -\infty < t < \infty.$$

SOLUTION: (a) From $x = t^2 - 4$: $t = \pm\sqrt{x + 4}$.

Substituting in $y = t^2 + t$ we have:

$$(1)\ y = (\sqrt{x+4})^2 + \sqrt{x+4} = x + 4 + \sqrt{x+4}$$

$$\text{and } (2)\ y = (-\sqrt{x+4})^2 - \sqrt{x+4} = x + 4 - \sqrt{x+4}$$

Both (1) and (2) represent functions with domain $[-4, \infty)$.

Turning to (1): At $x = -4$, $y = x+4+\sqrt{x+4}$ assumes the value of 0. Moreover, as x increases the function values clearly increase, leading us to the anticipated graph in Figure 10.2 (a) below.

$\underbrace{x+4}_{c}$ vs $\underbrace{\sqrt{x+4}}_{\sqrt{c}}$

(1) $0 < c < 1 \Rightarrow c < \sqrt{c}$

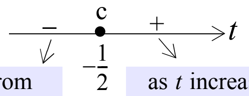
(2) $c > 1 \Rightarrow c > \sqrt{c}$

Turning to (2): At $x = -4$, $y = x+4-\sqrt{x+4}$ also assumes the value of 0. The values of y are negative immediately to the right of $x = -4$ [see margin (1)] and then are eventually positive [see margin (2)], bringing us to the anticipated graph in Figure 10.2(b).

We merged (a) and (b) to arrive at the curve in Figure 10.2 (c), which also displays the traversed direction as t progresses from $-\infty$ to ∞. Why that particular direction? Because:

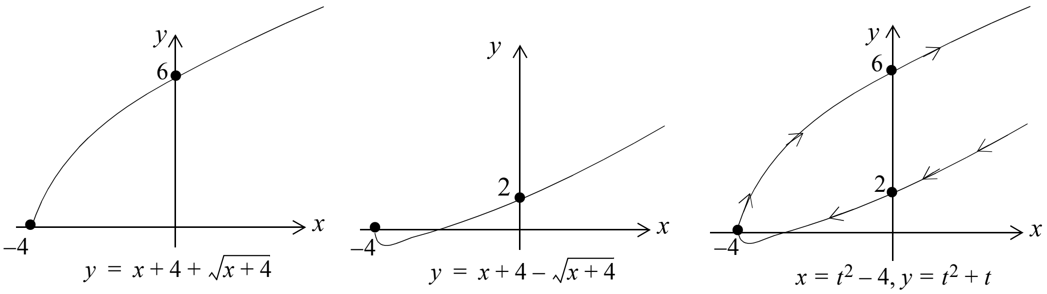
$$y = t^2 + t \Rightarrow \frac{dy}{dt} = 2t + 1$$

SIGN:



as t increases, from $-\infty$ to $-\frac{1}{2}$, y decreases

as t increases from $-\frac{1}{2}$ to ∞, y increases



(a) (b) (c)

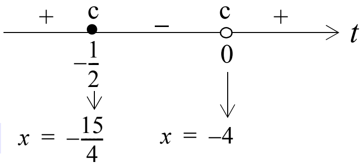
Figure 10.2

The lower portion of the curve in (c) corresponding to $t < 0$ appears to show a minimum just to the right of $x = -4$ $(t = 0)$. While we could certainly investigate the first derivative situation for the function of x depicted in (b) to verify this, we choose instead (see margin) to focus on the curve in (c).

Such an option need not be available for other parametrized curves. (See Example 10.4).

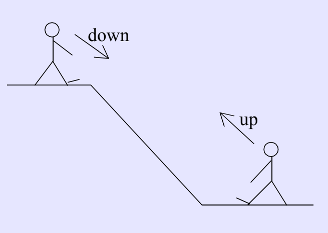
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t}$$

SIGN:

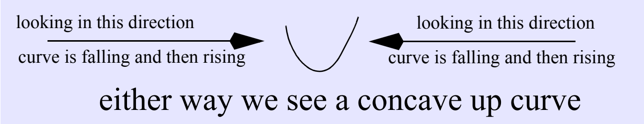


$x = t^2 - 4$: $x = -\frac{15}{4}$ $x = -4$

We see that $\frac{dy}{dx} = 0$ when $t = -\frac{1}{2}$ (at $x = -\frac{15}{4}$), and if you look at the above SIGN information you may conclude (incorrectly) that the graph achieves a maximum when $t = -\frac{1}{2}$ (at $x = -\frac{15}{4}$).



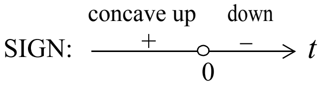
The problem, you see, is that $\frac{dx}{dt} = 2t$ is negative for $t < 0$, which tells us that x and t are heading in different directions: as t increases, x decreases; and as t decreases, x increases. It follows that a positive slope in one direction turns into a negative slope in the other direction (see margin). **FORTUNATELY**, concavity is direction-insensitive:

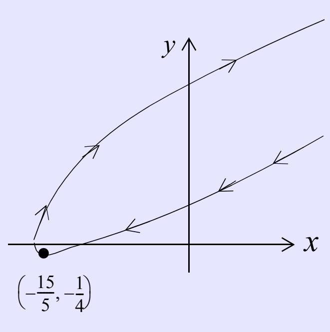


And so we turn to the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{2t} = \frac{\frac{d}{dt}\left(\frac{2t+1}{2t}\right)}{2t}$$

$$= \frac{\frac{2t(2) - (2t+1) \cdot 2}{4t^2}}{2t} = -\frac{1}{4t^3}$$





We already know that there is a horizontal tangent line when $t = -\frac{1}{2}$. Noting that $t = -\frac{1}{2}$ falls in a concave up region, we conclude that a minimum occurs at $(x(-\frac{1}{2}), y(-\frac{1}{2})) = (-\frac{15}{4}, -\frac{1}{4})$.

(Consider, also, Theorem 4.8, page 137)

CHECK YOUR UNDERSTANDING 10.3

Follow the procedure of Example 10.3 to sketch the curve with parametrization $x = t^2, y = t^3 - 3t$ for $-\infty < t < \infty$ of Example 10.2.

Answer: See page A-62.

EXAMPLE 10.4

Determine the points on the parametrized curve:

$$x = t^3 - t^2 + 1,\ y = t^3 + t^2 - 1\ (-\infty < t < \infty)$$

where local maxima/minima and inflection points occur.

SOLUTION: Turning to the first derivative, we have:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 2t}{3t^2 - 2t} = \frac{3t+2}{3t-2}$$

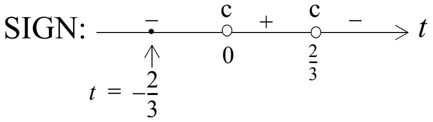
At this point we know that a horizontal tangent line occurs when $t = -\frac{2}{3}$; which is to say, at the point:

$$(x, y) = \left[\left(-\frac{2}{3}\right)^3 - \left(-\frac{2}{3}\right)^2 + 1, \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2 - 1\right] = \left(\frac{7}{27}, -\frac{23}{27}\right)$$

Turning to the second derivative, we have:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}\left(\dfrac{3t+2}{3t-2}\right)}{3t^2-2t} = \frac{\dfrac{3(3t-2)-3(3t+2)}{(3t-2)^2}}{t(3t-2)}$$

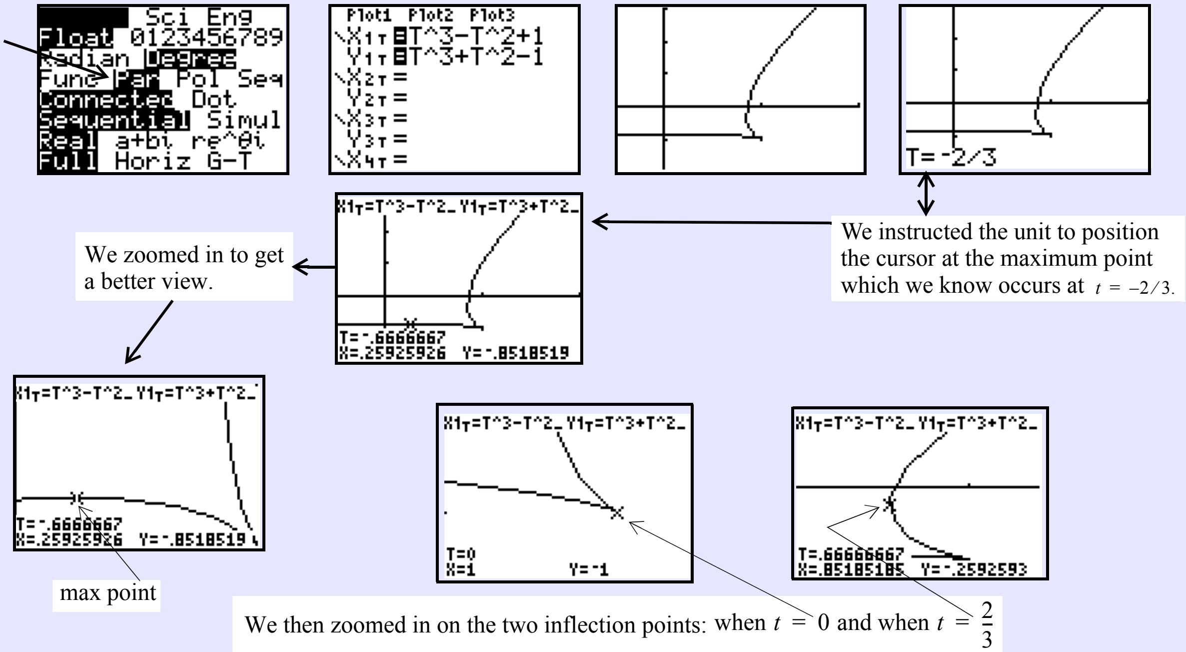
$$= -\frac{12}{t(3t-2)^3}$$



Since the second derivative is negative at $t = -\frac{2}{3}$, a local maximum occurs at $\left(\frac{7}{27}, -\frac{23}{27}\right)$ (Theorem 4.8, page 137). We also see that concavity changes about $t = 0$ and $t = \frac{2}{3}$, and even though the second derivative does not exist at those points, they do correspond to two inflection points on the curve; namely: $(1, -1)$ and $\left(\frac{23}{27}, -\frac{7}{27}\right)$, respectively (see margin).

Recall that:
$[x(t), y(t)]$
$= (t^3 - t^2 + 1, t^3 + t^2 - 1)$

The TI 84 has parametric graphing capabilities:



```
    Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

```
Plot1 Plot2 Plot3
\X1T=T^3-T^2+1
 Y1T=T^3+T^2-1
\X2T=
 Y2T=
\X3T=
 Y3T=
\X4T=
```

```
T=-2/3
```

We instructed the unit to position the cursor at the maximum point which we know occurs at $t = -2/3$.

```
X1T=T^3-T^2… Y1T=T^3+T^2…
T=-.6666667
X=.25925926  Y=-.8518519
```

We zoomed in to get a better view.

```
X1T=T^3-T^2… Y1T=T^3+T^2…
T=-.6666667
X=.25925926  Y=-.8518519
```

max point

```
X1T=T^3-T^2… Y1T=T^3+T^2…
T=0
X=1          Y=-1
```

```
X1T=T^3-T^2… Y1T=T^3+T^2…
T=.6666667
X=.85185185  Y=-.2592593
```

We then zoomed in on the two inflection points: when $t = 0$ and when $t = \frac{2}{3}$

CHECK YOUR UNDERSTANDING 10.4

Determine the points on the parametrized curve:

$$x = t^3 - t^2,\ y = t^2e^t + 3\ (-\infty < t < \infty)$$

where local maxima/minima and inflection points occur.

Answers: Local maximum at $\left(-12, \frac{4}{e^2} + 3\right)$. See page A-63 for inflection points.

ARC LENGTH

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

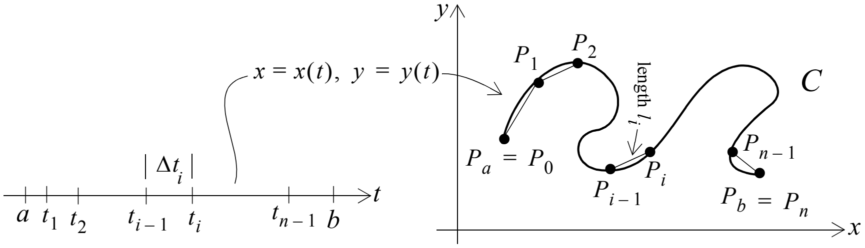
(Definition 5.7, page 209)

We previously developed a formula for the length of the graph of a function $y = f(x)$ from a to b (margin). That formula cannot be applied directly to curves which are not graphs of functions. We now develop a formula for the arc length of parametrically defined curves.

Consider a curve C defined by the parametric equations

$$x = x(t),\ y = y(t) \text{ for } a \le t \le b$$

for which $\frac{dx}{dt}$ and $\frac{dy}{dt}$ exist and are continuous on $[a, b]$. Partition the closed interval $[a, b]$ into n subintervals Δt_i as is indicated below.



Corresponding to the numbers $a, t_1, \ldots, t_{i-1}, t_i, \ldots, t_{n-1}, b$, are the points $P_a = (x(a), y(a))$, $P_1 = (x(t_1), y(t_1))$,..., $P_b = (x(b), y(b))$ on the curve C. Here is the length, l_i, of the line segment joining $P_{i-1} = (x(t_{i-1}), y(t_{i-1}))$ to $P_i = (x(t_i), y(t_i))$:

$$l_i = \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$$

The sum of the lengths of those n line segments serves to approximate the length, L, of the curve, C, in question:

$$L \approx \sum_{i=1}^{n} \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left[\frac{x(t_i) - x(t_{i-1})}{\Delta t_i}\right]^2 + \left[\frac{y(t_i) - y(t_{i-1})}{\Delta t_i}\right]^2}\,\Delta t_i$$

To make the approximation better and better, we simply let the Δt_i's get smaller and smaller; bringing us to:

$$L = \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\left[\frac{x(t_i) - x(t_{i-1})}{\Delta t_i}\right]^2 + \left[\frac{y(t_i) - y(t_{i-1})}{\Delta t_i}\right]^2}\, \Delta t_i$$

The problem is that the above sum is not quite a Riemann sum, a difficulty that is typically circumvented in an analysis course.

It may come as no surprise to find that the above limit can be expressed in the following integral form $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (see margin); bringing us to:

DEFINITION 10.1 Let $x(t)$ and $y(t)$ have continuous derivatives on a closed interval $a \le t \le b$. The length L of the parametrized curve

$$x = x(t),\ y = y(t) \text{ for } a \le t \le b$$

that is traversed exactly once as t increases from a to b is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note: When you apply the above formula:

THE LENGTH YOU SEE MAY NOT BE THE LENGTH YOU GET.

It will be, as long as no part of the curve in question is traced out more than once as t goes from a to b. A case in point:

The parametrization $x = \cos t,\ y = \sin t$ for $0 \le t < 2\pi$ traces out the unit circle centered at the origin (see Figure 10.1). The parametrization $x = \cos t,\ y = \sin t$ for $0 \le t < 4\pi$ will also trace out the unit circle, but **twice**! Applying the formula of Definition 10.1 we have:

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} dt = 2\pi$$

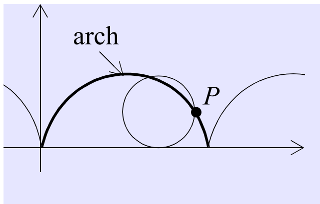
(the circumference of the circle)

On the other hand:

$$\int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{4\pi} dt = 4\pi$$

(not too surprising since we traced out the circle twice)

While we're at it, we also point out that the formula of Definition 10.1 yields the same result for any two parametrizations of the curve, as long as neither traces out any part of the curve more than once in the process.



EXAMPLE 10.5 As a circle rolls along a line in a plane, the curve described by a fixed point P on the circle is called a **cycloid** (see margin**)**. Find a parametrization for the curve and the length of one arch.

SOLUTION: We begin with a circle of radius r and P a point on the circle. Let the line on which the circle rolls be the x axis, with P at the origin [Figure 10.3(a)].

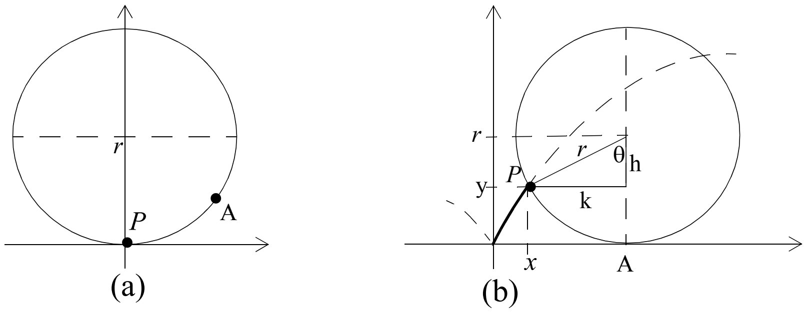


Figure 10.3

Figure 10.3(b) depicts the position of the point *P* after the circle has rolled a bit, and where θ denotes the angle through which it has rolled. Since the length of the arc on the circle joining *P* to *A,* namely $r\theta$, equals the distance between the point *A* and 0 on the *x*-axis, we have: $A = r\theta$. Referring to Figure 10.3(b) we then have:

$$x = A - k = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

and: $$y = r - h = r - r\cos\theta = r(1 - \cos\theta)$$

Bringing us to the following parametric equations for the cycloid:

$$x = r(\theta - \sin\theta),\quad y = r(1 - \cos\theta) \text{ for } -\infty < \theta < \infty$$

Noting that one arch of the cycloid comes from a complete rotation of the circle: $0 \le \theta \le 2\pi$, we appeal to Definition 10.1 to find the length, *L*, of one arch:

$$L = \int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi}\sqrt{[r(1-\cos\theta)]^2 + [r\sin\theta]^2}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\,d\theta$$

$$= r\int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta$$

$$= \sqrt{2}r\int_0^{2\pi}\sqrt{1 - \cos\theta}\,d\theta$$

$$\text{(see margin)} = 2r\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 2r\left(-2\cos\frac{\theta}{2}\right)\Bigg|_0^{2\pi} = 2r(2+2) = 8r$$

Theorem 1.5(vi), page 37:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Since $0 \le \frac{\theta}{2} \le \pi$, $\sin\frac{\theta}{2}$ cannot be negative. Thus:

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}\text{, or:}$$

$$\sqrt{1-\cos\theta} = \sqrt{2}\sin\frac{\theta}{2}$$

CHECK YOUR UNDERSTANDING 10.5

Determine the length of the curve:

$$x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t,\ 0 \le t \le \pi$$

Answer: 12.

EXERCISES

Exercises 1-12. Find the rectangular equation of the curve defined by the given parametric equations. Sketch the curve showing its orientation.

1. $x = \sqrt{t}+1, y = t^{3/2}, t \ge 1$
2. $x = 2t^2, \quad y = 2t^3 \quad , \quad 0 \le t \le 3$
3. $x = \sec t, y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$
4. $x = 2e^t, y = 1 - e^t, t \ge 0$
5. $x = 3\sin t - 2, \quad y = 2\cos t \quad, 0 \le t \le 2\pi$
6. $x = 1 + 2\sin t, y = 2 - \cos t, 0 \le t \le 2\pi$
7. $x = t^3, y = t^2, -\infty < t < \infty$
8. $x = e^t, y = \sin t, -\infty < t < \infty$
9. $x = t^2, y = \sqrt{t^4+1}, t \ge 0$
10. $x = \sqrt{t+1}, y = \sqrt{t}, t \ge 0$
11. $x = \cos 2t, y = \sin t, -\frac{\pi}{2} < t < \frac{\pi}{2}$
12. $x = t^2, y = 2\ln t, t \ge 0$

Exercises 13-18. Find the tangent to the given curve at the indicated point,

13. $x = t^4 + 1, \quad y = t^3 + t \quad, t = -1$
14. $x = 3t, y = 2t^2 - 1, t = 1$
15. $x = 3\cos t, y = 4\sin t, t = \frac{\pi}{4}$
16. $x = t^3 - 1, y = 2e^t, t = 2$
17. $x = t^2 - 1, y = 2e^t, t = 2$
18. $x = \sec^2 t - 1, y = \tan t, t = -\frac{\pi}{4}$

Exercises 19-24. Find the points on the given curve where the tangent line is horizontal or vertical.

19. $x = 3t^2, y = t^3 - 4t$
20. $x = 2t^3 + 3t^2 - 12t, y = 2t^3 + 3t^2 + 1$
21. $x = 2\cos t, y = \sin 2t$
22. $x = \frac{1}{t}, \quad y = 2t$
23. $x = \frac{1}{t}, y = t^2 + 3$
24. $x = 2 + 3\sin t, y = 3 - 2\cos t$

Exercises 25-28. Find the values of t for which the curve is increasing, and the values of t for which the curve is concave up.

25. $x = t^2 + 4, y = t^3 + t^2$
26. $x = t^3 - 12t, y = t^2 - 1$
27. $x = t - e^t, y = t + e^{-t}$
28. $x = t + \ln t, y = t - \ln t$

Exercises 29-32. Sketch the graph of the given parametrized curve. Label its (local) max/min points and its inflection points.

29. $x = 3t^2 + 1, y = 2t^2 + 4, -\infty < t < \infty$
30. $x = 3t^2 + t, \quad y = 2t^2 - t, -\infty < t < \infty$
31. $x = \frac{t^2}{2}, \quad y = \frac{t^3}{2} - 6t, 0 \le t \le 4$
32. $x = e^t - t, y = 2e^t, -\infty < t < \infty$

Exercises 33-36. Determine the length of the given curve

33. $x = 3t^2 + 1$, $y = 2t^3 + 4$, $0 \le t \le 1$

34. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 < t < 3$

35. $x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$, $0 \le t \le \pi$

36. $x = 2\sin t - 1$, $y = 2\cos t + 1$, $0 \le t \le 2\pi$

Exercises 37-39. Express the length of the given curve in integral form.

37. $x = t - t^2$, $y = 3t^{3/2}$, $0 \le t \le 2$

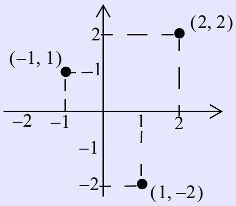
38. $x = \ln t$, $y = \sqrt{t} + 1$, $1 \le t \le 2$

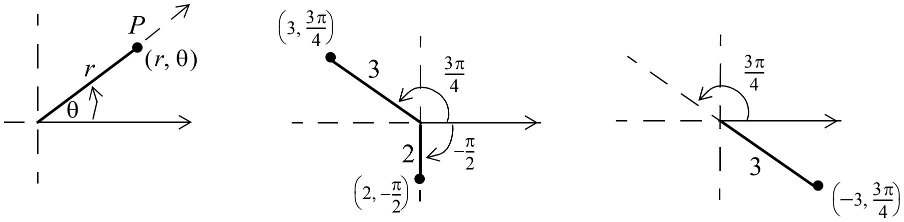
39. $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$

40.$x = 4e^{2t}$, $y = t^2$, $-1 \le t \le 1$

§2. POLAR COORDINATES

In the familiar Cartesian (or rectangular) coordinate system, points in the plane are represented by ordered pairs of numbers (x, y) (see margin. In the polar coordinate system, a point P in the plane is again represented by an ordered pair, (r, θ), where r is the directed distance from the origin to P, and θ is an angle in standard position, with terminal side the line segment connecting the origin to P [see Figure 10.4(a)]. In particular, $\left(3, \frac{3\pi}{4}\right)$ and $\left(2, -\frac{\pi}{2}\right)$ are displayed in Figure10.4(b). The origin's coordinates are $(0, \theta)$, for any angle θ.



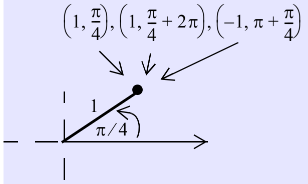


(a) (b) (c)

Figure 10.4

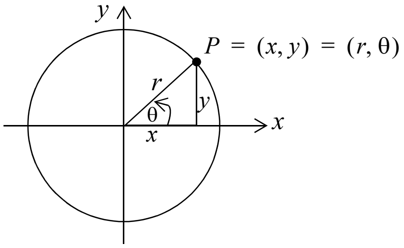
Just as the angle θ can assume both positive and negative values, it is also convenient to allow r to assume both positive and negative values. In general if r is positive, then the point $(-r, \theta)$ is obtained by reflecting the point (r, θ) about the origin, or, equivalently by plotting the point $(r, \theta + \pi)$ [to put it roughly: *walk r units in the "opposite direction."*]. In particular, the point $\left(-3, \frac{3\pi}{4}\right)$ appears in Figure 10.4(c).

Note that while in the rectangular coordinate system each point has exactly one pair of coordinates, that is not the case in the polar system. A case in point:



RECTANGULAR TO POLAR AND VICE VERSA

The adjacent figure reveals relations between the rectangular coordinates (x, y) and the polar coordinates (r, θ) of a point P in the plane; namely:



$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Observe that these relations hold independently of the quadrant in which P resides. In the second quadrant, for example:
x and $\cos\theta$ are both negative,
y and $\sin\theta$ are both positive,
$\frac{y}{x}$ and $\tan\theta$ are both negative.

EXAMPLE 10.6

(a) Find the rectangular coordinates of the point P with polar coordinates:

(i) $\left(2, \frac{3\pi}{4}\right)$ (ii) $\left(-3, \frac{\pi}{3}\right)$

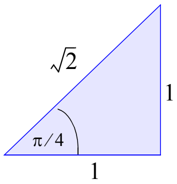
(b) Find all possible polar coordinates of the point P with rectangular coordinates $(-1, \sqrt{3})$.

SOLUTION: (a) We need the equations $x = r\cos\theta$, $y = r\sin\theta$.

(i) For $P = (r, \theta) = \left(2, \frac{3\pi}{4}\right)$:

$$x = 2\cos\frac{3\pi}{4} = 2\left(-\cos\frac{\pi}{4}\right) = -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

$$y = 2\sin\frac{3\pi}{4} = 2\left(\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

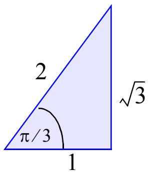


Conclusion: The point P has rectangular coordinates $(-\sqrt{2}, \sqrt{2})$.

(ii) For $P = (r, \theta) = \left(-3, \frac{\pi}{3}\right)$:

$$x = -3\cos\frac{\pi}{3} = (-3)\left(\frac{1}{2}\right) = -\frac{3}{2}$$

$$y = -3\sin\frac{\pi}{3} = -3\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$$



Conclusion: The point P has rectangular coordinates $\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$.

Note also that:

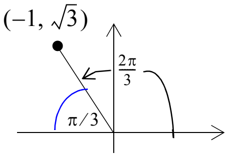
$$P = \left(-3, \frac{\pi}{3}\right) = \left(3, \pi + \frac{\pi}{3}\right)$$
$$= \left(3, \frac{4\pi}{3}\right)$$

and that:

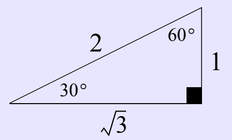
$$3\cos\frac{4\pi}{3} = -\frac{3}{2},$$
$$3\sin\frac{4\pi}{3} = -\frac{3\sqrt{3}}{2}$$

(b) Let $P = (-1, \sqrt{3})$. Turning to $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$ we have:

$$r^2 = (-1)^2 + (3^{1/2})^2 = 4, \text{ or } r = \pm 2, \quad \text{and} \quad \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$



There are infinitely many θ that can accommodate $\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. We know that we are dealing with a $60°$ reference angle (see margin). Since we can arrive at the above terminal side by rotating any multiple of 2π we have: $\theta = \frac{2\pi}{3} + 2k\pi$ for any integer k. It follows that P has infinitely many polar coordinate representations:



$$P = \left(2, \frac{2\pi}{3} + 2k\pi\right) \text{ for any integer } k$$

And if that isn't enough (see comments directly below Figure 10.4):

$$P = \left(-2, \left(\frac{2\pi}{3} + \pi\right) + 2k\pi\right)$$
$$= \left(-2, \frac{5\pi}{3} + 2k\pi\right) \text{ for any integer } k$$

CHECK YOUR UNDERSTANDING 10.6

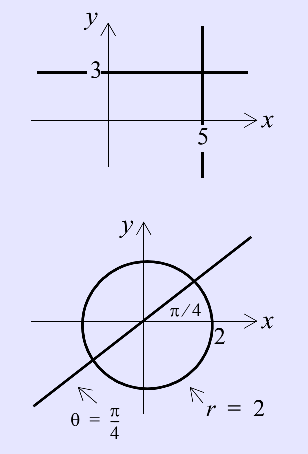
(a) Find the rectangular coordinates of the point with polar coordinates $\left(-2, -\frac{\pi}{6}\right)$.

(b) Find all possible polar coordinates of the point with rectangular coordinates $(1, -1)$.

Answers: (a) $(-\sqrt{3}, 1)$

(b) $\left(\sqrt{2}, -\frac{\pi}{4} + 2k\pi\right)$ and $\left(-\sqrt{2}, \frac{3\pi}{4} + 2k\pi\right)$ for any integer k.

POLAR CURVES



In the rectangular coordinate system, the curve in the plane associated with the equation $x = 5$ is the vertical line with x-intercept 5, while $y = 3$ is the horizontal line with y-intercept 3 (see margin).

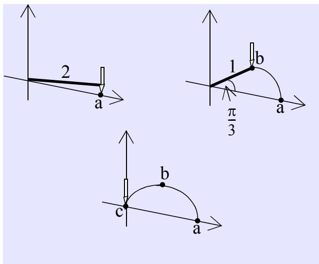
In the polar setting, the curve in the plane associated with the equation $r = 2$ is the circle of radius 2 centered at the origin, while $\theta = \frac{\pi}{4}$ is the line of slope 1 passing through the origin (see margin). In general:

Equation	Graph
$r = a$	Circle of radius $\lvert a \rvert$ centered at the origin.
$\theta = \theta_0$	Line containing the terminal side of the angle θ_0 in standard position.

When graphing a function $y = f(x)$ in the rectangular coordinate system, one generally "observes" what happens to y as x assumes different values. The same can be said about graphing a curve $r = g(\theta)$, except that now one tries to "observe" what happens to the radius r as θ assumes different values. Consider the following example.

EXAMPLE 10.7 Sketch, in the Cartesian plane, the curve with polar equation $r = 2\cos\theta$.

SOLUTION: (a) The graph of $r = 2\cos\theta$ appears in Figure 10.5(a) (note the labeling of the axis in that figure). The key that will enable us to transform that graph to the rectangular coordinate system is to realize that (r, θ), for $r \geq 0$, is associated with the point (x, y) in the Cartesian plane that lies on the terminal side of θ and is r units from the origin.



Let's position an adjustable ruler, with one end at the origin and a pencil at its other end. We start off with $\theta = 0$ and with a ruler of length $r = 2\cos 0 = 2$ to arrive at the point a in the margin [and in Figure 10.5(b)]. We then swing the ruler through an angle θ, diminishing its length in accordance with the formula $r = 2\cos\theta$. In particular, when $\theta = \frac{\pi}{3}$, the ruler is of length $r = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$ [see point b in margin and in Figure 10.5(b)]. Rotating further we get to $\theta = \frac{\pi}{2}$, at which time the ruler is of length 0 [see point c in margin and in Figure 10.5(b)].

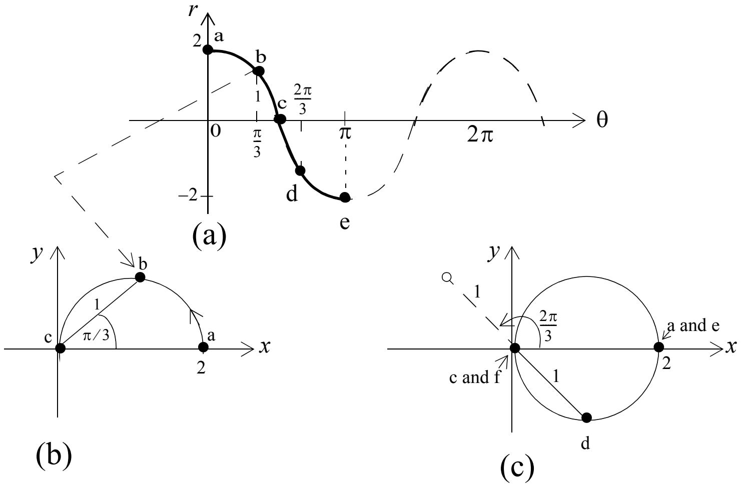


Figure 10.5

Rotating further to the angle $\theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ we are confronted with a ruler of "negative length," namely: $r = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$. Not to worry:

> To find the polar-point $\left(-1, \frac{2\pi}{3}\right)$ in the Cartesian plane we mark off 1 unit in the "opposite direction" of $\theta = \frac{2\pi}{3}$ [see *d* in Figure 10.5(c)].

Continuing the good fight, we come to the polar point $(-2, \pi)$, which brings us back to the point $(2, 0)$ in the Cartesian plane [see *e* in Figure 10.5(c)].

Note that while the point $d = \left(1, \frac{4\pi}{3}\right)$ lies on the curve in Figure 10.5(c), it fails to satisfy the equation $r = 2\cos\theta$:

$$2\cos\frac{4\pi}{3} = -1 \text{ (and not 1).}$$

Your turn: Find a polar point $(-1, \theta)$ satisfying $r = 2\cos\theta$ that corresponds to the point *d* in Figure 10.5(c).

Note that the circle in Figure 10.5(c) will be retraced, over and over again, as θ runs over intervals of length π. In particular, if θ runs from 0 to $\mathbf{2\pi}$, then the circle will be traced out **twice**.

The polar curve $r = 2\cos\theta$ of the previous example turned out to be the circle of radius 1 centered at $(1, 0)$ [Figure 10.5(c)]. As such, it has a nice Cartesian representation; namely: $(x-1)^2 + y^2 = 1$. Not all polar curves are that fortunate. Consider the following example.

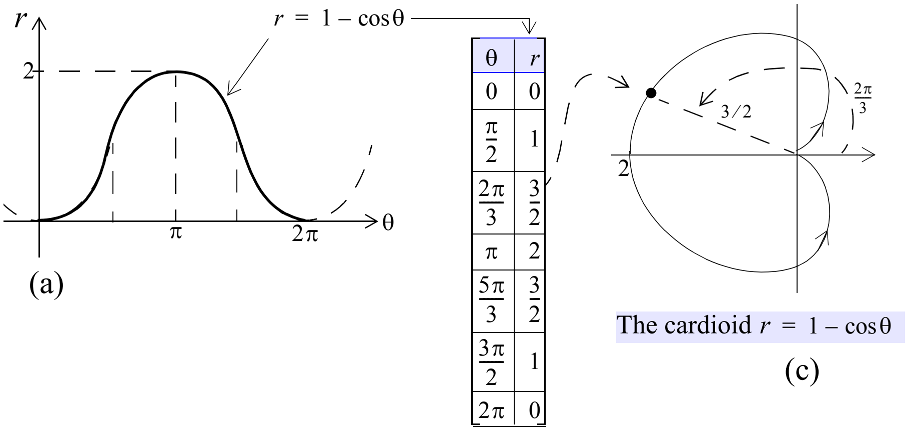
EXAMPLE 10.8 Sketch, in the Cartesian plane, the curve with given polar equation.

(a) $r = 1 - \cos\theta$ (cardioid).

(b) $r = \sin 2\theta$ (four-leaved rose).

(c) $r^2 = 4\cos 2\theta$ (lemniscate).

Solution: (a) Focusing on the graph of $r = 1 - \cos\theta$ in Figure 10.6(a) we observe that $r \geq 0$ throughout the interval $[0, 2\pi]$. Consequently, as you can see in Figure 10.6(c), the distance r associated with any θ is measured along the terminal side of that angle (as opposed to its “opposite direction”). To help us construct that heart-shaped curve (called a **cardioid**), we first plotted a few points [see table in Figure 10.6(b)].



θ	r
0	0
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2
$\frac{5\pi}{3}$	$\frac{3}{2}$
$\frac{3\pi}{2}$	1
2π	0

(a)

The cardioid $r = 1 - \cos\theta$

(c)

(b)

Figure 10.6

(b) The graph of $r = \sin 2\theta$, for $0 \leq \theta \leq 2\pi$, appears in Figure 10.7(a).

As θ sweeps from 0 to $\frac{\pi}{2}$: r starts at 0, reaches a maximum length of 1 at $\frac{\pi}{4}$, and then decreases back to 0 at $\frac{\pi}{2}$ [see (b) in figure].

As θ goes from $\frac{\pi}{2}$ to π: r starts at 0 and again returns to 0, but now r assumes negative values. As such, the length $|r|$ is measured in the opposite direction of the terminal side of θ [see (c) in figure].

For $\pi \leq \theta \leq \frac{3\pi}{2}$: r again goes from 0 to 0. Since r is nonnegative, it is now measured along the terminal side of θ [see (d) in figure].

For $\frac{3\pi}{2} \leq \theta \leq 2\pi$ (in the fourth quadrant), r is again negative. As such, its magnitude is measured in the opposite direction of the terminal side of θ (appearing in the second quadrant) [see (e) in figure].

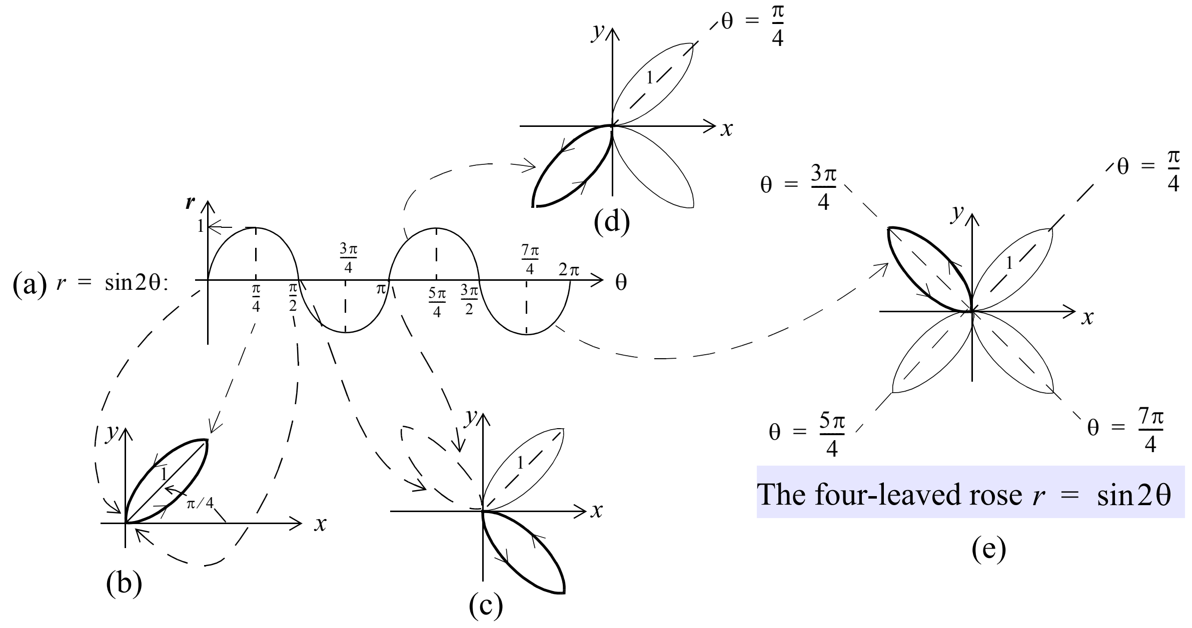


Figure 10.7

We point out that the graph of $r = \sin n\theta$ or $r = \cos n\theta$ is a rose with $2n$ leaves if n is even and n leaves if n is odd.

(c) Turning to $r^2 = 4\cos 2\theta$, we first note that r is undefined when $4\cos 2\theta$ is negative; namely, for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ [see Figure 10.8(a)]. As θ sweeps from 0 to $\frac{\pi}{4}$: r^2 decreases from 4 to 0. This will give rise to two branches of the curve: one corresponding to r increasing from -2 to 0, and the other for r increasing from 0 to 2 (see margin). These two branches are represented in Figure 10.8(b). You can see how the "r-positive-ruler" decreases from 2 to 0 as θ sweeps from 0 to $\frac{\pi}{4}$; as does the "r-negative-ruler," but now along the opposite direction of the terminal side of θ [see Figure 10.8(b)].

$0 \le r^2 \le 4$
$-2 \le r \le 2$

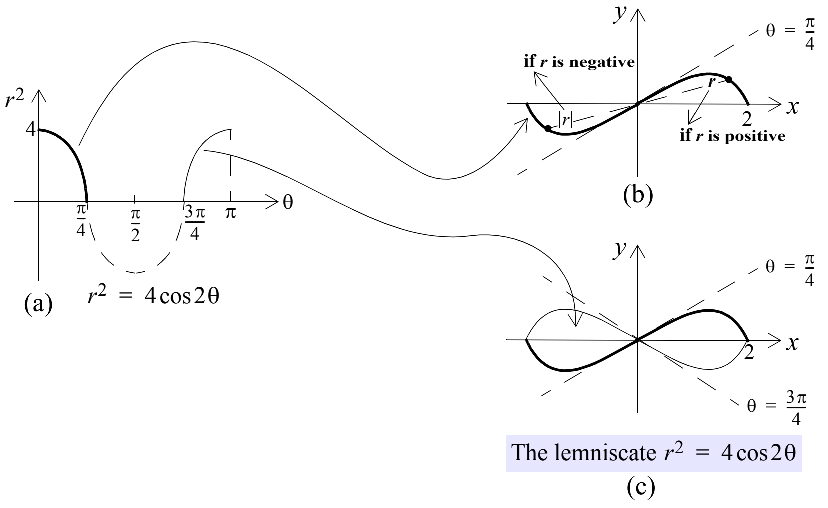
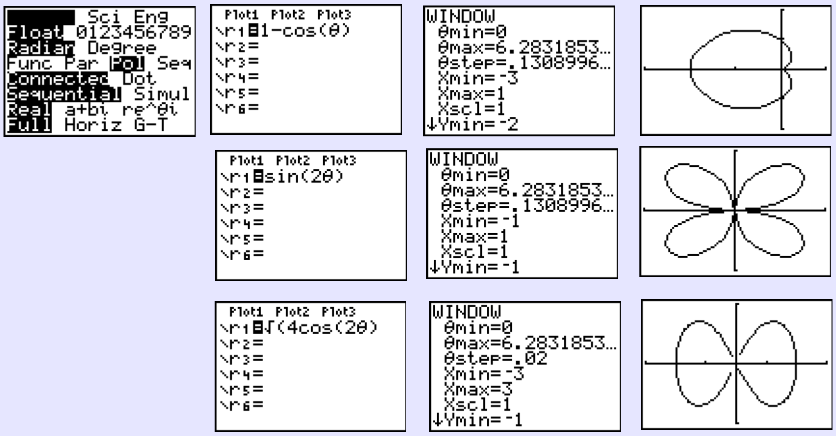


Figure 10.8

As θ sweeps from $\frac{3\pi}{4}$ to π: r^2 increases from 0 to 4. This will again give rise to two branches of the curve: one for $0 \le r \le 2$ (second quadrant of Figure 10.8(c)] and the other for $-2 \le r \le 0$ (fourth quadrant).

The TI84 has polar graphing capabilities:



```
      Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

```
 Plot1 Plot2 Plot3
\r1■1-cos(θ)
\r2=
\r3=
\r4=
\r5=
\r6=
```

```
WINDOW
 θmin=0
 θmax=6.2831853…
 θstep=.1308996…
 Xmin=-3
 Xmax=1
 Xscl=1
↓Ymin=-2
```

```
 Plot1 Plot2 Plot3
\r1■sin(2θ)
\r2=
\r3=
\r4=
\r5=
\r6=
```

```
WINDOW
 θmin=0
 θmax=6.2831853…
 θstep=.1308996…
 Xmin=-1
 Xmax=1
 Xscl=1
↓Ymin=-1
```

```
 Plot1 Plot2 Plot3
\r1■√(4cos(2θ)
\r2=
\r3=
\r4=
\r5=
\r6=
```

```
WINDOW
 θmin=0
 θmax=6.2831853…
 θstep=.02
 Xmin=-3
 Xmax=3
 Xscl=1
↓Ymin=-1
```

Answer: See page A-64.

CHECK YOUR UNDERSTANDING 10.7

Sketch, in the Cartesian plane, the spiral $r = \theta$ for $\theta \ge 0$.

DERIVATIVES IN POLAR COORDINATES

If $r = 1 + \sin\theta$ then:
$\frac{dr}{d\theta} = \cos\theta$

To find $\frac{dr}{d\theta}$ for $r = f(\theta)$ you would proceed in the usual manner (see margin). But this is of no help if you are interested in finding the slope of a tangent line to the graph of $r = f(\theta)$ in the *x,y*-plane. For that purpose, we need $\frac{dy}{dx}$, which we determine as follows:

The graph of $r = f(\theta)$ in the rectangular coordinate system can be defined by the following parametric equations (with parameter θ):

$$\underset{\uparrow}{x = } r\cos\theta = f(\theta)\cos\theta, \quad \underset{\uparrow}{y =} r\sin\theta = f(\theta)\sin\theta$$

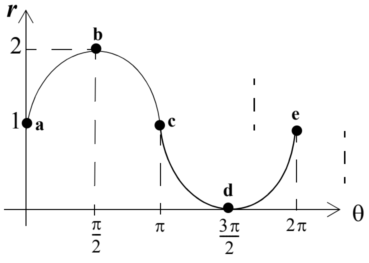
see page 405

As such:

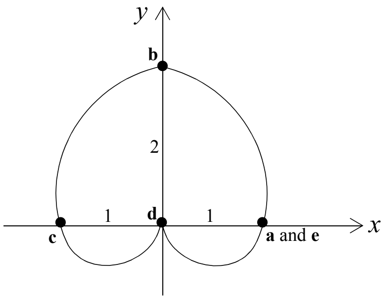
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{[f(\theta)\sin\theta]'}{[f(\theta)\cos\theta]'} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

EXAMPLE 10.9 Sketch, in the Cartesian plane, the curve with polar equation $r = 1 + \sin\theta$, labeling its (local) maxima and minima points.

SOLUTION: The graph:



$r = 1 + \sin\theta$



The cardioid $r = 1 + \sin\theta$

Glancing at the above cardioid we observe that a (local) maximum occurs at $b = (0, 2)$ (when $\theta = \frac{\pi}{2}$) and at $d = (0, 0)$ (when $\theta = \frac{3\pi}{2}$). We also see that a minimum occurs at some x between points c and d (when θ is somewhere between π and $\frac{3\pi}{2}$) and at some x between d and a (when θ is somewhere between $\frac{3\pi}{2}$ and 2π).

Let's find them:

For $r = f(\theta) = 1 + \sin\theta$:

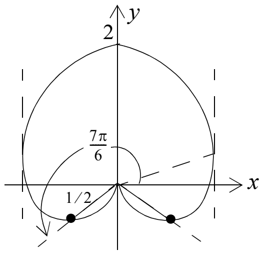
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta(2\sin\theta + 1)}{\cos^2\theta - \sin\theta - \sin^2\theta}$$

The numerator, $\cos\theta(2\sin\theta + 1)$, is zero if either $\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$; which, for $0 \le \theta < 2\pi$, occurs at:

$$\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}, \theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$$

We already spotted the maximum points on the cardioid associated with $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, namely $(0, 2)$ and $(0, 0)$. We now know that the two minimum points occur when $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$, with corresponding lengths:



$$r = 1 + \sin\theta = 1 + \sin\frac{7\pi}{6} = \frac{1}{2} \text{ and } r = 1 + \sin\frac{11\pi}{6} = \frac{1}{2}.$$

Note that while a maximum occurs at the origin, the derivative is not zero at that point. Indeed, a vertical tangent line occurs when $\theta = \frac{3\pi}{2}$, as well as when with $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ — a consequence of the fact that the denominator of $\frac{dy}{dx}$ is 0 for those values of θ:

$$\cos^2\theta - \sin\theta - \sin^2\theta = 0$$
$$(1 - \sin^2\theta) - \sin\theta - \sin^2\theta = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}$

$\theta = \frac{3\pi}{2}$

see figure

You can get to the rectangular coordinates associated with those minimum points by using the bridges:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta .$$

In particular, $\left(-\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)$ is associated with $\theta = \frac{7\pi}{6}$:

$$x = \frac{1}{2}\cos\frac{7\pi}{6} = \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4}$$

$$y = \frac{1}{2}\sin\frac{7\pi}{6} = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$$

and $\left(\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)$ is associated with $\theta = \frac{11\pi}{6}$:

$$x = \frac{1}{2}\cos\frac{11\pi}{6} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

$$y = \frac{1}{2}\sin\frac{11\pi}{6} = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$$

CHECK YOUR UNDERSTANDING 10.8

Find the rectangular coordinates of the (local) maxima and minima points of the cardioid $r = 1 - \cos\theta$ of Example 10.8(a).

Answer: Local maximum:

$$(x, y) = \left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$$

Local minimum:

$$(x, y) = \left(-\frac{3}{4}, -\frac{3\sqrt{3}}{4}\right)$$

EXERCISES

Exercises 1-8. Find the rectangular coordinates of the point with given polar coordinates.

1. $(2, 0)$
2. $(-2, 0)$
3. $(-3, \pi)$
4. $\left(2, \frac{\pi}{3}\right)$
5. $\left(-2, -\frac{\pi}{3}\right)$
6. $\left(2, \frac{2\pi}{3}\right)$
7. $\left(-2, \frac{3\pi}{4}\right)$
8. $\left(5, \tan^{-1}\frac{4}{3}\right)$

Exercises 9-12. Find the polar coordinates (r, θ), with $0 \le \theta < 2\pi$, of the point with give rectangular coordinates.

9. $(4, -4)$
10. $(2, -2)$
11. $(0, 1)$
12. $(1, 0)$

Exercises 13-16. Find all possible polar coordinates of the point with given polar coordinates.

13. $(-1, \sqrt{3})$
14. $(\sqrt{3}, -1)$
15. $(4\sqrt{3}, 4)$
16. $(3, -3\sqrt{3})$

Exercises 17-22. Find a polar equation for the curve represented by the give rectangular equation.

17. $x^2 + y^2 = 9$
18. $x = 5$
19. $x = -y^2$
20. $y^2 = 8x$
21. $x^2 + y^2 + 4x = 0$
22. $x^2 + 4y^2 = 4$

Exercises 23-28. Find a rectangular equation for the curve represented by the give polar equation.

23. $r = 4$
24. $\theta = \frac{\pi}{4}$
25. $r = 3\sin\theta$
26. $r\cos\theta - 1 = 0$
27. $r = \csc\theta$
28. $r = \tan\theta\sec\theta$

Exercises 29-40. Sketch, in the Cartesian plane, the curve with the given polar equation.

29. $r = 4$
30. $\theta = \frac{3\pi}{4}$
31. $r = 6\sin\theta$
32. $r = \frac{1}{2} + \cos\theta$
33. $r = 2 - 2\cos\theta$
34. $r = 4 + 4\cos\theta$
35. $r = 4\sin 3\theta$
36. $r = \frac{1}{2} + \sin\theta$
37. $r = 2 + 4\cos\theta$
38. $r = 1 - 2\sin\theta$
39. $r = \frac{3}{2} + \cos\theta$
40. $r = \cos 3\theta$
41. $r = 2\tan\theta$
42. $r = 2\sec\theta$
43. $r^2 = 9\sin 2\theta$
44. $r^2 = 9\cos 2\theta$
45. $r = 2\theta$ for $0 \le \theta \le 3\pi$
46. $r = 1 + 2\theta$ for $0 \le \theta \le 3\pi$

Exercises 47-52. Find, at the given point, the slope of the tangent line to the curve in the Cartesian plane with the given polar equation.

47. $r = 2\sin\theta,\ \theta = \frac{\pi}{6}$
48. $r = \cos\frac{\theta}{2},\ \theta = \frac{\pi}{3}$
49. $r = \cos 2\theta,\ \theta = \frac{\pi}{4}$
50. $r = 1 - \sin\theta,\ \theta = \pi$
51. $r = \frac{1}{\theta},\ \theta = \pi$
52. $r = 1 + 2\cos\theta,\ \theta = \frac{\pi}{3}$

Exercises 53-58. Determine the local maxima or minima points of the curve in the Cartesian plane with the given polar equation.

53. $r = 4 + 4\cos\theta$
54. $r = 1 - \sin\theta$
55. $r = \cos^2\theta$
56. $r = \sin^2\theta$
57. $r = 1 + 2\cos\theta$
58. $r = 2 - 3\sin\theta$

Exercises 59-62. Find the intersection points of the given polar equations in the Cartesian plane.

59. $r = \sin\theta, \quad r = -\cos\theta$
60. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$
61. $r = 1 - \cos\theta, \quad r = \cos\theta$
62. $r = 2\sin\theta, \quad r = 2\sin 2\theta$

63. Prove that the polar equation $r = a\sin\theta + b\cos\theta$ represents a circle,

64. Give a polar coordinate formula for the distance between two point in the Cartesian plane with polar coordinates (r_1, θ_1) and (r_2, θ_2).

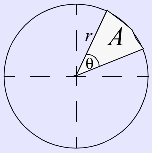
65. Show that the graphs of $r = \cos\theta$ and $r = \sin\theta$ intersect at right angles.

66. Prove that the area of a triangle with polar vertex coordinates $(0, 0), (r_1, \theta_1)$, and (r_2, θ_2) is $A = \frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$.

§3. AREA AND LENGTH

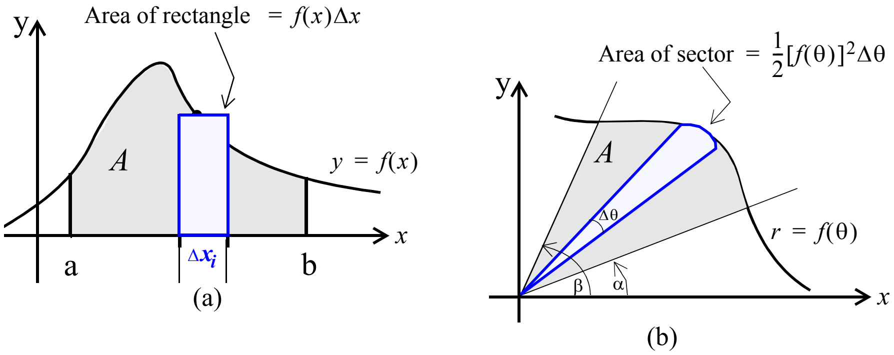
The procedure for finding the area of a region enclosed by the graph of a polar equation $r = f(\theta)$ that lies between the terminal sides of two angles α and β, is very similar to that discussed in Section 5.2 for finding the area of a region bounded above by the graph of a positive function $y = f(x)$ over an interval $[a, b]$. The main difference is that in the $y = f(x)$-case one approximates the desired area by summing areas of rectangles [see Figure 10.9(a)], while in the $r = f(\theta)$-case one sums areas of sectors [see margin and Figure 10.9(b)].

A **sector** is a portion of a circle bounded by two rays:



Noting that the area A of the sector is to the angle θ as the area of the circle is to a complete revolution brings us to:

$$\frac{A}{\theta} = \frac{\pi r^2}{2\pi} \text{ or: } A = \frac{1}{2}r^2\theta$$



(a) (b)

Figure 10.9

And just as the area of the region in Figure 10.9(a) for a continuous function f is given by $A = \int_a^b f(x)dx = \lim_{\Delta x \to 0}\sum_a^b f(x)\Delta x$, so then:

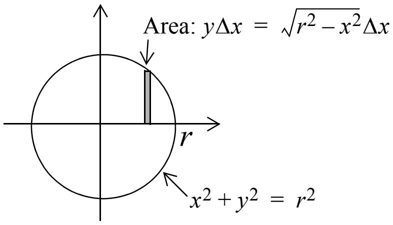
THEOREM 10.1 If $r = f(\theta)$ is continuous and nonnegative for $\alpha \le \theta \le \beta$, then the area A of the region enclosed by the polar curve lying between the terminal sides of α and β is given by:

$$A = \int_\alpha^\beta \frac{1}{2}r^2 d\theta = \lim_{\Delta\theta \to 0}\frac{1}{2}\sum_\alpha^\beta [f(\theta)]^2\Delta\theta$$

A point of comparison:

Finding the area of a circle of radius r

Rectangular Approach



$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx$$

by symmetry

$x = r\sin\theta$
$dx = r\cos\theta\, d\theta$
$0 \le x \le r \Rightarrow 0 \le \theta \le \frac{\pi}{2}$

$$\longrightarrow = 4\int_0^{\frac{\pi}{2}} [\sqrt{r^2 - r^2\sin^2\theta}]\, r\cos\theta\, d\theta$$

$$= 4r^2\int_0^{\frac{\pi}{2}} [\sqrt{1 - \sin^2\theta}]\cos\theta\, d\theta$$

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

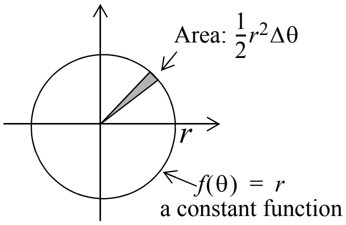
Theorem 1.5(ix), page 37:

$$= 4r^2\int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= 2r^2\left(\theta + \frac{\sin 2\theta}{2}\Bigg|_0^{\frac{\pi}{2}}\right)$$

$$= 2r^2\left(\frac{\pi}{2}\right) = \pi r^2$$

Polar Approach



$$A = \int_0^{2\pi} \frac{1}{2}r^2\, d\theta$$

$$= \frac{1}{2}r^2\int_0^{2\pi} d\theta$$

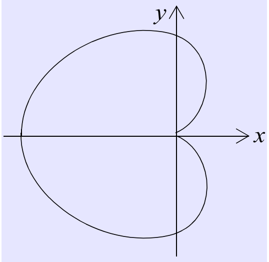
$$= \frac{1}{2}r^2(\theta|_0^{2\pi})$$

$$= \frac{1}{2}r^2(2\pi)$$

$$= \pi r^2$$

Clearly the above polar approach is the better choice, but that is due to the circular nature of the region in question. If you want a contest that dramatically favors the rectangular approach, just try to find the area of a rectangle using the polar approach.

EXAMPLE 10.10 Find the area of the region in the plane enclosed by the cardioid $r = 1 - \cos\theta$.



$r = 1 - \cos\theta$

SOLUTION: The graph of $r = 1 - \cos\theta$ was constructed in Example 10.8, page 408(a) (margin). By virtue of symmetry, we double the area of the top half of the cardioid (the area obtained by letting θ run from 0 to π):

$$
\begin{aligned}
A &= 2\int_0^{\pi}\frac{1}{2}(1-\cos\theta)^2\,d\theta = \int_0^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta \\
&= (\theta-2\sin\theta)\Big|_0^{\pi} + \int_0^{\pi}\cos^2\theta\,d\theta \\
&= \pi + \int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta \\
&= \pi + \left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Bigg|_0^{\pi} \\
&= \pi + \frac{\pi}{2} = \frac{3\pi}{2}
\end{aligned}
$$

Theorem 1.5(ix), page 37

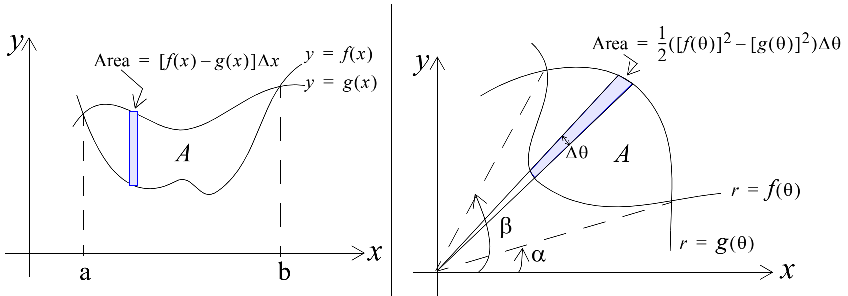
CHECK YOUR UNDERSTANDING 10.9

(a) Find the area of the four-leaved rose $r = \sin 2\theta$ of Example 10.8(b), page 408.

(b) Find the area within the inner loop of the (limacon) $r = 2\cos\theta + 1$.

Answer: (a) $\frac{\pi}{2}$ (b) $\pi - \frac{3\sqrt{3}}{2}$

The procedure for finding the area of a region enclosed by the graphs of two polar equations is very similar to that for finding the area of a region enclosed by the graphs of two functions. The main difference is that instead of summing the enclosed areas of rectangles [see Figure 10.10(a)], we sum the enclosed areas of sectors [see Figure 10.10(b)].



$$A = \int_a^b [f(x) - g(x)]\,dx$$

(a)

$$A = \int_\alpha^\beta \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2)\,d\theta$$

(b)

Figure 10.10

Just as the limits of integration in Figure 10.10(a) are the x-coordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$, so then are those of Figure (b) the θ-coordinates of the points of intersection of the curves $r = f(\theta)$ and $r = g(\theta)$.

Please note that while both points of intersection in Figure 10.10(a) satisfy the equation $f(x) = g(x)$, not all points of intersection of two polar equations $r = f(\theta)$ and $r = g(\theta)$ need satisfy the corresponding equation $f(\theta) = g(\theta)$. This is because every point in the plane has infinitely many pairs of polar coordinates, and a point of intersection may have no single pair of polar coordinates satisfying both equations. Consider the following example:

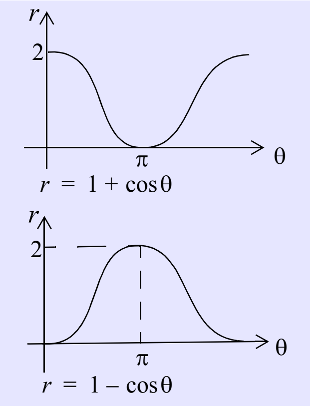
EXAMPLE 10.11 Find the points of intersection of the two cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

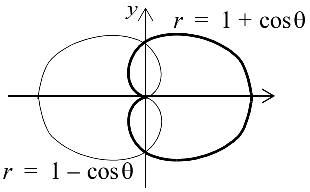
SOLUTION: Equating the two r-expressions we have:

$$1 + \cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = -\cos\theta \Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} + k\pi$$

(for any integer k)

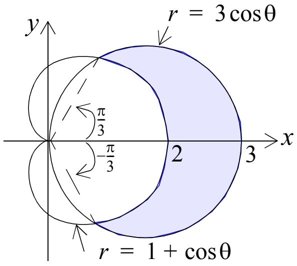




Though the above solution does reveal two points of intersection of the cardioids, it fails to reveal the third intersection point: the origin! Why was that point missed? Because the origin is not on the two curves **for a common value of** θ. Specifically, with respect to the curve $r = 1 + \cos\theta$, the origin is reached when $\theta = \pi + 2k\pi$, while it is reached when $\theta = 2k\pi$ on the curve $r = 1 - \cos\theta$ (see margin).

Moral: When determining the intersection points of polar coordinate curves, sketch the curves to spot their anticipated number.

EXAMPLE 10.12 Find the area of the region that lies inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.



SOLUTION: We want to find the area of the shaded region in the adjacent figure. Here is how we found the θ values of the indicated points of intersections of the curves:

$$3\cos\theta = 1 + \cos\theta$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$$

Taking advantage of symmetry, we calculate the area lying above the x-axis and multiply by two:

$$\begin{aligned}
A &= 2\int_0^{\frac{\pi}{3}}\frac{1}{2}[(3\cos\theta)^2-(1+\cos\theta)^2]d\theta \\
&= \int_0^{\frac{\pi}{3}}(8\cos^2\theta-2\cos\theta-1)d\theta \\
&= \int_0^{\frac{\pi}{3}}\left[8\left(\frac{1+\cos 2\theta}{2}\right)-2\cos\theta-1\right]d\theta \\
&= (4\theta+2\sin 2\theta-2\sin\theta-\theta)\Big|_0^{\frac{\pi}{3}} = \frac{4\pi}{3}-\frac{\pi}{3} = \pi
\end{aligned}$$

CHECK YOUR UNDERSTANDING 10.10

Find the area common to the region bounded by the cardioid $r = 1-\cos\theta$ and the circle $r = \frac{1}{2}$.

Answer: $\frac{2\pi}{3}-\frac{7\sqrt{3}}{8}$

ARC LENGTH

A polar curve $r = f(\theta)$ is but a parametrized curve with parameter $\alpha \le \theta \le \beta$, and parametric equations:

$$x = r\cos\theta, \quad y = r\sin\theta \quad (*)$$

As such (Definition 10.1, page 401):

$$L = \int_\alpha^\beta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta$$

Applying the product rule in (*) we have:

$$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta}-r\sin\theta, \quad \frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta}+r\cos\theta$$

Leading us to:

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2 &= \left[\cos\theta\frac{dr}{d\theta}-r\sin\theta\right]^2+\left[\sin\theta\frac{dr}{d\theta}+r\cos\theta\right]^2 \\
&= \cos^2\theta\left(\frac{dr}{d\theta}\right)^2\cancel{-2r\sin\theta\cos\theta\frac{dr}{d\theta}}+r^2\sin^2\theta \\
&\quad +\sin^2\theta\left(\frac{dr}{d\theta}\right)^2\cancel{+2r\sin\theta\cos\theta\frac{dr}{d\theta}}+r^2\cos^2\theta \\
&= (\cos^2\theta+\sin^2\theta)\left(\frac{dr}{d\theta}\right)^2+r^2(\sin^2\theta+\cos^2\theta)
\end{aligned}$$

Theorem 1.5(i), page 37: $= \left(\frac{dr}{d\theta}\right)^2+r^2$

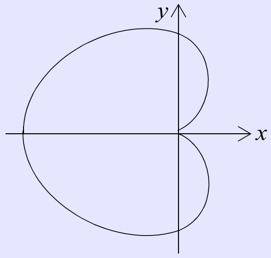
Bringing us to:

THEOREM 10.2 If the curve $r = f(\theta)$ is traced out exactly once as $\alpha \le \theta \le \beta$, and if f has a continuous first derivative on $[\alpha, \beta]$, then the length L of the curve is given by:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXAMPLE 10.13 Find the total arc length of the cardioid $r = 1 - \cos\theta$.

See Example 10.8, page 408.



$r = 1 - \cos\theta$

SOLUTION:

$$L = \int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta$$

Theorem 1.5(iiiv) page 37 → $1 - \cos 2x = 2\sin^2 x$

$1 - \cos x = 2\sin^2\frac{x}{2}$:

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\left(\sin^2\frac{\theta}{2}\right)}\, d\theta$$

Note that since $\sin\frac{\theta}{2} \ge 0$ for $0 \le \theta \le 2\pi$, $\sqrt{\left(\sin\frac{\theta}{2}\right)^2} = \sin\frac{\theta}{2}$:

$$= 2\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= 2\left(-2\cos\frac{\theta}{2}\Big|_0^{2\pi}\right) = 8$$

CHECK YOUR UNDERSTANDING 10.11

Express the total length of the four-leaved rose $r = \sin 2\theta$ [Example 10.8(b), page 404] in integral form, and then use a graphing calculator to approximate its value to two decimal places.

Answer:

$$4\int_0^{\frac{\pi}{2}} \sqrt{\sin^2 2\theta + (2\cos 2\theta)^2}\, d\theta$$

EXERCISES

Exercises 1-6. Sketch the curve with given polar equation and find the area it encloses.

1. $r = 1 + \cos\theta$
2. $r = 3\cos\theta$
3. $r = 2\cos 2\theta$
4. $r = 2 - \sec\theta$
5. $r^2 = 4\sin^2\theta$
6. $r^2 = \sin^2 2\theta$

Exercises 7-12. Sketch the curve with given polar equation and find the specified area.

7. $r = \theta^2, 0 \le \theta \le \frac{\pi}{4}$.
8. $r = 3\sin\theta, 0 \le \theta \le \frac{\pi}{4}$.
9. $r = \tan 2\theta, 0 \le \theta \le \frac{\pi}{8}$.
10. $r = \sqrt{\sin\theta}, 0 \le \theta \le \pi$.
11. $r = \sqrt{1 - \cos\theta}, \frac{\pi}{2} \le \theta \le \pi$.
12. $r = e^{\theta/2}, \pi \le \theta \le 2\pi$.

Exercises 13-20. Sketch the two curves with given polar equations and find the area of the region common to those curves.

13. $r = 2, r = 2(1 - \cos\theta)$.
14. $r = 3\cos\theta, r = 1 + \cos\theta$.
15. $r = \cos\theta, r = \sin\theta$.
16. $r = 2(1 - \cos\theta), r = 2(1 + \cos\theta)$.
17. $r = -4\sin\theta, r = 4(1 + \cos\theta)$.
18. $r = \sin 2\theta, r = \cos 2\theta$.
19. $r^2 = \sin 2\theta, r^2 = \cos 2\theta$.
20. $r = a\sin\theta, r = b\cos\theta, a > 0, b > 0$.

21. Find the area of the region that is inside the cardioid $r = 4(1 + \cos\theta)$ and outside the circle $r = 6$.

22. Find the area of the region that is inside the circle $r = \sin\theta$ and outside the cardioid $r = 1 + \cos\theta$.

23. Find the area within the inner loop of the limacon $r = 1 - 2\sin\theta$.

24. Find the area of the region outside the inner loop and inside the limacon $r = 2 + 4\cos\theta$.

25. Find the area of the region that is inside the limacon $r = 4 + \cos\theta$ and outside the limacon $r = 2 + \cos\theta$.

26. Find the area of the region that is inside the circle $r = 8\cos\theta$ and to the right of the line $r = 2\sec\theta$.

27. Find the area of the region that is outside the circle $r = 2$ and inside the cardioid $r = 4(1 + \cos\theta)$.

28. Find the area of the region that is outside the circle $r = 3$ and inside the cardioid $r = 2(1+\cos\theta)$.

29. Find the area of the region that is inside the lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \sqrt{2}$.

Exercises 30-35. Find the length of the given polar curve.

30. The circle $r = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{3}$.

31. The cardioid $r = 1+\cos\theta$.

32. The spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$.

33. The curve $r = \cos^3\left(\frac{\theta}{3}\right)$, $0 \le \theta \le \frac{\pi}{4}$.

34. The spiral $r = e^{3\theta}$, $0 \le \theta \le 2$.

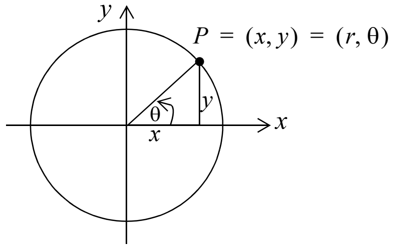
35. The parabolic arc $r = \frac{6}{1+\cos\theta}$, $0 \le \theta \le \frac{\pi}{2}$.

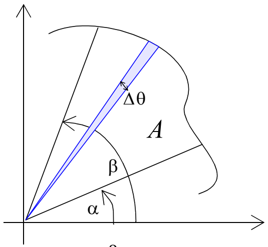
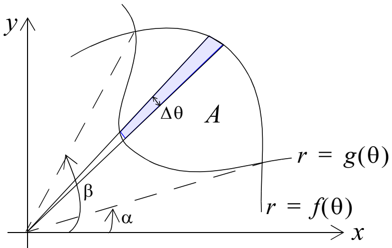
Exercises 36-37. Use a graphing calculator to approximate the length of the given polar curve to two decimal places.

36. The three-leaved rose $r = 2\cos 3\theta$.

37. The four-leaved rose $r = 4\cos 2\theta$

CHAPTER SUMMARY

PARAMETRIZED CURVE	Let $x(t)$ and $y(t)$ be a pair of functions defined on some interval I. To each t in I we associate the point $(x(t), y(t))$ in the plane. As t ranges over I, the point $(x(t), y(t))$ traces out a path (curve) in the plane. Such a curve is said to be a **parametrized curve**, and the variable t is said to be a **parameter**.
FIRST DERIVATIVE	For the curve $C = \{(x(t), y(t)) \text{ for } a \le t \le b\}$: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ Providing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ exist, and $\frac{dx}{dt} \neq 0$
SECOND DERIVATIVE	$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$ Providing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are differentiable, and $\frac{dx}{dt} \neq 0$
ARC LENGTH	Let $x(t)$ and $y(t)$ have continuous derivatives on a closed interval $a \le t \le b$. The length L of the parametrized curve $x = x(t),\ y = y(t)$ for $a \le t \le b$ is given by: $L = \int_a^b \sqrt{\left(\frac{dx}{\mathrm{d}t}\right)^2 + \left(\frac{dy}{\mathrm{d}t}\right)^2}\, dt$
POLAR VERSUS RECTANGULAR COORDINATES	 $x = r\cos\theta$ $y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \dfrac{y}{x}$

AREA	If $r = f(\theta)$ is continuous and nonnegative for $\alpha \le \theta \le \beta$, then the area A of the region enclosed by the polar curve lying between the terminal sides of α and β is given by: $$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$ The area of a region enclosed by two continuous polar curves $r = f(\theta)$ and $r = g(\theta)$ for $\alpha \le \theta \le \beta$ is given by: $$A = \int_{\alpha}^{\beta} \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2) d\theta$$  $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$  $A = \int_{\alpha}^{\beta} \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2) d\theta$
ARC LENGTH	If the curve $r = f(\theta)$ is traced out exactly once as $\alpha \le \theta \le \beta$, and if f has a continuous first derivative on $[\alpha, \beta]$, then the length L of the curve is given by: $$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

CHECK YOUR UNDERSTANDING SOLUTIONS

CHAPTER 1: PRELIMINARIES

CYU 1.1 (a) $f(-2) = 3(-2)-5 = -11$ (b) $f(t+1) = 3(t+1)-5 = 3t-2$

(c) $f(-2x+1) = 3(-2x+1)-5 = -6x-2$

(d) $f\left(-\frac{2}{x}\right) = 3\left(-\frac{2}{x}\right)-5 = -\frac{6}{x}-5 = \frac{-6-5x}{x}$

CYU 1.2
$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h} = \frac{\frac{(x+h)(x+1)-x(x+h+1)}{(x+h+1)(x+1)}}{h}$$
$$= \frac{x^2+x+hx+h-(x^2+xh+x)}{(x+h+1)(x+1)h}$$
$$= \frac{h}{(x+h+1)(x+1)h} = \frac{1}{(x+h+1)(x+1)}$$

CYU 1.3 (a) For $f(x) = \sqrt{x+3}$ to be defined, $x+3 \geq 0 \Rightarrow x \geq -3$, so that the domain is $D_f = [-3, \infty)$. The values of the square root are the nonnegative numbers, which means that the range is $R_f = D_f = [0, \infty)$.

(b) The function $g(x) = \frac{1}{(x+1)(x-2)}$ is defined except at -1 and 2 where the denominator is 0. Thus $D_g = (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.

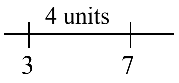
CYU 1.4 Since $-1 < 0$, $f(x) = -4x+1 \Rightarrow f(-1) = -4(-1)+1 = 5$.

For $x = 1$ or 5, $f(x) = x^2 \Rightarrow f(1) = 1^2 = 1$ and $f(5) = 25$.

For $x = 7$, $f(x) = -2x \Rightarrow f(7) = -2(7) = -14$.

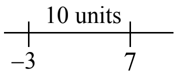
The function f is not defined at 10.

CYU 1.5 (a)



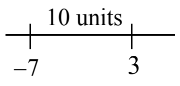
$|3-7| = |4| = 4$

(b)



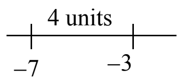
$|-3-7| = |-10| = 10$

(c)



$|-7-3| = |-10| = 10$

(d)



$|-7-(-3)| = |-7+3| = 4$

CYU 1.6 For $f(x) = x-3$ and $g(x) = \dfrac{1}{x-3}$:

$$(f+g)(x) = x-3+\frac{1}{x-3} = \frac{x^2-6x+10}{x-3} \quad \text{Domain: all numbers except 3: } (-\infty, 3) \cup (3, \infty).$$

$$(f-g)(x) = x-3-\frac{1}{x-3} = \frac{x^2-6x+8}{x-3} \quad \text{Domain: } (-\infty, 3) \cup (3, \infty).$$

$$(fg)(x) = (x-3)\left(\frac{1}{x-3}\right) = 1 \quad \text{Domain: } (-\infty, 3) \cup (3, \infty).$$

$$\left(\frac{f}{g}\right)(x) = \frac{x-3}{\dfrac{1}{x-3}} = (x-3)^2 \quad \text{Domain: } (-\infty, 3) \cup (3, \infty).$$

$$(5g)(x) = 5\cdot\frac{1}{x-3} = \frac{5}{x-3} \quad \text{Domain: } (-\infty, 3) \cup (3, \infty).$$

CYU 1.7 (a) For $f(x) = x^2+2x-2$ and $g(x) = 4x+3$:

(i) $(f\circ g)(-2) = f[g(-2)] = f[4(-2)+3] = f(-5)$

$$= (-5^2)+2(-5)-2 = 25-12 = 13$$

(ii) $(f\circ g)(x) = f[g(x)] = f[4x+3] = (4x+3)^2+2(4x+3)-2$

$$= 16x^2+24x+9+8x+6-2 = 16x^2+32x+13$$

(b) From $h(x) = \dfrac{x^2}{x^2+3} = (g\circ f)(x) = g[f(x)]$, we see one possibility:

$$f(x) = x^2 \quad \text{and} \quad g(x) = \frac{x}{x+3}$$

CYU 1.8 $f(x) = \dfrac{x}{x+1}$ is one-to-one:

$$f(a) = f(b) \Rightarrow \frac{a}{a+1} = \frac{b}{b+1} \Rightarrow a(b+1) = b(a+1) \Rightarrow ab+a = ba+b \Rightarrow a = b$$

CYU 1.9 For $f(x) = \dfrac{x}{x+1}$:

Start with: $f[f^{-1}(x)] = x$

For notational convenience, substitute t for $f^{-1}(x)$: $f(t) = x$

Since $f(x) = \dfrac{x}{x+1}$: $\dfrac{t}{t+1} = x$

Solve for t:

$$\begin{aligned} t &= (t+1)x \\ t &= tx + x \\ t - tx &= x \\ t(1-x) &= x \\ t &= \frac{x}{1-x} \end{aligned}$$

Substituting $f^{-1}(x)$ back for t: $f^{-1}(x) = \dfrac{x}{1-x}$

Verifying that $(f \circ f^{-1})(x) = x$:

$$(f \circ f^{-1})x = f[f^{-1}(x)] = f\left(\frac{x}{1-x}\right)$$

$$= \frac{\frac{x}{1-x}}{\frac{x}{1-x} + 1} = \frac{x}{x + (1-x)} = x$$

Verifying that $(f^{-1} \circ f)(x) = x$:

$$(f^{-1} \circ f)x = f^{-1}[f(x)] = f^{-1}\left(\frac{x}{x+1}\right)$$

$$= \frac{\frac{x}{x+1}}{1 - \frac{x}{x+1}} = \frac{x}{(x+1) - x} = x$$

CYU 1.10 $f(x) = \sqrt{x} - 2$ is one-to-one:

$$f(a) = f(b) \Rightarrow (\sqrt{a} - 2 = \sqrt{b} - 2 \Rightarrow \sqrt{a} = \sqrt{b} \Rightarrow a = b)$$

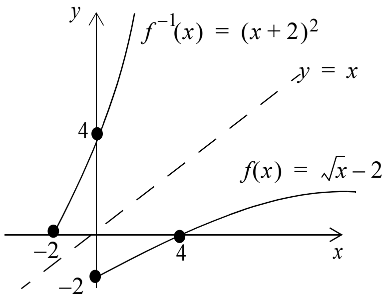
Finding f^{-1}:

$$f[f^{-1}(x)] = x$$

Let $t = f^{-1}(x)$: $f(t) = x \Rightarrow \sqrt{t} - 2 = x \Rightarrow \sqrt{t} = x + 2 \Rightarrow t = (x+2)^2 \Rightarrow f^{-1}(x) = (x+2)^2$

$D_f = [0, \infty)$ and $R_f = [-2, \infty)$. Consequently, $D_{f^{-1}} = [-2, \infty)$ and $R_{f^{-1}} = [0, \infty)$.

The graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$:



CYU 1.11

$$\begin{aligned} \frac{3x}{5} - \frac{2+5x}{3} - 1 &= \frac{-x-1}{15} \\ 15\left(\frac{3x}{5} - \frac{2+5x}{3} - 1\right) &= 15\left(\frac{-x-1}{15}\right) \\ 9x - 5(2+5x) - 15 &= -x - 1 \\ 9x - 10 - 25x - 15 &= -x - 1 \\ -15x &= 24 \\ x &= -\frac{24}{15} = -\frac{8}{5} \end{aligned}$$

$$\begin{aligned} \frac{3x}{5} - \frac{2+5x}{3} - 1 &< \frac{-x-1}{15} \\ 15\left(\frac{3x}{5} - \frac{2+5x}{3} - 1\right) &< 15\left(\frac{-x-1}{15}\right) \\ 9x - 5(2+5x) - 15 &< -x - 1 \\ 9x - 10 - 25x - 15 &< -x - 1 \\ -15x &< 24 \\ x &> -\frac{24}{15} = -\frac{8}{5} \end{aligned}$$

CYU 1.12 (a)

$$2x^2 + 7x - 4 = 0$$
$$(2x - 1)(x + 4) = 0$$
$$x = \frac{1}{2}, x = -4$$

(b)

$$3x^2 - 4x - 3 = 0$$
$$x = \frac{4 \pm \sqrt{4^2 - 4(3)(-3)}}{2 \cdot 3} = \frac{4 \pm 2\sqrt{13}}{2 \cdot 3} = \frac{2 \pm \sqrt{13}}{3}$$

quadratic formula

(c)

$$x^3 + x^2 - 16x = 16$$
$$x^3 + x^2 - 16x - 16 = 0$$
$$x^2(x + 1) - 16(x + 1) = 0 \Rightarrow (x^2 - 16)(x + 1) = 0 \Rightarrow (x + 4)(x - 4)(x + 1) = 0$$

Solution: $x = \pm 4, x = -1$

CYU 1.13 Since 1 is a zero of the polynomial $p(x) = x^3 + 2x^2 - 7x + 4$, $(x - 1)$ is a factor.

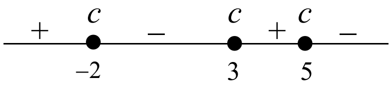
Dividing: $x - 1 \overline{\big) x^3 + 2x^2 - 7x + 4}$ with quotient $x^2 + 3x - 4$ reveals the fact that:

$$x^3 + 2x^2 - 7x + 4 = (x - 1)(x^2 + 3x - 4) = (x - 1)(x - 1)(x + 4)$$

Returning to our equation we have:

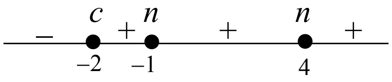
$$x^3 + 2x^2 - 7x + 4 = 0$$
$$(x - 1)^2(x + 4) = 0 \Rightarrow x = 1, x = -4$$

CYU 1.14 (a) SIGN $(x - 3)(x + 2)(-x + 5)$:



$$(x - 3)(x + 2)(-x + 5) < 0:$$
$$(-2, 3) \cup (5, \infty)$$

(b) SIGN $(x + 1)^2(x + 2)^3(x - 4)^2$:



$$(x + 1)^2(x + 2)^3(x - 4)^2 \geq 0:$$
$$[-2, \infty)$$

CYU 1.15 $(x^2 + x - 2)(x^2 - x + 5)(-x^2 + x + 5) = (x + 2)(x - 1)(x^2 - x + 5)(-x^2 + x + 5)$

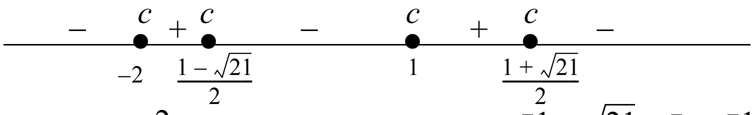
Since the discriminant of the quadratic polynomial $x^2 - x + 5$ is negative: $[b^2 - 4ac = (-1)^2 - 4(1)(5)]$, it has no zeros. As such, its signs cannot change, and since it is positive at $x = 0$, it must be positive everywhere.

The polynomial $-x^2 + x + 5$ does have zeros:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{21}}{-2} = \frac{1 \pm \sqrt{21}}{2}$$

Bringing us to:

SIGN $(x + 2)(x - 1)(x^2 - x + 5)(-x^2 + x + 5)$:



$$(x^2 + x - 2)(x^2 - x + 5)(-x^2 + x + 5) \leq 0: \ (-\infty, -2] \cup \left[\frac{1 - \sqrt{21}}{2}, 1\right] \cup \left[\frac{1 + \sqrt{21}}{2}, \infty\right)$$

CYU 1.16 (a) $\frac{x-2}{x^2-4}-\frac{5}{4}=\frac{1}{x-3}\Rightarrow\frac{x-2}{(x+2)(x-2)}-\frac{5}{4}=\frac{1}{x-3}\Rightarrow\frac{1}{x+2}-\frac{5}{4}=\frac{1}{x-3}$

clear denominators:
$$\begin{aligned} 4(x-3)-5(x+2)(x-3) &= 4(x+2)\\ 4x-12-5(x^2-x-6) &= 4x+8\\ -5x^2+5x+10 &= 0\\ x^2-x-2 &= 0\\ (x-2)(x+1) &= 0 \end{aligned}$$

not a solution: $\Rightarrow \cancel{x=2}$ or $x=-1$

(b) $3\left(\frac{x}{x^2+1}\right)+\left(\frac{x^2+1}{x}\right)=4$

Let $y=\frac{x}{x^2+1}$: $3y+\frac{1}{y}=4\Rightarrow 3y^2+1=4y\Rightarrow 3y^2-4y+1=0\Rightarrow(3y-1)(y-1)=0$

$$\begin{aligned} y &= \frac{1}{3}\\ \frac{x}{x^2+1} &= \frac{1}{3}\\ x^2+1 &= 3x\\ x^2-3x+1 &= 0\\ x=\frac{3\pm\sqrt{9-4}}{2} &= \frac{3\pm\sqrt{5}}{2} \end{aligned}$$

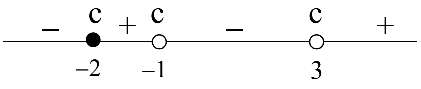
both are solutions

$$\begin{aligned} y &= 1\\ \frac{x}{x^2+1} &= 1\\ x^2+1 &= x\\ x^2-x+1 &= 0 \end{aligned}$$

no solution: negative discriminant

CYU 1.17 (a) $\frac{x+2}{x^2-2x-3}\ge 0\Rightarrow\frac{x+2}{(x-3)(x+1)}\ge 0$

SIGN $\frac{x+2}{(x-3)(x+1)}$:



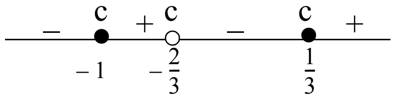
$\frac{x+2}{(x-3)(x+1)}\ge 0$: $[-2,-1)\cup(3,\infty)$

(b) $x<\frac{1}{3x+2}\Rightarrow x-\frac{1}{3x+2}<0$

$\Rightarrow\frac{3x^2+2x-1}{3x+2}<0$

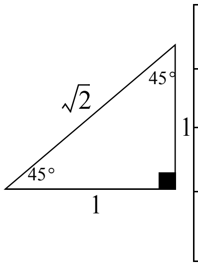
$\Rightarrow\frac{(3x-1)(x+1)}{3x+2}<0$

SIGN $\frac{(3x-1)(x+1)}{3x+2}$:

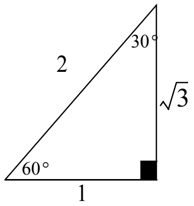


$x<\frac{1}{3x+2}$: $(-\infty,-1)\cup(-\frac{2}{3},\frac{1}{3})$

CYU 1.18



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$



CYU 1.19 (a) $\theta = 120° = 120° \cdot \frac{\pi}{180°} = \frac{2\pi}{3}$ (b) $\theta = \frac{\pi}{6} = \frac{\pi}{6} \cdot \frac{180°}{\pi} = 30°$

CYU 1.20

θ		$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
degrees	radians						
0°	0	0	1	0	undef	1	undef
90°	$\frac{\pi}{2}$	1	0	undef	1	undef	0
180°	π	0	–1	0	undef	–1	undef
270°	$\frac{3\pi}{2}$	–1	0	undef	–1	undef	0

CYU 1.21 Since $-\frac{29\pi}{7} + 4\pi = -\frac{\pi}{7}$, $\theta = -\frac{29\pi}{7}$ is coterminal with $-\frac{\pi}{7}$ which lies in the fourth quadrant. Therefore:

$$\sin\theta < 0,\ \cos\theta > 0,\ \tan\theta < 0,\ \csc\theta = \frac{1}{\sin\theta} < 0,\ \sec\theta = \frac{1}{\cos\theta} > 0,\ \text{and } \cot\theta = \frac{1}{\tan\theta} < 0$$

CYU 1.22 (a) Since $-840° + 720° = -120°$, $-840°$ is coterminal with $-120°$. Since the reference angle of $-120°$ is $60°$, and since $-120°$ lies in the third quadrant: $\sin(-840°) = -\sin 60° = -\frac{\sqrt{3}}{2}$.

(b) Since $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$, $\frac{11\pi}{4}$ is coterminal with $\frac{3\pi}{4}$. Since the reference angle of $\frac{3\pi}{4}$ is $\frac{\pi}{4}$ and since $\frac{3\pi}{4}$ lies in the second quadrant: $\cot\frac{11\pi}{4} = \cot\frac{3\pi}{4} = -\cot\frac{\pi}{4} = -1$

(c) Since $-\frac{25\pi}{6} + 4\pi = -\frac{\pi}{6}$, $-\frac{25\pi}{6}$ is coterminal with $-\frac{\pi}{6}$. Since the reference angle of $-\frac{\pi}{6}$ is $\frac{\pi}{6}$ and since $-\frac{25\pi}{6}$ lies in the fourth quadrant:

$$\sec\left(-\frac{25\pi}{6}\right) = \sec\left(-\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}}.$$

CHAPTER 2: LIMITS AND CONTINUITY

CYU 2.1 (a) As x approaches -1, $4x^2+x$ approaches 3 ($4x^2$ tends to 4, and x to -1). Thus: $\lim\limits_{x\to -1}(4x^2+x) = 3$

(b) As x approaches 2, $x+3$ approaches 5 and $x+2$ approaches 4. Thus: $\lim\limits_{x\to 2}\dfrac{x+3}{x+2} = \dfrac{5}{4}$

(c) As x approaches 3, $x(3x^2+1)$ approaches $3(3\cdot 3^2+1) = 84$. Thus: $\lim\limits_{x\to 3}[x(3x^2+1)] = 84$

CYU 2.2 (a)
$$\lim_{x\to 1}\frac{x^2+3x-4}{x^2-1} = \lim_{x\to 1}\frac{(x+4)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \lim_{x\to 1}\frac{(x+4)}{(x+1)} = \frac{5}{2}$$

(b)
$$\lim_{x\to 2}\frac{x^3-2x^2+2x-4}{x^2-x-2} = \lim_{x\to 2}\frac{x^2(x-2)+2(x-2)}{(x-2)(x+1)}$$
$$= \lim_{x\to 2}\frac{\cancel{(x-2)}(x^2+2)}{\cancel{(x-2)}(x+1)} = \lim_{x\to 2}\frac{(x^2+2)}{(x+1)} = \frac{6}{3} = 2$$

CYU 2.3 (a)
$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right) = \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x(x+1)}\right)$$
$$= \lim_{x\to 0}\left(\frac{(x+1)-1}{x(x+1)}\right) = \lim_{x\to 0}\left(\frac{1}{x+1}\right) = \frac{1}{0+1} = 1$$

(b)
$$\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x^2-4} = \lim_{x\to 2}\frac{\sqrt{x+2}-2}{x^2-4}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$$
$$= \lim_{x\to 2}\frac{(\sqrt{x+2})^2-2^2}{(x^2-4)(\sqrt{x+2}+2)}$$
$$= \lim_{x\to 2}\frac{x+2-4}{(x+2)(x-2)(\sqrt{x+2}+2)}$$
$$= \lim_{x\to 2}\frac{(x-2)}{(x+2)(x-2)(\sqrt{x+2}+2)}$$
$$= \lim_{x\to 2}\frac{1}{(x+2)(\sqrt{x+2}+2)} = \frac{1}{4(\sqrt{4}+2)} = \frac{1}{16}$$

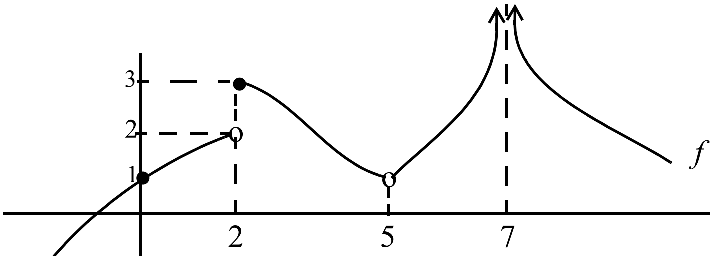
CYU 2.4 (a) $\lim\limits_{x \to 3} \frac{x^2-9}{x-3} = \lim\limits_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = \lim\limits_{x \to 3}(x+3) = 6$

(b) $\lim\limits_{x \to 3} \frac{x-3}{x^2-6x+9} = \lim\limits_{x \to 3} \frac{\cancel{x-3}}{\cancel{(x-3)}(x-3)} = \lim\limits_{x \to 3} \frac{1}{(x-3)}$ DNE (denominator goes to 0 while numerator does not)

(c) Does not exist: $\lim\limits_{x \to 3^-} f(x) = 5$ while $\lim\limits_{x \to 3^+} f(x) = 8$

(d) Since $\lim\limits_{x \to 3^-} g(x) = 2$ and $\lim\limits_{x \to 3^+} g(x) = 2$, $\lim\limits_{x \to 3} g(x) = 2$

CYU 2.5



(a) As you approach 0 from either side, the function values approach 1. Thus: $\lim\limits_{x \to 0} f(x) = 1$.

(b) As you approach 2 from the left, the function values approach 2, but as you approach from the right, the function values approach 3. Thus: $\lim\limits_{x \to 2} f(x)$ does not exist.

(c) As you approach 5 from either side, the function values approach 1 (never mind that the function is not defined at 5; for the limit does not care what happens there—it is only concerned about what happens as you **approach** 5. Thus: $\lim\limits_{x \to 5} f(x) = 1$.

(d) As you approach 7 from either side, the function values get larger and larger, and cannot tend to any number. Thus: $\lim\limits_{x \to 7} f(x)$ does not exist, but we can write $\lim\limits_{x \to 5} f(x) = \infty$.

CYU 2.6 (a) For $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 1.001 & \text{if } x \ge 2 \end{cases}$: $\lim\limits_{x \to 2^-} f(x) = 2 + 1 = 3$ and

$\lim\limits_{x \to 2^+} f(x) = 4 - 1.001 = 2.999$. The limit does not exist at 2 as f has a jump discontinuity at that point.

(b) For $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ 25 & \text{if } x = 2 \\ x^2 - 1 & \text{if } x > 2 \end{cases}$: $\lim\limits_{x \to 2^-} f(x) = 2 + 1 = 3$ and

$\lim\limits_{x \to 2^-} f(x) = 2^2 + 1 = 3$. Hence: $\lim\limits_{x \to 2} f(x) = 3$. Since $f(2) = 25$ (and not 3), f has a removable discontinuity at 2.

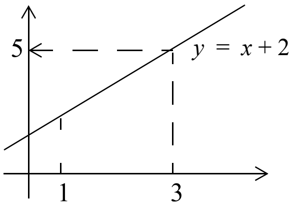
CYU 2.7 To prove that $\lim\limits_{x \to 4} 5x + 1 = 21$, for given $\varepsilon > 0$ we are to find $\delta > 0$ such that:

$$\left.\begin{array}{l} 0 < |x-4| < \delta \Rightarrow |f(x) - 21| < \varepsilon \\ 0 < |x-4| < \delta \Rightarrow |(5x+1) - 21| < \varepsilon \\ 0 < |x-4| < \delta \Rightarrow |5x - 20| < \varepsilon \\ 0 < |x-4| < \delta \Rightarrow 5|x-4| < \varepsilon \\ 0 < |x-4| < \delta \Rightarrow |x-4| < \dfrac{\varepsilon}{5} \end{array}\right] \text{same} \qquad \text{choose } \delta = \frac{\varepsilon}{5}$$

CYU 2.8 To prove that $\lim\limits_{x \to 2} (x^2 + 1) = 5$, for given $\varepsilon > 0$ we are to find $\delta > 0$ such that:

$$0 < |x-2| < \delta \Rightarrow |(x^2+1) - 5| < \varepsilon$$
$$0 < |x-2| < \delta \Rightarrow |(x+2)(x-2)| < \varepsilon$$
$$0 < |x-2| < \delta \Rightarrow |x+2||x-2| < \varepsilon \ (*)$$

Since we are interested in what happens near $x = 2$, we decide to focus on the interval: $(1, 3) = \{x \mid |x-2| < 1\}$. **Within that interval** $|x+2| < 5$. Consequently, within that interval: $|x+2||x-2| < 5|x-2|$.



We observe that (*) is satisfied for $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$:

$$0 < \underbrace{|\mathbf{x}-2| < \delta} \Rightarrow \overbrace{|x+2|}|\mathbf{x}-2| < 5\delta < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$$

CYU 2.9 Let $\lim\limits_{x \to c} [f(x)] = L$. To prove that $\lim\limits_{x \to c} [af(x)] = aL$, for given $\varepsilon > 0$ we are to find $\delta > 0$ such that: $0 < |x-c| < \delta \Rightarrow |af(x) - aL| < \varepsilon$. In the event that $a = 0$, then any $\delta > 0$ will surely work, since $|af(x) - aL| = 0$. If $a \neq 0$, choose $\delta > 0$ such that $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \frac{\varepsilon}{|a|}$. Consequently, for $0 < |x-c| < \delta$:

$$|af(x) - aL| = |a(f(x) - L)| = |a||f(x) - L| < |a|\frac{\varepsilon}{|a|} = \varepsilon$$

CYU 2.10 $\lim\limits_{x \to c} (f+g)(x) = \lim\limits_{x \to c} [f(x) + g(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$

$$= f(c) + g(c) = (f+g)(c)$$

CYU 2.11 For given $\varepsilon > 0$ we are to find $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f[g(x)] - f(b)| < \varepsilon$.

Since f is continuous at b, we can find $\delta_1 > 0$ such that $0 < |y-b| < \delta_1 \Rightarrow |f(y) - f(b)| < \varepsilon$.

Since $\lim\limits_{x \to a} g(x) = b$ there exists $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |g(x) - b| < \delta_1$.

Consequently: $0 < |x-a| < \delta \Rightarrow |g(x) - b| < \delta_1 \Rightarrow |f[g(x)] - f(b)| < \varepsilon$.

CYU 2.12

(a) $\lim\limits_{x \to c^-} f(x) = \infty$: For any given $M > 0$ there exists $\delta > 0$ such that $c - \delta < x < c \Rightarrow f(x) > M$.

(b) $\lim\limits_{x \to c^+} f(x) = \infty$: For any given $M > 0$ there exists $\delta > 0$ such that $c < x < c + \delta \Rightarrow f(x) > M$.

(c) $\lim\limits_{x \to -\infty} f(x) = c$: For any $\varepsilon > 0$ there exists $N < 0$ such that $x < N \Rightarrow |f(x) - c| < \varepsilon$.

(d) $\lim\limits_{x \to \infty} f(x) = -\infty$: For any $M < 0$ there exists $N > 0$ such that $x > N \Rightarrow f(x) < M$.

(e) $\lim\limits_{x \to -\infty} f(x) = \infty$: For any $M > 0$ there exists $N < 0$ such that $x < N \Rightarrow f(x) > M$.

(f) $\lim\limits_{x \to -\infty} f(x) = -\infty$: For any $M < 0$ there exists $N < 0$ such that $x < N \Rightarrow f(x) < M$.

CHAPTER 3: THE DERIVATIVE

CYU 3.1 In Example 3.2 we showed that if $f(x) = \frac{x}{2x+1}$, then $f'(x) = \frac{1}{(2x+1)^2}$.

In particular, the slope of the tangent line to the graph of f at $x = -1$ is

$$m = f'(-1) = \frac{1}{[2(-1)+1]^2} = 1.$$

To find the y-intercept of our tangent line $y = 1 \cdot x + b$ we need a point on that line. Since the tangent line touches the graph at that point, the point

$(-1, f(-1)) = \left(-1, \frac{-1}{2(-1)+1}\right) = (-1, 1)$ on the graph of f also lies on the tangent line.

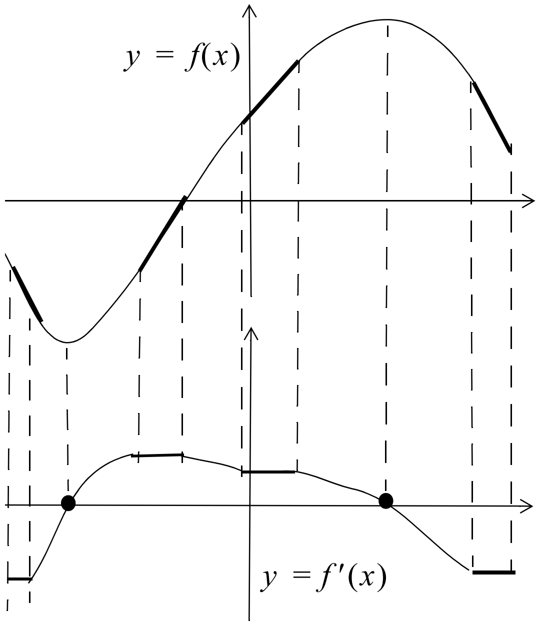
Substituting -1 for x and 1 for y in the equation $y = x + b$ we can determine b:
$1 = -1 + b \Rightarrow b = 2$. Tangent line: $y = x + 2$.

CYU 3.2 For $y = f(x) = 3x^2 - x + 1$:

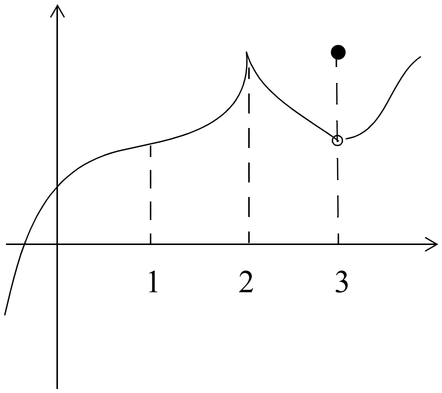
$$\begin{aligned}\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - (x+\Delta x) + 1 - (3x^2 - x + 1)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 - x - \Delta x + 1 - 3x^2 + x - 1}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{\Delta x(3\Delta x + 6x - 1)}{\Delta x} = \lim_{\Delta x \to 0} (3\Delta x + 6x - 1) = 6x - 1\end{aligned}$$

and: $\left.\frac{dy}{dx}\right|_{x=2} = 6 \cdot 2 - 1 = 11$

CYU 3.3



CYU 3.4



CYU 3.5 (a) Since the tangent line at every point on the line $y = x$ equals the line itself, and since the line $y = x$ has slope 1: $x' = 1$.

(b) For $f(x) = x$: $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{x + h - x}{h} = \lim\limits_{h \to 0} \frac{h}{h} = 1$.

CYU 3.6 (a) $(3x^4 - 2x + 5 - 5x^{-4})' = 3(4x^3) - 2 + 0 - 5(-4x^{-5}) = 12x^3 - 2 + \frac{20}{x^5}$

$$\text{(b) } \left(\frac{4x^2 - 525}{x + 3}\right)' = \frac{(x+3)(4x^2 - 525)' - (4x^2 - 525)(x+3)'}{(x+3)^2}$$

$$= \frac{(x+3)(8x) - (4x^2 - 525)(1)}{(x+3)^2} = \frac{4x^2 + 24x + 525}{(x+3)^2}$$

CYU 3.7 Since horizontal tangent lines have a slope of **0**, the points on the graph of the function $f(x) = 2x^4 - 4x^2 + 1$ with horizontal tangent lines occur where$f'(x) = \mathbf{0}$:

$$(2x^4 - 4x^2 + 1)' = \mathbf{0} \Rightarrow 8x^3 - 8x = 0 \Rightarrow 8x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$$

Conclusion: Horizontal tangent lines occur at

$$[0, f(0)], [1, f(1)], [-1, f(-1)]: (0, 1), (1, -1), (-1, -1)$$

CYU 3.8 For $f(x) = x^{1/3}$: $f'(x) = (x^{1/3})' = \frac{1}{3x^{2/3}}$. Thus:

$$(8.1)^{2/3} = f(8 + 0.1) \approx f(8) + f'(8)(0.1) = 8^{1/3} + \frac{0.1}{3 \cdot 8^{2/3}} = 2 + \frac{0.1}{12} \approx 2.008$$

CYU 3.9 We find an approximation for the area error ΔA resulting from a change of a radius measurement from 50 to $50 + \Delta r$ cm, where $\Delta r = 0.5$ cm.

For $A(r) = \pi r^2$: $A'(r) = 2\pi r$. Thus: $\Delta A \approx A'(50)\Delta r = 2\pi(50)(0.5) = 50\pi \text{ cm}^2$.

$$\text{Relative area error: } \frac{\Delta A}{A} \approx \frac{50\pi}{\pi(50)^2} = \frac{1}{50} = 0.02.$$

CYU 3.10 If $n = 0$ then $(x^0)' = 1' = 0$ and nx^{n-1} also equals 0: $0 \cdot x^{0-1} = \frac{0}{x} = 0$.

For $n < 0$, $x^n = \frac{1}{x^{-n}}$ where $-n$ is a positive integer. Applying the quotient rule and the result of Example 3.10 we have:

$$(\boldsymbol{x^n})' = \left(\frac{1}{x^{-n}}\right)' = \frac{x^{-n}\cdot(1)' - 1\cdot(x^{-n})'}{(x^{-n})^2} = \frac{nx^{-n-1}}{x^{-2n}} = nx^{-n-1+2n} = nx^{n-1}$$

CYU 3.11 (a) $(x^5)''' = (5x^4)'' = (20x^3)' = 60x^2$

(b) Looking at $(x^{-1})' = -x^{-2}$, $(x^{-1})^{(2)} = 2x^{-3}$, $(x^{-1})^{(3)} = -2\cdot 3x^{-4}$, $(x^{-1})^{(4)} = 2\cdot 3\cdot 4x^{-5}$, and $(x^{-1})^{(5)} = -2\cdot 3\cdot 4\cdot 5x^{-6}$, suggests that: $(x^{-1})^{(n)} = (-1)^n n! x^{-n-1}$

(c) Let $P(n)$ be the proposition that $(x^{-1})^{(n)} = (-1)^n n! x^{-n-1}$.

I. Since $(x^{-1})^{(1)} = -x^{-2} = (-1)^1 1! x^{-1-1}$, the proposition holds at $n = 1$.

II. Assume $P(k)$ is true; that is: $(x^{-1})^{(k)} = (-1)^k k! x^{-k-1}$.

III. We show that $P(k+1)$ is true, namely that $(x^{-1})^{(k+1)} = (-1)^{k+1}(k+1)! x^{-k-2}$:

$$\begin{aligned}(x^{-1})^{(k+1)} = [(x^{-1})^{(k)}]' \underset{\substack{\uparrow \\ \text{by } \mathbf{II}}}{=} [(-1)^k k! x^{-k-1}]' &= (-1)^k k!(-k-1)x^{-k-1-1} \\ &= (-1)^k k!(-1)(k+1)x^{-k-2} \\ &= (-1)^{k+1}(k+1)! x^{-k-2}\end{aligned}$$

CYU 3.12 Since $\lim\limits_{x\to 0}\left(1 - \frac{x^2}{4}\right) = \lim\limits_{x\to 0}\left(1 + \frac{x^2}{4}\right) = 1$ and $1 - \frac{x^2}{4} \le h(x) \le 1 + \frac{x^2}{2}$: $\lim\limits_{x\to 0} h(x) = 1$.

CYU 3.13 $\lim\limits_{x\to 0}\tan x = \lim\limits_{x\to 0}\frac{\sin x}{\cos x} = \frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}\cos x} = \frac{0}{1} = 0$.

CYU 3.14

$$\lim_{x\to 0}\frac{\cos x-1}{x} = \lim_{x\to 0}\frac{\cos x-1}{x}\cdot\frac{\cos x+1}{\cos x+1}$$

$$= \lim_{x\to 0}\frac{\cos^2 x-1}{x\cos(x+1)} = \lim_{x\to 0}\frac{-\sin^2 x}{x\cos(x+1)} = -\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{\cos(x+1)}$$

$$= -1\cdot\frac{\sin 0}{\cos 1} = -1\cdot\frac{0}{\cos 1} = 0$$

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CYU 3.15 $$(\cos x)' = \lim_{h\to 0}\frac{\mathbf{\cos(x+h)}-\cos x}{h} = \lim_{h\to 0}\frac{\mathbf{\cos x\cos h-\sin x\sin h}-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$= \cos x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)-\sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$

Theorem 3.5, pgae 90
and CYU 3.14, page 91

$$= \cos x(0)-\sin x(1) = -\sin x$$

CYU 3.16

(a) $$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x(\cos x)'-\cos x(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x-\cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

(b) $$(\csc x)' = \left(\frac{1}{\sin x}\right)' = \frac{\sin x\cdot(1)'-1\cdot(\sin x)'}{(\sin x)^2} = \frac{\sin x\cdot 0-\cos x}{(\sin x)^2}$$

$$= -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x\cot x$$

(c) $$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cos x\cdot(1)'-1\cdot(\cos x)'}{(\cos x)^2} = \frac{\cos x\cdot 0-1\cdot(-\sin x)}{(\cos x)^2}$$

$$= \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x\tan x$$

CYU 3.17 (a) $$(x\sec x+\tan x)' = (x\sec x)'+\sec^2 x = x\sec x\tan x+\sec x+\sec^2 x$$

$$= \sec x(x\tan x+1+\sec x)$$

(b) $$\left(\frac{\sin x\cos x}{x^2}\right)' = \frac{x^2(\mathbf{\sin x\cos x})'-\sin x\cos x(x^2)'}{x^4}$$

Theorem 1.5(iv), page 37

$$= \frac{x^2(\mathbf{\cos^2 x-\sin^2 x})-2x\sin x\cos x}{x^4} = \frac{x^2(\mathbf{\cos 2x})-x\sin 2x}{x^4}$$

$$= \frac{x\cos 2x-\sin 2x}{x^3}$$

CYU 3.18 (a)

$$\left[x\tan\left(\frac{x}{x+1}\right)\right]' = x'\tan\left(\frac{x}{x+1}\right)+x\left[\tan\left(\frac{x}{x+1}\right)\right]'$$

$$= \tan\left(\frac{x}{x+1}\right)+x\sec^2\left(\frac{x}{x+1}\right)\left(\frac{x}{x+1}\right)'$$

$$= \tan\left(\frac{x}{x+1}\right)+x\sec^2\left(\frac{x}{x+1}\right)\left(\frac{x+1-x}{(x+1)^2}\right)$$

$$= \tan\left(\frac{x}{x+1}\right)+\frac{x}{(x+1)^2}\cdot\sec^2\left(\frac{x}{x+1}\right)$$

(b)

$$\left(\frac{\sin x^2}{\cos x}\right)' = \frac{\cos x(\sin x^2)'-\sin x^2(\cos x)'}{\cos^2 x} = \frac{\cos x\cos x^2\cdot 2x-\sin x^2(-\sin x)}{\cos^2 x}$$

$$= \frac{2x\cos x\cos x^2+\sin x\sin x^2}{\cos^2 x}$$

CYU 3.19 (a)

$$[(x^4+\sin x)^3]' = 3(x^4+\sin x)^2(x^4+\sin x)' = 3(x^4+\sin x)^2(4x^3+\cos x)$$

(b)

$$(\sin^3 x^3)' = [(\sin x^3)^3]' = 3(\sin x^3)^2(\sin x^3)'$$

$$= 3(\sin x^3)^2\cos x^3\cdot 3x^2 = 9x^2\sin^2 x^3\cos x^3$$

(c)

$$\left[(\sin x^2)^{\frac{1}{2}}\right]' = \frac{1}{2}(\sin x^2)^{\frac{1}{2}-1}(\mathbf{\sin x^2})' = \frac{1}{2}(\sin x^2)^{-\frac{1}{2}}(\mathbf{\cos x^2})(\mathbf{x^2})'$$

$$= \frac{1}{2\sqrt{\sin x^2}}(\cos x^2)(2x) = \frac{x\cos x^2}{\sqrt{\sin x^2}}$$

CYU 3.20 With the chain rule:

$$(g\circ f)'(x) = \mathbf{g'[f(x)]}\cdot f'(x) = \mathbf{g'(2x^2-5)}\cdot 4x$$

$g'(x)=2x$:

$$= 2(2x^2-5)\cdot 4x = 16x^3-40x$$

Without the chain rule:

$$(g\circ f)(x) = g[f(x)] = g(2x^2-5) = (2x^2-5)^2+2 = 4x^4-20x^2+27$$

Differentiating: $(g\circ f)'(x) = [4x^4-20x^2+27]' = 16x^3-40x$

CYU 3.21 $x^2+xy+2y^2 = 8 \xrightarrow{\text{differentiate}} 2x+xy'+y+4yy' = 0$

$$xy'+4yy' = -2x-y \Rightarrow y' = -\frac{2x+y}{x+4y}$$

Slope of tangent line at $(2, 1)$: $m = -\frac{2\cdot 2+1}{2+4\cdot 1} = -\frac{5}{6}$. Equation: $y = -\frac{5}{6}x+b$.

$$1 = -\frac{5}{6}\cdot 2+b \Rightarrow b = \frac{8}{3}$$

So: $y = -\frac{5}{6}x+\frac{8}{3}$

Slope of tangent line at $(2, -2)$: $m = -\frac{2\cdot 2+(-2)}{2+4(-2)} = \frac{1}{3}$. Equation: $y = \frac{1}{3}x+b$

$$-2 = \frac{2}{3}+b \Rightarrow b = -\frac{8}{3}$$

So: $y = \frac{1}{3}x-\frac{8}{3}$

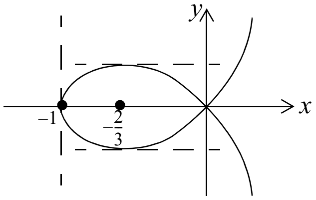
CYU 3.22 $(y^3+2x)' = (y)' \Rightarrow 3y^2y'+2 = y' \Rightarrow 3y^2y'-y' = -2 \Rightarrow y' = -\frac{2}{3y^2-1}$

Then: $y'' = \left(-\frac{2}{3y^2-1}\right)' = [-2(3y^2-1)^{-1}]' = 2(3y^2-1)^{-2}(3y^2-1)'$

$$= \frac{2(6y\mathbf{y'})}{(3y^2-1)^2}$$

Since $y' = -\frac{2}{3y^2-1}$: $= \frac{12y\left(-\frac{\mathbf{2}}{\mathbf{3y^2-1}}\right)}{(3y^2-1)^2} = -\frac{24y}{(3y^2-1)^3}$

CYU 3.23 $2y^2-x^3-x^2 = 0 \xrightarrow{\text{differentiate}} 4\text{yy}'-3x^2-2x = 0 \Rightarrow y' = \frac{3x^2+2x}{4y}$.



From $y' = \frac{3x^2+2x}{4y} = \frac{x(3x+2)}{y}$. The two points on the curve with x-coordinate $-\frac{2}{3}$ do have horizontal tangent lines. The y-coordinate of those points can be found from the equation $2y^2-x^3-x^2 = 0$: $y = \pm\sqrt{\frac{x^3+x^2}{2}}$; specifically: $y = \pm\sqrt{\frac{\left(-\frac{2}{3}\right)^3+\left(-\frac{2}{3}\right)^2}{2}} = \pm\sqrt{\frac{2}{27}}$, so the two points are $\left(-\frac{2}{3}, -\sqrt{\frac{2}{27}}\right)$ and $\left(-\frac{2}{3}, \sqrt{\frac{2}{27}}\right)$.

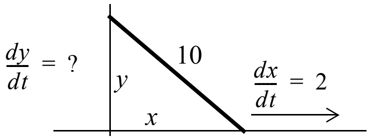
From the above figure you can see that the curve cannot be approximated by a function near the point $(0, 0)$, so that the implicit differentiation method is not applicable when $x = 0$. Moreover, since $y' = \frac{3x^2+2x}{4y}$ is not defined at $y = 0$, a tangent line at any point on the curve with that y-coordinate (if it exists) must be vertical. Substituting 0 for y in the equation $2y^2 - x^3 - x^2 = 0$ we are able to find the corresponding x values:

$$0 - x^3 - x^2 = 0 \Rightarrow x^3 + x^2 = 0 \Rightarrow x^2(x+1) = 0 \Rightarrow x = 0 \text{ (irrelevant) and } x = -1.$$

Conclusion: The tangent line is vertical at $(-1, 0)$.

CYU 3.24 We are given that $\frac{dr}{dt} = 3\frac{\text{cm}}{\text{min}}$ and want to find $\left.\frac{dC}{dt}\right|_{r=12}$, where C denotes the circumference of the circle. From $C = 2\pi r$ we have: $\frac{dC}{dt} = 2\pi\frac{dr}{dt}$. It follows that, independent of r: $\frac{dC}{dt} = 2\pi \cdot 3 = 6\pi\frac{\text{cm}}{\text{min}}$.

CYU 3.25 We find a relation between x and y: $x^2 + y^2 = 10^2$. Differentiating both sides with respect to t we have:

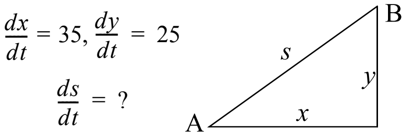


$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

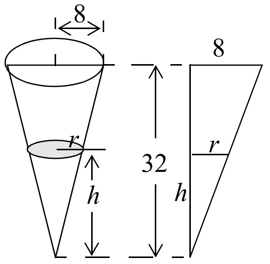
We are given that $\frac{dx}{dt} = 2$, and are asked to find $\frac{dy}{dt}$ when $x = 6$. Turning to the Pythagorean Theorem we find the corresponding value of y: $y^2 = 10^2 - 6^2 = 64 \Rightarrow y = 8$.

Conclusion: At $x = 6$, $\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt} = -\frac{6}{8} \cdot 2 = -\frac{3}{2}$; which is to say that the top of the ladder is falling at a rate of $\frac{3}{2}$ feet per second.

CYU 3.26 At 4 PM, ship A has traveled 140 km and will be 20 km west of ship B. That being the case, x will be increasing with respect to time; specifically: $\frac{dx}{dt} = 35$. A 4 PM: $x = 20$, $y = 25 \cdot 4 = 100$, and $s = \sqrt{20^2 + 100^2} = 20\sqrt{26}$. From Step 3 of Example 3.21:



$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s} = \frac{20 \cdot 35 + 100 \cdot 25}{20\sqrt{26}} = \frac{160}{\sqrt{26}} \approx 31.4\frac{\text{km}}{\text{hr}}$$



CYU 3.27 We are to find the constant *c*, given that $\frac{dV}{dt} = -c$ and that $\frac{dh}{dt} = -2\frac{\text{in}}{\text{min}}$ when $h = 4\,\text{in}$. We need to find a relation between *V* and *h*. Turning to the equation $V = \frac{1}{3}\pi r^2 h$, we set our sight on expressing *r* in terms of *h*. From the similar triangles in the adjacent figure we have:

$\frac{h}{32} = \frac{r}{8} \Rightarrow r = \frac{h}{4}$. Thus: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$. Differentiating:

$$\frac{dV}{dt} = \frac{\pi}{48} \cdot 3h^2\frac{dh}{dt}$$

$$-c = \frac{\pi h^2}{16}\frac{dh}{dt}$$

Substituting -2 for $\frac{dh}{dt}$ and 4 for *h* we have: $-c = \frac{\pi \cdot 4^2}{16}(-2) = -2\pi$.

Conclusion: Water is leaking out at a rate of 2π cubic inches per minute.

CHAPTER 4: THE MEAN-VALUE THEOREM AND APPLICATIONS

CYU 4.1 (a) For $f(x) = x^4 - 2x^2$: $f'(x) = 0 \Rightarrow 4x^3 - 4x = 0 \Rightarrow 4x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$. Conclusion: The graph of *f* has horizontal tangent lines at $x = 0$ and at $x = \pm 1$. Note that all three numbers fall within the interval $(-\sqrt{2}, \sqrt{2})$.

(b) The Mean-Value Theorem assures us that in the interval $[-2, 3]$ there will be at least one point on the graph of the function $f(x) = x^3 - x^2$ where the tangent line has slope:

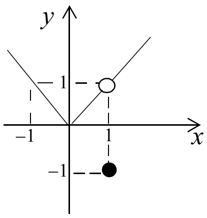
$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(-2)}{3 - (-2)} = \frac{3^3 - 3^2 - [(-2)^3 - (-2)^2]}{5} = 6$$

Turning to the equation $(x^3 - x^2)' = 6$ we have:

$$3x^2 - 2x = 6 \Rightarrow 3x^2 - 2x - 6 = 0 \Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2 \cdot 3} = \frac{1 \pm \sqrt{19}}{3}$$

Note that both of the numbers $\frac{1 \pm \sqrt{19}}{3}$ are contained in the interval $[-2, 3]$.

CYU 4.2



(a) Since $\lim_{x \to 1} f(x) \neq f(1)$, *f* is not continuous on [−1, 1], and is not differentiable at 0 in $(-1, 1)$ (Example 3.4, page 71).

(b) $\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-1 - 1}{2} = -1$ and $f'(c) = -1$ for any $c < 0$. In particular:

$$f'(-\tfrac{1}{2}) = \frac{f(1) - f(-1)}{1 - (-1)}$$

CYU 4.3 (Proof by contradiction.) Assume that the function $h(x) = f(x) - g(x)$ is not constant on the open interval I. It follows that there are two points in I, say a and b with $a < b$, such that $h(a) \neq h(b)$. The Mean-Value Theorem assures us of the existence of some $a < c < b$ such that $h'(c) = \frac{h(b) - h(a)}{b - a}$ which is not zero since $h(a) \neq h(b)$. But $h'(x) = f'(x) - g'(x)$ must be zero throughout I, since we are told that $f'(x) = g'(x)$ in I — a contradiction.

CYU 4.4 Let $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} < 0$. Letting $-f'(c) > 0$ play the role of ε in the definition of the limit, we can find $\delta > 0$ such that:

$$0 < |h| < \delta \Rightarrow \left| \frac{f(c+h) - f(c)}{h} - f'(c) \right| < -f'(c)$$

$$\Rightarrow f'(c) < \frac{f(c+h) - f(c)}{h} - f'(c) < -f'(c)$$

$$\Rightarrow 2f'(c) < \frac{f(c+h) - f(c)}{h} < 0$$

In particular, $f(c+h) - f(c)$ must be negative for any $0 < h < \delta$, while $f(c+h) - f(c)$ must be positive for any $-\delta < h < 0$

CYU 4.5 Suppose that f has a local minimum at c. Can $f'(c)$ be positive? No, for if it were positive then there would be x's immediately to the left of c with function values smaller than $f(c)$ [Theorem 4.4(a)]. Can $f'(c)$ be negative? No, for if it were negative then there would be x's immediately to the right of c with function values smaller than $f(c)$ [Theorem 4.4(b)]. Since $f'(c)$ exists and cannot be positive or negative, it must be 0.

CYU 4.6 (Proof by contradiction.) Assume that $2x^4 - x + 10 = 0$ has solutions x_1, x_2, x_3, with $x_1 < x_2 < x_3$. Consequently, for $p(x) = 2x^4 - x + 10$: $p(x_1) = p(x_2) = p(x_3) = 0$. By Rolle's Theorem (applied twice), there must exist $x_1 < y_1 < x_2$ and $x_2 < y_2 < x_3$, such that $p'(y_1) = p'(y_2) = 0$. But this cannot happen since the equation $p'(x) = 0$ has but one solution: $p'(x) = 0 \Rightarrow 8x^3 - 1 = 0 \Rightarrow x^3 = \frac{1}{8} \Rightarrow x = \frac{1}{2}$.

CYU 4.7 Consider the continuous function $h(x) = f(x) - g(x)$. Since $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$, the exists $a < c < b$ such that $h(c) = 0$; which is to say: that $f(c) = g(c)$.

CYU 4.8 Assume that f is negative at some point in $[a, b]$. Theorem 4.7 assures us that f assumes its minimum value at some $c \in [a, b]$. Indeed, c must be contained in (a, b), for f is assumed to take on negative values in $[a, b]$, and $f(a) = f(b) = 0$. Now proceed, as before, to show that $f'(c)$ cannot be either positive or negative, and that therefore $f'(c)$ must be zero.

CYU 4.9. (a) Since the leading term of $f(x) = x^3 - x$ is x^3 the graph of f will resemble that of the cubic polynomial x^3 as $x \to \pm\infty$.

(b) Since $f(x) = (2x^3 + x)(x^2 - 5x + 1)(x - 1) = 2x^6 + \cdots$, the graph of f will resemble that of the polynomial $2x^6$ as $x \to \pm\infty$ (or more simply x^6, for the two polynomials are the same shape as $x \to \pm\infty$).

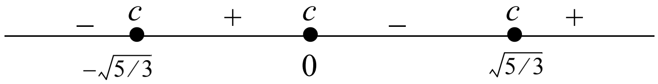
CYU 4.10. **(a)** Graphing the function $f(x) = 3x^5 - 5x^3$

Step 1. Factor: $f(x) = 3x^5 - 5x^3 = x^3(3x^2 - 5) = x^3(\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5})$

Step 2. y-intercept: $f(0) = 0$.

x-intercepts: $f(x) = x^3(\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5}) = 0$: at $x = 0$, $x = \pm\sqrt{\frac{5}{3}}$.

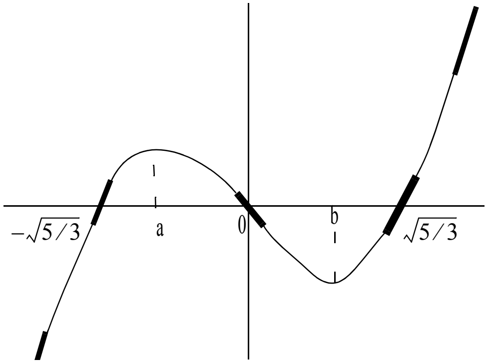
SIGN f:



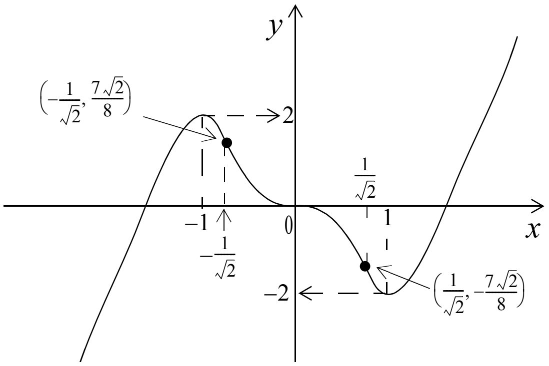
Step 3. As $x \to \pm\infty$: The graph resembles the graph of $g(x) = 3x^5$.

Step 4. From the above information, we have a pretty good sense of the graph of the function, and can sketch its anticipated graph:

Anticipated Graph:



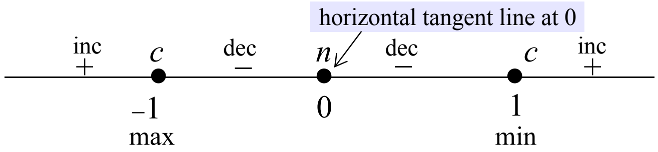
Graph:



Step 5. (a) [Increasing, Decreasing; Maximum and Minimums] Differentiating, we have:

$$f'(x) = (3x^5 - 5x^3)' = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x + 1)(x - 1)$$

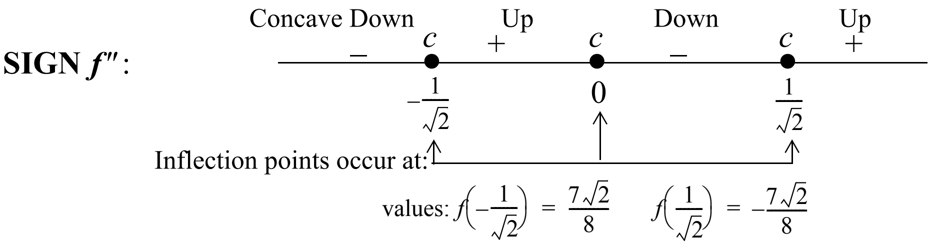
SIGN f':



Values: $f(-1) = 3(-1)^5 - 5(-1)^3 = 2$ $f(1) = 3(1)^5 - 5(1)^3 = -2$

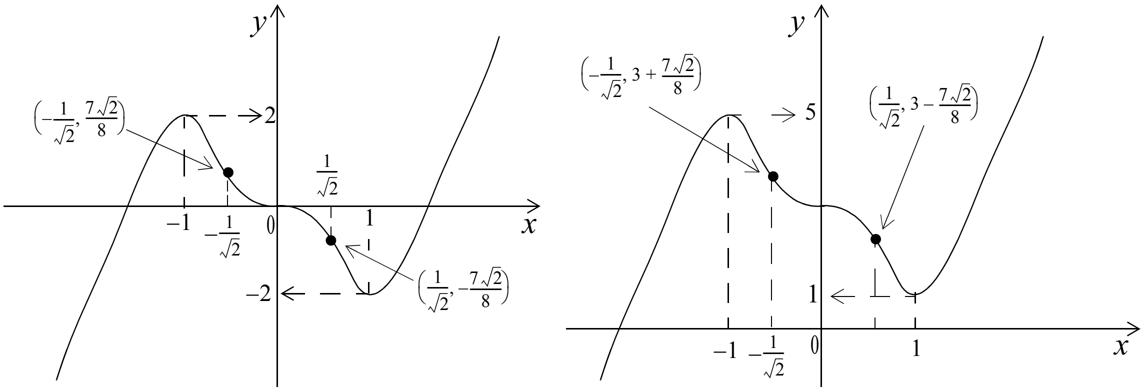
Step 5 (b) [Concavity and Inflection Points] Taking the second derivative, we have:

$$f''(x) = (15x^4 - 15x^2)' = 60x^3 - 30x = 30x(2x^2 - 1) = 30x(\sqrt{2}x + 1)(\sqrt{2}x - 1)$$



The above information is reflected in the final graph of the function depicted above.

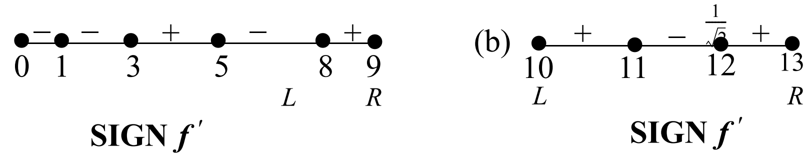
(b) We can simply take the graph of $f(x) = 3x^5 - 5x^3$ and lift it 3 units to arrive at the graph of $g(x) = 3x^5 - 5x^3 + 3$:



Graph of $f(x) = 3x^5 - 5x^3$

Graph of $g(x) = 3x^5 - 5x^3 + 3$

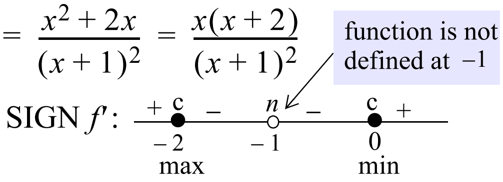
CYU 4.11. (a)



(a) Since the graph is falling immediately to the left of 3 and rising to its right, f has a local minimum at 3. It also has a local minimum at 8. Since the graph is rising immediately to the left of 5 and falling to its right, f has a local maximum at 5. The graph is falling immediately to the right of the endpoint 0, so the graph has an endpoint maximum there. It also has an endpoint maximum at the 9, since the graph is increasing to the left of that point.

(b) Since the graph is rising immediately to the left of 11 and falling to its right, f has a local maximum at 11. A local minimum occurs at 12, since the graph falls to the left of 12 and rises to its right. The graph is rising immediately to the right of the endpoint 10, so the graph has an endpoint minimum there. It has a maximum at the endpoint 13, since the graph is increasing to the left of that point.

CYU 4.12. First derivative test: $f'(x) = \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$



Second derivative test:

$$f'(x) = \frac{(x+1)(2x)-x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2} \quad \text{Critical Points: } 0, -2, -1$$

$$f''(x) = \frac{(x+1)^2(2x+2)-(x^2+2x)2(x+1)}{(x+1)^4}$$

$$= \frac{(x+1)(2x+2)-2(x^2+2)}{(x+1)^3} = \frac{4x-2}{(x+1)^3}$$

$f''(0) = -2$: maximum at 0

$f''(-2) = \dfrac{2-12}{(-1)^3} = 10$: minimum at -2

$f''(-1)$ is undefined

CYU 4.13. (a) As $x \to \pm\infty$, $\dfrac{3x^4+2x}{6x^4-5} \to \dfrac{3}{6} = \dfrac{1}{2}$: a horizontal asymptote, with equation: $y = \dfrac{1}{2}$.

(b) As $x \to \pm\infty$, $\dfrac{3x^4+2x}{6x^5-5} \to \dfrac{3}{6x} \to 0$: a horizontal asymptote, with equation: $y = 0$.

(c) As $x \to \pm\infty$, the graph of $f(x) = \dfrac{4x^3-1}{2x^2+x}$ will resemble, **in shape** a line of slope 2 ($\dfrac{4x^3}{2x^2} = 2x$).

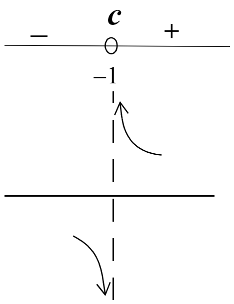
The actual oblique asymptote is the line $y = 2x-1$. Why? Because:

$$\begin{array}{r} 2x-1 \\ 2x^2+x\,\overline{)\,4x^3 \qquad\quad -1} \\ \underline{4x^3+2x^2 \qquad} \\ -2x^2 \quad -1 \\ \underline{-2x^2-x} \\ x-1 \end{array}$$

$$\Rightarrow \frac{4x^3-1}{2x^2+x} = \mathbf{2x-1} + \underbrace{\frac{x-1}{2x^2+x}}_{\to 0 \text{ as } x \to \infty}$$

(d) As $x \to \pm\infty$, the graph of $f(x) = \dfrac{6x^6-5}{3x^4+2x}$ will resemble, **in shape** that of the parabola $y = \dfrac{6x^6}{3x^4} = 2x^2$.

CYU 4.14. A glance at SIGN *f* about -1:



reveals the nature of the vertical asymptote at -1

CYU 4.15 Graphing $f(x) = \dfrac{x^2}{x^2-4}$.

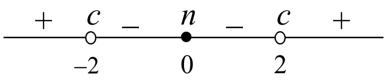
Step 1. Factor: $f(x) = \dfrac{x^2}{x^2-4} = \dfrac{x^2}{(x-2)(x+2)}$.

Step 2. y-intercept: $y = f(0) = 0$.

x-intercepts: $f(x) = 0$: $x = 0$.

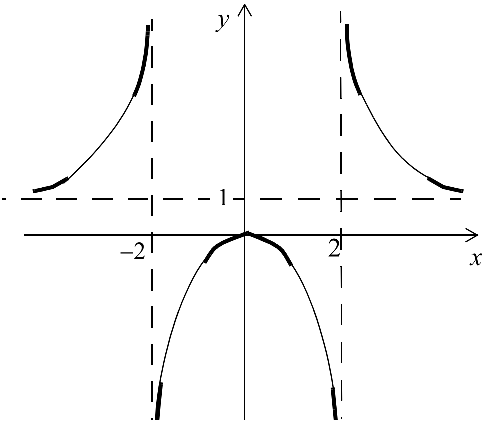
Vertical Asymptotes: The lines $x = -2$ and $x = 2$.

SIGN $f(x) = \dfrac{x^2}{(x-2)(x+2)}$:



Step 3. As $x \to \pm\infty$: $y = \dfrac{x^2}{x^2} = 1$ is the horizontal asymptote for the graph.

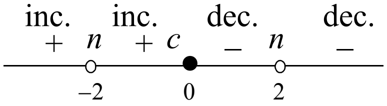
Step 4. Sketch the anticipated graph:



Step 5: Turning to the calculus:

$$f'(x) = \left(\frac{x^2}{x^2-4}\right)' = \frac{(x^2-4)\cdot 2x - (x^2 \cdot 2x)}{(x^2-4)^2} = \frac{-8x}{(x+2)^2(x-2)^2}$$

SIGN f':



Conforms with anticipated graph

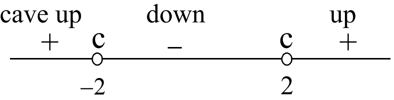
$$f''(x) = \left(\frac{-8x}{(x^2-4)^2}\right)' = \frac{(\mathbf{x^2-4})^2(\mathbf{-8}) - (\mathbf{-8}x)[2(\mathbf{x^2-4})\cdot 2x]}{(x^2-4)^4}$$

pull out the common factor $\mathbf{8(x^2-4)}$:

$$= \frac{8(x^2-4)[-(x^2-4)+4x^2]}{(x^2-4)^4}$$

$$= \frac{8(3x^2+4)}{(x^2-4)^3} = \frac{8(3x^2+4)}{(x+2)^3(x-2)^3}$$

SIGN f'':



Conforms with anticipated graph

<u>**CYU 4.16**</u>. Graphing $f(x) = (x-2)^{1/3}$.

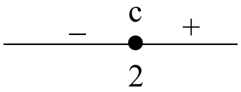
Step 1. Factor: Already in factored form.

Step 2. y-intercept: $y = f(0) = -2^{1/3}$.

x-intercepts: $f(x) = 0: x = 2$.

Vertical Asymptotes: None.

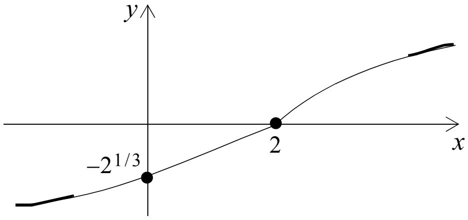
SIGN $f(x) = (x-2)^{1/3}$:



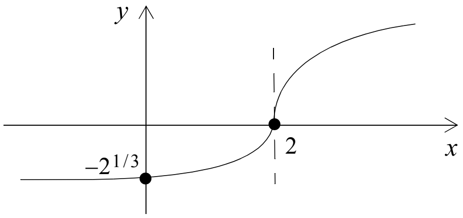
Step 3. As $x \to \infty$: $f(x) \approx x^{1/3}$

Step 4. Sketch the anticipated graph:

Anticipated Graph:



Graph:



Step 5: Turning to the calculus:

$$f'(x) = [(x-2)^{1/3}]' = \frac{1}{3}(x-2)^{-2/3} = \frac{1}{3(x-2)^{2/3}}$$

SIGN f': $+\ \overset{n}{\underset{2}{\circ}}\ +$

even ↓ note the 2/3 power ↓

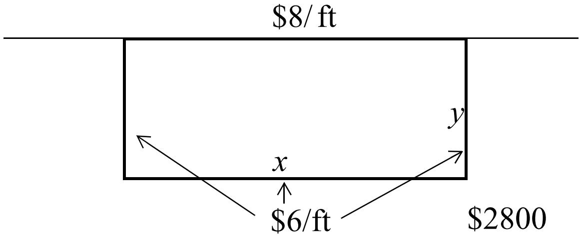
At this point we know that the graph is increasing everywhere (which we anticipated) but now find that since the derivative is not defined at $x = 2$, a vertical tangent line must occur at that point. Note also the concavity nature of the indicated graph (as is supported by the second derivative):

$$f''(x) = \left(\frac{1}{3}(x-2)^{-2/3}\right)' = -\frac{2}{9}(x-2)^{-5/3} = -\frac{2}{9(x-2)^{5/3}}$$

SIGN f'': $\overset{\text{concave up}}{+}\ \underset{2}{\circ}\ \overset{\text{down}}{-}$

inflection point

CYU 4.17 **See the Problem:**

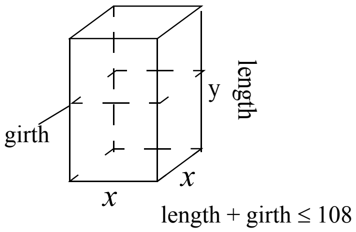


$$A = x\mathbf{y}$$
$$8x + 6(x + 2y) = 2800$$
$$14x + 12y = 2800$$
$$\mathbf{y} = \frac{2800 - 14x}{12} = \mathbf{\frac{700}{3} - \frac{7}{6}x}$$

So: $A = x\left(\mathbf{\frac{700}{3} - \frac{7}{6}x}\right) = \frac{700}{3}x - \frac{7}{6}x^2$ (*)

Then: $\left(\frac{700}{3}x - \frac{7}{6}x^2\right)' = 0 \Rightarrow \frac{700}{3} - \frac{7}{3}x = 0 \Rightarrow \mathbf{x = 100}$. At this point we know that a maximum area occurs when $x = 100$. Substituting in (*) we find the maximum area: $\frac{700}{3}(100) - \frac{7}{6}(100)^2 = \frac{35000}{3}\text{ ft}^2$.

CYU 4.18 **SEE THE PROBLEM:**



We want to maximize $V = x^2y$.

Clearly the greatest volume can only be achieved when we allow the sum of *length-plus-girth* to be as large as is allowed:

$$y + 4x = 108 \Rightarrow y = 108 - 4x \quad (*)$$

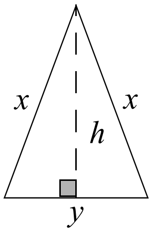
Bringing us to:

$$V = x^2(108 - 4x) = 108x^2 - 4x^3$$

Then: $V' = (108x^2 - 4x^3)' = 0 \Rightarrow 216x - 12x^2 = 0 \Rightarrow 12x(18 - x) = 0 \Rightarrow \boxed{\cancel{x = 0}}$ or $x = 18$. For maximum volume the length of a side of the square base should be 18 in. and the height [see (*)]: $y = 108 - 4(18) = 36$ in.

CYU 4.19 SEE THE PROBLEM:

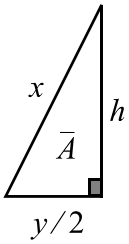
$2x + y = 12$



Due to symmetry, the area, A, of the triangle at the left is twice that of the area, $\bar{A}$, of the adjacent right triangle:

$$\bar{A} = \left[\frac{1}{2} \cdot \frac{y}{2} \cdot h\right] = \frac{yh}{4} \quad (*)$$

one-half base times height

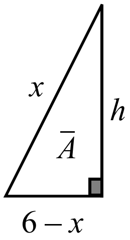


Expressing y in terms of x: $2x + y = 12 \Rightarrow y = 12 - 2x$ we can replace $y/2$ in the above right triangle with $6 - x$
And this enables us to also express h in terms of x:

$$x^2 = (6 - x)^2 + h^2$$

$$h = \sqrt{x^2 - (6 - x)^2}$$

$$= 2\sqrt{3x - 9}$$



From (*): $\bar{A} = \frac{(12 - 2x)2\sqrt{3x - 9}}{4}$, and therefore: $A = 2\bar{A} = (12 - 2x)\sqrt{3x - 9}$.

Then: $A' = [(12 - 2x)\sqrt{3x - 9}]' = 0 \Rightarrow (12 - 2x) \cdot \frac{1}{2\sqrt{3x - 9}} \cdot 3 + \sqrt{3x - 9}(-2) = 0$

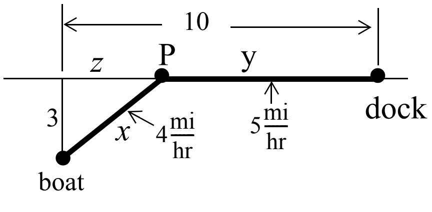
$$\frac{18 - 3x}{\sqrt{3x - 9}} = 2\sqrt{3x - 9}$$

$$18 - 3x = 2(3x - 9)$$

$$x = 4$$

Maximum Area: $= (12 - 2 \cdot 4)\sqrt{3 \cdot 4 - 9} = 4\sqrt{3}$ in.^{2}

CYU 4.20 SEE THE PROBLEM:



Total time for the trip: $T = \frac{x}{4} + \frac{y}{5}$ hours.

We will replace both x and y with the indicated variable z:

$$y = 10 - z \quad \text{and} \quad x = \sqrt{3^2 + z^2}$$

Bringing us to:

$$T = \frac{\sqrt{3^2 + z^2}}{4} + \left(\frac{10 - z}{5}\right) = \frac{\sqrt{9 + z^2}}{4} - \frac{z}{5} + 2$$

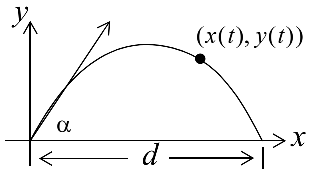
$$\text{Then: } T' = \left(\frac{\sqrt{9+z^2}}{4} - \frac{z}{5} + 2\right)' = 0 \Rightarrow \frac{z}{4\sqrt{9+z^2}} - \frac{1}{5} = 0 \Rightarrow 5z = 4\sqrt{9+z^2}$$

$$25z^2 = 16(9+z^2)$$

$$9z^2 = 16 \cdot 9 \Rightarrow z = 4$$

Conclusion: The point P should be $10 - z = 10 - 4 = 6$ miles from the dock.

CYU 4.21 **SEE THE PROBLEM:**



$x(t) = (V_0 \cos\alpha)t$

$y(t) = (V_0 \sin\alpha)t - 16t^2$

(a) We find the time it takes for the object to hit the ground:

$$y(t) = (V_0 \sin\alpha)t - 16t^2 = 0 \Rightarrow 16t = V_0 \sin\alpha \Rightarrow t = \frac{V_0}{16}\sin\alpha$$

At that point in time:

$$x(t) = (V_0\cos\alpha)\frac{V_0}{16}\sin\alpha = \frac{V_0^2}{16}\cos\alpha\sin\alpha$$

$$= \frac{V_0^2}{32} \cdot 2\sin\alpha\cos\alpha = \frac{V_0^2}{32}\sin 2\alpha$$

and since $\sin 2\alpha$ is maximum when $2\alpha = 90°$: $\alpha = 45°$.

(b) $y(t) = (V_0 \sin 45°)t - 16t^2 = \frac{V_0}{\sqrt{2}}t - 16t^2$. Then:

$$\left(\frac{V_0}{\sqrt{2}}t - 16t^2\right)' = \frac{V_0}{\sqrt{2}} - 32t = 0 \Rightarrow t = \frac{V_0}{32\sqrt{2}}$$

$$\text{Maximum height: } y\left(\frac{V_0}{32\sqrt{2}}\right) = \frac{V_0}{\sqrt{2}}\left(\frac{V_0}{32\sqrt{2}}\right) - 16\left(\frac{V_0}{32\sqrt{2}}\right)^2 = \frac{V_0^2}{128} \text{ ft}$$

CYU 4.22 **SEE THE PROBLEM:**

units sold

price per unit

$p = 50 - \frac{x^2}{6000}$

cost: \$30 for each unit

By definition: Profit = Revenue – Cost ($P = R - C$). Since Revenue is equal to the number of units sold times the price per unit: $R = x\left(50 - \frac{x^2}{6000}\right)$, and since Cost is equal to the cost per unit times the number of units produced: $C = 30x$.

Bringing us to the profit function $P = x\left(50 - \frac{x^2}{6000}\right) - 30x$.

$$\text{Then: } \left[x\left(50 - \frac{x^2}{6000}\right) - 30x\right]' = 0 \Rightarrow \left(20x - \frac{x^3}{6000}\right)' = 0 \Rightarrow 20 - \frac{x^2}{2000} = 0 \Rightarrow x = 200.$$

Conclusion: To maximize profit for the company, the company should produce 200 units.

CYU 4.23 **SEE THE PROBLEM:**

price per unit

units sold

$$n = 200 - \frac{p}{1000}$$

cost: 100,000+75,000 n

$$P = 200p - \frac{p^2}{1000} - 100{,}000 - 75{,}000\left(200 - \frac{p}{1000}\right)$$

$$= 200p - \frac{p^2}{1000} - 100{,}000 - 15{,}000{,}000 + 75p$$

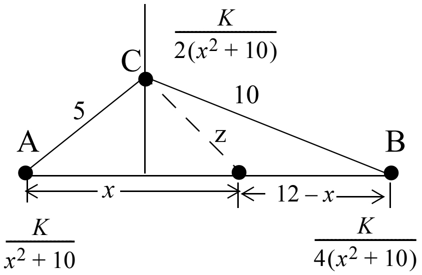
$$\text{Then: } P' = 200 - \frac{p}{500} + 75 = 0 \Rightarrow \frac{p}{500} = 275$$

$$\Rightarrow p = 275(500)$$

$$\text{So: } 200 - \frac{p}{1000} = 200 - \frac{275(500)}{1000} = \frac{125}{2} = 62.5$$

Since the company can only sell complete boats, the number produced to maximize profit will be either 62 or 63 boats. A direct calculation in (*) shows that the profit is the same for both options.

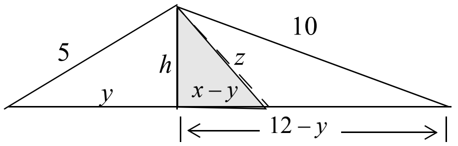
CYU 4.24 **SEE THE PROBLEM:**



Combined Pollution Count (P) at a point that is x units from A:

$$P = \frac{K}{x^2+10} + \frac{K}{4[(12-x)^2+10]} + \frac{K}{2(z^2+10)}$$

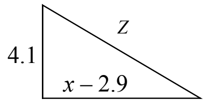
The main task is to express z in terms of x. With this in mind we turn to:



(You may choose to invoke the Law of Cosines instead of the Pythagorean Theorem))

Noting that $h^2 = 5^2 - y^2$ and that $h^2 = 10^2 - (12-y)^2$ enables us to solve for y and h:

$$25 - y^2 = 100 - 144 + 24y - y^2 \Rightarrow y = \frac{69}{24} \approx 2.9 \text{ and } h = \sqrt{5^2 - y^2} \approx \sqrt{5^2 - (2.9)^2} \approx 4.1$$



On the left we augmented the shaded right triangle in the above figure. We can now express z in terms of x: $z \approx \sqrt{(4.1)^2 + (x-2.9)^2}$, bringing us to:

$$P \approx K\left(\frac{1}{x^2+10} + \frac{1}{4[(12-x)^2+10]} + \frac{1}{2[(4.1)^2+(x-2.9)^2+10]}\right)$$

The constant K has no effect on the value of x which will minimize P. Turning to a graphing calculator you will find that the minimum solution count occurs at $x \approx 8.8$. So, the point is about 8.8 miles from A.

CHAPTER 5: INTEGRATION

CYU 5.1 The most “obvious” antiderivative of $f(x) = 8x^7$ is x^8. Adding any constant to x^8 will yield another antiderivative.

CYU 5.2 (a) $\int 5x^4\,dx = 5\cdot\frac{x^5}{5}+C = x^5+C$ (b) $\int -4x^{-5}dx = -4\cdot\frac{x^{-5+1}}{-5+1}+C = x^{-4}+C$

(c) $\int(2x^5+4x^3-\frac{1}{3}x^2+2)dx = 2\cdot\frac{x^6}{6}+4\cdot\frac{x^4}{4}-\frac{1}{3}\cdot\frac{x^3}{3}+2x+C = \frac{x^6}{3}+x^4-\frac{x^3}{9}+2x+C$

CYU 5.3

(a) $\int[(3x^2-2x+1)(x-5)]dx = \int(3x^3-17x^2+11x-5)dx = \frac{3}{4}x^4-\frac{17}{3}x^3+\frac{11}{2}x^2-5x+C$

(b) $\int\frac{x^4-2x-6}{x^4}dx = \int\left(\frac{x^4}{x^4}-2\frac{x}{x^4}-\frac{6}{x^4}\right)dx = \int(1-2x^{-3}-6x^{-4})dx$

$$= x-2\cdot\frac{x^{-2}}{-2}-6\cdot\frac{x^{-3}}{-3}+C = x+\frac{1}{x^2}+\frac{2}{x^3}+C$$

CYU 5.4 (a) $\int(\sin x+2\cos x)dx = -\cos x+2\sin x+C$

(b) $\int(x^2-\sec x\tan x)dx = \frac{x^3}{3}-\sec x+C$

CYU 5.5 $f(x) = \int(5x^4-2)dx = x^5-2x+C$. Since $f(0) = 1$: $1 = 0^5-2\cdot 0+C$, or $C = 1$.

Thus: $f(x) = x^5-2x+1$

CYU 5.6 (a) Differentiating the position function $s(t) = -16t^2+64t+80$ gives us the velocity function $v(t) = -32t+64$. Setting velocity to zero we determine the time it takes for the stone to reach its maximum height: $-32t+64 = 0$, or $t = 2$. Evaluating the position function at $t = 2$ yields the maximum height: $s(2) = -16\cdot 2^2+64\cdot 2+80 = 144$ feet.

(b) Setting the position function to zero (ground level) we determine the time it takes for the stone to hit the ground:

$$\begin{aligned}-16t^2+64t+80 &= 0\\ -16(t^2-4t-5) &= 0\\ -16(t-5)(t+1) &= 0\\ t = 5 \quad\text{or}\quad t &= -1\end{aligned}$$

Knowing it takes 5 seconds for the stone to hit the ground, we can determine its velocity at impact: $v(5) = -32\cdot 5+64 = -96\frac{\text{ft}}{\text{sec}}$. Since speed is the magnitude of velocity, we conclude that the stone hits the ground at a speed of 96 feet per second.

CYU 5.7

Object-one	Object-two
$v_1(t) = -32t + 32$	$v_2(t) = -32t + v_0$
$s_1(t) = -16t^2 + 32t + 128$	$s_2(t) = -16t^2 + v_0 t$

First object reaches maximum height when $v_1(t) = -32t + 32 = 0$: at $t = 1$ second at which time it is $s_1(1) = -16 + 32 + 128 = 144$ feet from the ground. We need to find v_0 such $s_2(1) = -16 + v_0 = 144$. Answer: $v_0 = 144 + 16 = 160$ feet per second.

CYU 5.8 Applying the Principal Theorem of Calculus:

$$T'(x) = \left[\int_3^x (3t^2+2)^7 dt\right]' = (3x^2+2)^7$$

CYU 5.9 Choosing $g(x) = x^3 + 2x + 100$ as an antiderivative of $f(x) = 3x^2 + 2$, we again have:

$$\int_1^2 (3x^2+2)dx = (x^3 + 2x + 100)\Big|_1^2 = (2^3 + 2\cdot 2 + 100) - (1^3 + 2\cdot 1 + 100) = 9.$$

CYU 5.10 (a-i) $\displaystyle\int_0^1 (x^3 + x - 1)dx = \frac{x^4}{4} + \frac{x^2}{2} - x\Big|_0^1 = \left(\frac{1}{4} + \frac{1}{2} - 1\right) - (0) = -\frac{1}{4}$

$$\text{(a-ii)} \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\, dx = -\cos x\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{2}} = -\left[\cos\frac{\pi}{2} - \cos\left(-\frac{\pi}{4}\right)\right] = -\left(0 - \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

(b) Noting that the graph of the function $f(x) = (x^2+1)(x^2+3)$ lies above the x-axis, we conclude that the area A bounded by its graph over the interval $[-1, 1]$ is:

$$\int_{-1}^1 (x^2+1)(x^2+3)dx = \int_{-1}^1 (x^4 + 4x^2 + 3)dx = \frac{x^5}{5} + \frac{4x^3}{3} + 3x\Big|_{-1}^1$$

$$= \left(\frac{1}{5} + \frac{4}{3} + 3\right) - \left(-\frac{1}{5} - \frac{4}{3} - 3\right) = \frac{136}{15}$$

CYU 5.11 For F and G antiderivatives of f and g, respectively, $F - G$ is an antiderivative of $f - g$, so:

$$\int_a^b [f(x) - g(x)]dx = [F(x) - G(x)]\Big|_a^b = [F(b) - G(b)] - [F(a) - G(a)]$$

$$= [F(b) - F(a)] + [G(b) - G(a)] = \int_a^b f(x)dx - \int_a^b g(x)dx$$

For any number c, cF is an antiderivative of cf, so:

$$\int_a^b cf(x)dx = [cF(x)]\Big|_a^b = [cF(b) - cF(a)] = c[F(b) - F(a)] = c\int_a^b f(x)dx$$

CYU 5.12 If $\int_a^c f(x)dx = 5$, $\int_c^b f(x)dx = -3$ and $\int_a^b g(x)dx = 7$, then:

(a) $\int_a^c -2f(x)dx + \int_b^a g(x)dx = -2\int_a^c f(x)dx - \int_a^b g(x)dx = -2(5)-(7) = -17$

(b) $\int_a^b f(x)dx + \int_a^b 2g(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx + 2\int_a^b g(x)dx = 5+(-3)+2(7) = 16$

CYU 5.13 Barrels produced: $\int_0^{30}\left(75 - \frac{t}{2500}\right)dt = \left(75t - \frac{t^2}{5000}\right)\Bigg|_0^{30} = 75\cdot 30 - \frac{30^2}{5000} \approx 2250$

Income: $\$(85\cdot 2250) = \$191{,}250$

CYU 5.14 (a) $$\int \frac{x}{5(x^2-10)^5}dx \underset{\uparrow}{=} \frac{1}{2}\int\frac{du}{5u^5} = \frac{1}{10}\int u^{-5}du = \frac{1}{10}\left(\frac{u^{-4}}{-4}\right)+C = -\frac{1}{40u^4}+C$$

$u = x^2-10 \Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$

$$= -\frac{1}{40(x^2-10)^4}+C$$

(b) $$\int\frac{\cos x}{\sin^2 x}dx \underset{\uparrow}{=} \int\frac{du}{u^2} = \int u^{-2}du = \frac{u^{-1}}{-1}+C = -\frac{1}{u}+C = -\frac{1}{\sin x}+C$$

$u = \sin x \Rightarrow du = \cos x dx$

CYU 5.15 $$\int x\sqrt{x+1}\,dx \underset{\uparrow}{=} \int (u-1)u^{\frac{1}{2}}du = \int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$$

$u = x+1 \Rightarrow du = dx$
$\Downarrow$
$x = u-1$

$$= \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} + C$$

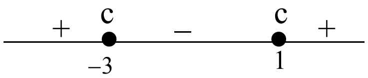
CYU 5.16 (a) $$\int_1^{\sqrt{2}} x\sqrt{x^2-1}\,dx \underset{\uparrow}{=} \frac{1}{2}\int_0^1 u^{\frac{1}{2}}du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\Bigg|_0^1 = \frac{1}{3}(1^{3/2}-0) = \frac{1}{3}$$

$u = x^2-1 \Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$
$\Downarrow$
$x = 1 \Rightarrow u = 0$ and $x = \sqrt{2} \Rightarrow u = 1$

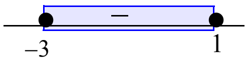
(b) $$\int_0^1 \frac{x}{(x^2+1)^2}dx \underset{\uparrow}{=} \frac{1}{2}\int_1^2\frac{du}{u^2} = \frac{1}{2}\int_1^2 u^{-2}du = \frac{1}{2}\cdot\frac{u^{-1}}{-1}\Bigg|_1^2 = -\frac{1}{2}\cdot\frac{1}{u}\Bigg|_1^2 = -\frac{1}{2}\left(\frac{1}{2}-\frac{1}{1}\right) = \frac{1}{4}$$

$u = x^2+1$ | $x = 1 \Rightarrow u = 2$
$du = 2xdx$ | $x = 0 \Rightarrow u = 1$
$xdx = \frac{1}{2}du$

CYU 5.17 SIGN $f(x) = x^2 + 2x - 3 = (x+3)(x-1)$:



(a)

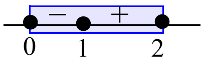


$$A = \int_{-3}^{1}|x^2+2x-3|\,dx = -\int_{-3}^{1}(x^2+2x-3)\,dx$$

$$= -\left[\left(\frac{x^3}{3}+x^2-3x\right)\Big|_{-3}^{1}\right]$$

$$= -\left[\left(\frac{1}{3}+1-3\right)-\left(\frac{-3^3}{3}+9+9\right)\right] = \frac{32}{3}$$

(b)

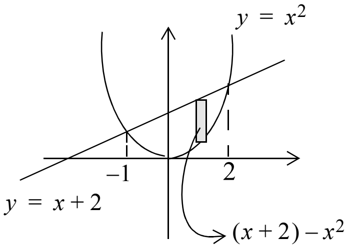


$$A = \int_{0}^{2}|x^2+2x-3|\,dx = -\int_{0}^{1}(x^2+2x-3)\,dx + \int_{1}^{2}(x^2+2x-3)\,dx$$

$$= -\left[\left(\frac{x^3}{3}+x^2-3x\right)\Big|_{0}^{1}\right] + \left(\frac{x^3}{3}+x^2-3x\right)\Big|_{1}^{2}$$

$$= -\left[\left(\frac{1}{3}+1-3\right)\right] + \left[\left(\frac{8}{3}+4-6\right)-\left(\frac{1}{3}+1-3\right)\right] = 4$$

CYU 5.18



$$x^2 = x+2 \Rightarrow x^2-x-2 = 0 \Rightarrow (x-2)(x+1) = 0$$

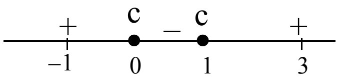
$$\Rightarrow x = -1, 2$$

$$A = \int_{-1}^{2}[(x+2)-x^2]\,dx = \frac{x^2}{2}+2x-\frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right) = \frac{9}{2}$$

CYU 5.19

SIGN $f(x)-g(x) = (x^4-x^2)-(x^3+x^2) = x^4-x^3 = x^3(x-1)$:



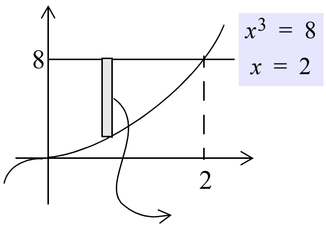
$$A\int_{-1}^{3}|f(x)-g(x)|\,dx = \int_{-1}^{0}(x^4-x^3)\,dx - \int_{0}^{1}(x^4-x^3)\,dx + \int_{1}^{3}(x^4-x^3)\,dx$$

$$= \left(\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_{-1}^{0} - \left(\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_{0}^{1} + \left(\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_{1}^{3}$$

$$= -\left(-\frac{1}{5}-\frac{1}{4}\right)-\left(\frac{1}{5}-\frac{1}{4}\right)+\left[\left(\frac{3^5}{5}-\frac{3^4}{4}\right)-\left(\frac{1}{5}-\frac{1}{4}\right)\right] = \frac{289}{10}$$

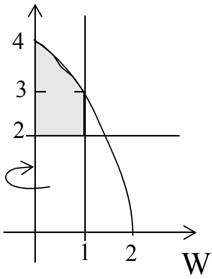
CYU 5.20 $V = \pi\int_{1}^{2}(x^3)^2\,dx = \pi\int_{1}^{2}x^6\,dx = \pi\left(\frac{x^7}{7}\Big|_{1}^{2}\right) = \pi\left(\frac{2^7}{7}-\frac{1}{7}\right) = \frac{127\pi}{7}$

CYU 5.21



$$V = \pi\int_0^2 [8^2 - (x^3)^2]dx = \pi\left(64x - \frac{x^7}{7}\Big|_0^2\right) = \frac{768}{7}\pi$$

CYU 5.22



Shell: $$V = 2\pi\int_0^1 x[(-x^2+4)-2]dx = 2\pi\int_0^1 (-x^3+2x)dx$$

$$= 2\pi\left[\left(-\frac{x^4}{4}+x^2\right)\Big|_0^1\right] = \frac{3\pi}{2}$$

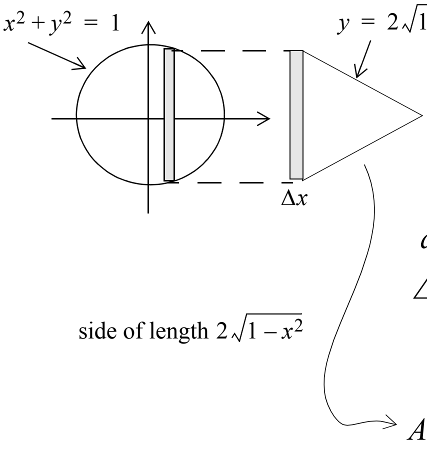
Washer: $$V = \pi\int_2^3 1^2dy + \pi\int_3^4 (\sqrt{4-y})^2dy = \pi(x|_2^3) + \pi\int_3^4 (4-y)dy$$

$$= \pi + \pi\left(4y - \frac{y^2}{2}\Big|_3^4\right) = \frac{3\pi}{2}$$

CYU 5.23

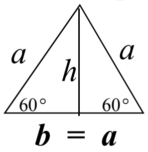
Top half of circle: $y = \sqrt{1-x^2}$, bottom half: $y = -\sqrt{1-x^2}$

So, side of triangle has length $(\sqrt{1-x^2} - \sqrt{1-x^2}) = 2\sqrt{1-x^2}$



answer $\rightarrow$ $V = \int_{-1}^{1} A(x)dx$

The main task is to determine the area of an equilateral triangle with side of length a:



$\boldsymbol{b} = \boldsymbol{a}$

$$A = \frac{1}{2}\boldsymbol{b}h = \frac{1}{2}\boldsymbol{a}h$$

$$\sin 60° = \frac{h}{a} \Rightarrow h = \frac{\sqrt{3}a}{2} \Rightarrow A = \frac{\sqrt{3}a^2}{4}$$

$$A(x) = \frac{\sqrt{3}}{4}(2\sqrt{1-x^2})^2 = \sqrt{3}(1-x^2)$$

So: $$V = \int_{-1}^{1} A(x)dx = \sqrt{3}\int_{-1}^{1}(1-x^2)dx = \sqrt{3}\left(x - \frac{x^3}{3}\Big|_{-1}^{1}\right) = \frac{4\sqrt{3}}{3}$$

CYU 5.24 $$L = \int_1^5 \sqrt{1+\left[\left(\sqrt{x}+\frac{1}{x}\right)'\right]^2}\,dx = \int_1^5 \sqrt{1+\left(\frac{1}{2x^{1/2}}-\frac{1}{x^2}\right)^2}\,dx = \int_1^5 \sqrt{1+\frac{1}{4x}-\frac{1}{x^{5/2}}+\frac{1}{x^4}}\,dx$$

CYU 5.25 $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \Rightarrow \frac{dy}{dx} = 2\sqrt{2}x^{1/2} \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+8x}$

$$\text{Then: } L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+8x}\,dx \underset{\uparrow}{=} \frac{1}{8}\int_1^9 u^{1/2}du = \frac{1}{8}\left(\frac{2}{3}u^{3/2}\Big|_1^9\right)$$

$u = 1 + 8x \Rightarrow du = 8dx \Rightarrow dx = \frac{1}{8}du$
$\downarrow$
$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 9$

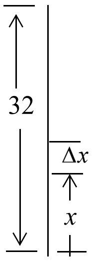
$$= \frac{1}{12}(9^{3/2} - 1) = \frac{13}{6}$$

CYU 5.26 force = k(displacement): $1 = k \cdot \left(\frac{1}{2}\right) \Rightarrow k = 2\ \frac{\text{lb}}{\text{ft}}$

$$\text{(a) } W = \int_0^{1/4} f(x)dx = \int_0^{1/4} 2xdx = x^2\Big|_0^{1/4} = \frac{1}{16}\text{ ft-lb}$$

$$\text{(b) } W = \int_{1/4}^{3/4} f(x)dx = \int_{1/4}^{3/4} 2xdx = x^2\Big|_{1/4}^{3/4} = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{2}\text{ ft-lb}$$

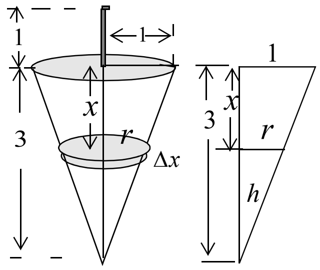
CYU 5.27 The bag is lifted a total of $4\frac{\text{ft}}{\text{sec}} \cdot 8\text{ sec} = 32\text{ ft}$. Partition that distance in Δx pieces. The weight of the bag when lifted through the indicated distance Δx is the original weight of 100 pounds minus the weight of sand that leaked out in reaching that height: $100\text{ lb} - \left(\frac{x\text{ ft}}{4\text{ ft/sec}}\right)\left(1\frac{\text{lb}}{\text{sec}}\right) = \left(100 - \frac{x}{4}\right)\text{lb}$. So:



$$W = \int_0^{32}\left(100 - \frac{x}{4}\right)dx = 100x - \frac{x^2}{8}\Big|_0^{32} = 100 \cdot 32 - \frac{32^2}{8} = 3072\text{ ft-lb}$$

If the bag did not have a hole in it, then the work in lifting the bag would be $100 \cdot 32 = 3200\text{ ft-lb}$. If the bag's weight were constant and equal to its weight of 92 pounds at the end of its journey, then the work would be $92 \cdot 32 = 2944\text{ ft-lb}$. Note that W is the average (mean) of those two extreme situations: $\frac{3200 + 2944}{2} = 3072\text{ ft-lb}$.

CYU 5.28 The work required to lift the shaded water-disk is approximately equal to $\Delta W = (\pi r^2 \Delta x)(1000)(9.8)(x + 1)$.
force $\cdot$ distance



From the two represented similar triangles we have:

$$\frac{1}{3} = \frac{\boldsymbol{r}}{h} \Rightarrow \boldsymbol{r} = \frac{h}{3} \Rightarrow \boldsymbol{r} = \frac{3-x}{3} = \mathbf{1} - \frac{\boldsymbol{x}}{\mathbf{3}}$$

Consequently:

$$W = 1000(9.8)\pi\int_0^3 (x+1)\boldsymbol{r}^2 dx = 9800\pi\int_0^3 (x+1)\left(\mathbf{1} - \frac{\boldsymbol{x}}{\mathbf{3}}\right)^2 dx$$

$$= 9800\pi\int_0^3\left(\frac{x^3}{9} - \frac{5x^2}{9} + \frac{x}{3} + 1\right)dx = 9800\pi\left(\frac{x^4}{36} - \frac{5x^3}{27} + \frac{x^2}{6} + x\Big|_0^3\right) = 17{,}150\pi\text{ J}$$

CHAPTER 6: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

CUY 6.1 (a)
$$\begin{aligned}(x^3\ln x^2)' &= x^3(\ln x^2)' + \ln x^2(x^3)' = x^3\left[\frac{1}{x^2}\cdot(x^2)'\right] + \ln x^2(3x^2)\\ &= x^3\left(\frac{2x}{x^2}\right) + 3x^2\ln x^2 = 2x^2 + 3x^2\ln x^2\end{aligned}$$

(b)
$$\begin{aligned}\left(\frac{\tan x}{\ln 2x}\right)' &= \frac{\ln 2x(\tan x)' - \tan x(\ln 2x)'}{(\ln 2x)^2} = \frac{\ln 2x(\sec^2 x) - \tan x\left(\frac{2}{2x}\right)}{(\ln 2x)^2}\\ &= \frac{\ln 2x(\sec^2 x) - \frac{\tan x}{x}}{(\ln 2x)^2} = \frac{x\sec^2 x\ln 2x - \tan x}{x(\ln 2x)^2}\end{aligned}$$

(c)
$$[\ln(\ln x)]' = \frac{1}{\ln x}(\ln x)' = \frac{1}{\ln x}\left(\frac{1}{x}\right) = \frac{1}{x\ln x}$$

CYU 6.2 (a)
$$\int\frac{7}{5x+2}dx \underset{\substack{\uparrow\\ \boxed{u = 5x+2 \Rightarrow du = 5dx \Rightarrow dx = \frac{1}{5}du}}}{=} \frac{7}{5}\int\frac{1}{u}du = \frac{7}{5}\ln|u| + C = \frac{7}{5}\ln|5x+2| + C$$

(b)
$$\int_e^5\frac{dx}{x\ln x} \underset{\substack{\uparrow\\ \boxed{\begin{gathered}u = \ln x \Rightarrow du = \frac{dx}{x}\\ \downarrow\\ x = e \Rightarrow u = \ln e = 1,\ x = 5 \Rightarrow u = \ln 5\end{gathered}}}}{=} \int_1^{\ln 5}\frac{du}{u} = \ln|u|\Big|_1^{\ln 5} = \ln|\ln 5| - \ln 1 = \ln(\ln 5) - 0 = \ln(\ln 5)$$

CYU 6.3
$$\ln\frac{x}{y} = \ln xy^{-1} \underset{\substack{\uparrow\\ \text{Theorem 6.4(a)}}}{=} \ln x + \ln y^{-1} \underset{\substack{\uparrow\\ \text{Theorem 6.4(c)}}}{=} \ln x - \ln y$$

CYU 6.4 (a)
$$\begin{aligned}\int\cot x\,dx &= \int\frac{\cos x}{\sin x}dx = \int\frac{1}{u}du \underset{\substack{\uparrow\\ \boxed{\begin{gathered}u = \sin x\\ du = \cos x\,dx\end{gathered}}}}{=} \ln|u| + C = \ln|\sin x| + C = \ln|(\csc x)^{-1}| + C\\ &= \ln|\csc x|^{-1} + C\\ &= -\ln|\csc x| + C\end{aligned}$$

(b)
$$\begin{aligned}\int\csc x\,dx &= \int\csc x\cdot\frac{\csc x + \cot x}{\csc x + \cot x}dx \qquad \boxed{\begin{gathered}u = \csc x + \cot x\\ du = -(\csc x\cot x + \csc^2 x)dx\end{gathered}}\\ &= \int\frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}dx \overset{\swarrow}{=} -\int\frac{du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C\end{aligned}$$

CYU 6.5
$$\begin{aligned}V = \pi\int_1^e\left[\left(\frac{1}{\sqrt{x}} + 1\right)^2 - 1^2\right]dx = \pi\int_1^e\left(\frac{1}{x} + \frac{2}{\sqrt{x}}\right)dx &= \pi(\ln|x| + 4\sqrt{x})\Big|_1^e\\ &= \pi[(\ln e + 4\sqrt{e}) - (\ln 1 + 4)]\\ &= \pi[(1 + 4\sqrt{e}) - (0 + 4)]\\ &= \pi(4\sqrt{e} - 3)\end{aligned}$$

CYU 6.6 If $f'(x) = \frac{1}{2x+1} + 2x + 1$, then:

$$f(x) = \int\left(\frac{1}{2x+1} + 2x + 1\right)dx = \frac{1}{2}\ln|2x+1| + x^2 + x + C$$

Since $f(0) = 1: 1 = \frac{1}{2}\ln|2\cdot 0 + 1| + 0^2 + 0 + C = C$. So: $f(x) = \frac{1}{2}\ln|2x+1| + x^2 + x + 1$.

CYU 6.7 (a) $\left(\frac{e^x}{\ln x} + 1\right)' = \frac{\ln x(e^x)' - e^x(\ln x)'}{(\ln x)^2} + 0 = \frac{\ln x \cdot e^x - \frac{e^x}{x}}{(\ln x)^2} = \frac{xe^x\ln x - e^x}{x(\ln x)^2}$

$$= \frac{e^x(x\ln x - 1)}{x(\ln x)^2}$$

(b) $(\sqrt{xe^x})' = \left[(xe^x)^{\frac{1}{2}}\right]' = \frac{1}{2}(xe^x)^{-\frac{1}{2}}(xe^x)' = \frac{1}{2(xe^x)^{1/2}}(xe^x + e^x)$

$$= \frac{e^x(x+1)}{2x^{1/2}(e^x)^{1/2}} = \frac{\sqrt{e^x}(x+1)}{2\sqrt{x}}$$

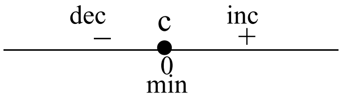
(c) $(xe^{\sqrt{x}})' = x(e^{\sqrt{x}})' + e^{\sqrt{x}}(x') = x[e^{\sqrt{x}}(\sqrt{x})'] + e^{\sqrt{x}} = \frac{xe^{\sqrt{x}}}{2\sqrt{x}} + e^{\sqrt{x}} = \frac{1}{2}e^{\sqrt{x}}(\sqrt{x} + 2)$

CYU 6.8 (a) $\int e^x \sin e^x dx \underset{\substack{\uparrow \\ u = e^x \Rightarrow du = e^x dx}}{=} \int \sin u\, du = -\cos u + C = -\cos e^x + C$

(b) $\int_0^{\pi} \cos x e^{\sqrt{\sin x}} dx \underset{\substack{\uparrow \\ u = \sin x \Rightarrow du = \cos x dx \\ x = 0 \Rightarrow u = \sin 0 = 0,\, x = \pi \Rightarrow u = \sin\pi = 0}}{=} \int_0^0 e^{\sqrt{u}} du = 0$

CYU 6.9 Since the exponential function only assumes positive values, the function $f(x) = e^{x^2-1}$ also assumes only positive values. We also know that the graph of f has y-intercept $f(0) = e^{0-1} = \frac{1}{e}$. Turning to the calculus we find that:

$f'(x) = (e^{x^2-1})' = e^{x^2-1}(x^2-1)' = 2xe^{x^2-1}$ SIGN f':

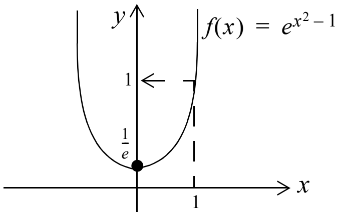


note that e^{x^2-1} is always positive

$$f''(x) = (2xe^{x^2-1})' = 2[x(e^{x^2-1})' + e^{x^2-1}(x)'] = 2[x(e^{x^2-1}\cdot 2x) + e^{x^2-1}]$$

$$= 2e^{x^2-1}(2x^2+1) \leftarrow$$ always positive concave up

Taking the above into account:



CYU 6.10 By Theorem 6.4(b), page 226: $\ln\left(\frac{e^a}{e^b}\right) = \ln e^a - \ln e^b = a - b$.

In addition: $\ln e^{a-b} = a - b$. Since $\ln\left(\frac{e^a}{e^b}\right) = \ln e^{a-b}$ and since the natural logarithmic function is one-to-one: $\frac{e^a}{e^b} = e^{a-b}$.

CYU 6.11 We begin with the exponential decay formula $A(t) = A_0 e^{kt}$. Since the substance loses $\frac{1}{3}$ of its mass in four days, $\frac{2}{3}$ of the initial amount A_0 will be present when $t = 4$:

$$\frac{2}{3}A_0 = A_0 e^{4k} \Rightarrow e^{4k} = \frac{2}{3} \Rightarrow 4k = \ln\frac{2}{3} \Rightarrow k = \frac{\ln(2/3)}{4}$$

Here, then, is the specific exponential formula for the substance at hand: $A(t) = A_0 e^{\left[\frac{\ln(2/3)}{4}\right]t}$. To find the time it will take for the substance to decay to $\frac{1}{10}$ of its original mass we solve the following equation for t:

$$\frac{A_0}{10} = A_0 e^{\left[\frac{\ln(2/3)}{4}\right]t}$$

$$\frac{1}{10} = e^{\left[\frac{\ln(2/3)}{4}\right]t} \Rightarrow \ln\frac{1}{10} = \left[\frac{\ln(2/3)}{4}\right]t \Rightarrow t = \frac{4\ln(1/10)}{\ln(2/3)} \approx 22.7 \text{ days}$$

CYU 6.12 We consider the exponential growth formula $A(t) = A_0 e^{kt}$. Since the population increases from 500 to $500 + \frac{15}{100} \cdot 500 = 575$ in 9 years, we have:

$$575 = 500e^{9k} \Rightarrow e^{9k} = \frac{23}{20} \Rightarrow 9k = \ln\frac{23}{20} \Rightarrow k = \frac{\ln(23/20)}{9}$$

To find the time it will take for the population to triple we solve the following equation for t:

$$1500 = 500e^{\left[\frac{\ln(23/20)}{9}\right]t}$$

$$3 = e^{\left[\frac{\ln(23/20)}{9}\right]t} \Rightarrow \ln 3 = \left[\frac{\ln(23/20)}{9}\right]t \Rightarrow t = \frac{9\ln 3}{\ln(23/20)} \approx 70.75 \text{ years}$$

CYU 6.13 $f'(x) = (x^{e+1})' = (e+1)x^{e+1-1} = (e+1)x^e$

$$f''(x) = [(e+1)x^e]' = (e+1)ex^{e-1} = (e^2+e)x^{e-1}$$

CYU 6.14 $(x^{\sin x})' = [e^{\sin x \ln x}]' = \underset{\substack{\uparrow\\ x^{\sin x}}}{e^{\sin x\ln x}}[\sin x \ln x]' = \underset{\uparrow}{x^{\sin x}}\left[\frac{\sin x}{x} + \ln x\cos x\right]$

CYU 6.15 (a) $\int \frac{2^{\sqrt{x}}}{\sqrt{x}}dx \underset{\uparrow}{=} 2\int 2^u du = 2\cdot\frac{2^u}{\ln 2} + C = \frac{2^{u+1}}{\ln 2} + C = \frac{2^{\sqrt{x}+1}}{\ln 2} + C$

$$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{dx}{\sqrt{x}} = 2du$$

(b) $\int_1^e \frac{5^{\ln x}}{x}dx \underset{\uparrow}{=} \int_0^1 5^u du = \left.\frac{5^u}{\ln 5}\right|_0^1 = \frac{1}{\ln 5}(5^1 - 5^0) = \frac{4}{\ln 5}$

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

$$\downarrow$$

$$x = 1 \Rightarrow \ln x = \ln(1) = 0, x = e \Rightarrow \ln x = \ln e = 1$$

CYU 6.16 (a) Since $a^{\log_a xy} = xy$ and since $xy = a^{\log_a x}a^{\log_a y} = a^{\log_a x + \log_a y}$ [Theorem 6.10(a)], and since the function a^x is one-to-one: $\log_a xy = \log_a x + \log_a y$.

(b) Since $a^{\log_a (x/y)} = \frac{x}{y}$ and since $\frac{x}{y} = \frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a x - \log_a y}$, and since the function a^x is one-to-one: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

CYU 6.17 (a) $(\log_3 x^2)' = \frac{1}{\ln 3\cdot x^2}\cdot(x^2)' = \frac{2x}{\ln 3\cdot x^2} = \frac{2}{x\ln 3}$

(b) $[\sqrt{\sin(\log_5 x)}]' = \frac{1}{2}[\sin(\log_5 x)]^{-1/2}[\sin(\log_5 x)]'$

$$= \frac{1}{2\sqrt{\sin(\log_5 x)}}\cdot\cos(\log_5 x)\cdot(\log_5 x)' = \frac{\cos(\log_5 x)}{2\sqrt{\sin(\log_5 x)}}\cdot\frac{1}{x\ln 5}$$

CYU 6.18 First: $[\cos(\cos^{-1}x)]' = x' \Rightarrow -\sin(\cos^{-1}x)\cdot(\cos^{-1}x)' = 1$

$$(\cos^{-1}x)' = -\frac{1}{\sin(\cos^{-1}x)} \qquad (*)$$

$$\text{Then:}\quad \begin{aligned}\sin^2(\cos^{-1}x)+\cos^2(\cos^{-1}x) &= 1\\ \sin^2(\cos^{-1}x) &= 1-[\cos(\cos^{-1}x)]^2\\ \sqrt{\sin^2(\cos^{-1}x)} &= \sqrt{1-[\cos(\cos^{-1}x)]^2}\\ \sin(\cos^{-1}x) &= \sqrt{1-x^2}\end{aligned}$$

$$\text{Finally:}\quad (\cos^{-1}x)' \underset{(*)}{\overset{}{\underset{\uparrow}{=}}} -\frac{1}{\sin(\cos^{-1}x)} = -\frac{1}{\sqrt{1-x^2}}$$

CYU 6.19 (a) Since the domain of $\cos^{-1}x$ is $-1 \le x \le 1$, the domain of $f(x) = \cos^{-1}(\ln x)$ is $-1 \le \ln x \le 1$. Applying the (increasing) function e^x we arrive at the domain of f: $e^{-1} \le x \le e$.

Employing the chain rule we have: $[\cos^{-1}(\ln x)]' = -\dfrac{1}{\sqrt{1-(\ln x)^2}} \cdot \dfrac{1}{x}$.

In order for $\dfrac{1}{\sqrt{1-(\ln x)^2}} \cdot \dfrac{1}{x}$ to be defined we must have:

$$\begin{gathered}1-(\ln x)^2 > 0 \text{ and } x > 0\\ (\ln x)^2 < 1\\ -1 < \ln x < 1\\ e^{-1} < x < e\end{gathered}$$

Conclusion: $\left(\dfrac{1}{e}, e\right)$ is the domain of $[\cos^{-1}(\ln x)]'$

(b) Since the domain of $\ln x$ is $x > 0$, the domain of $g(x) = \ln(\tan^{-1}x)$ consists of those x's for which $\tan^{-1}x > 0$, which is to say [see Figure 6.6(b)]: $(0, \infty)$.

Employing the chain rule we have: $[\ln(\tan^{-1}x)]' = \dfrac{1}{\tan^{-1}x} \cdot \dfrac{1}{1+x^2}$.

In order for $\dfrac{1}{\tan^{-1}x} \cdot \dfrac{1}{1+x^2}$ to be defined we must have: $\tan^{-1}x > 0$.

Conclusion: $(0, \infty)$ is the domain of $[\ln(\tan^{-1}x)]'$

CYU 6.20 (a) $\displaystyle\int_0^1 \frac{1}{x^2+1}dx = \tan^{-1}x\Big|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

(b) $$\int\frac{dx}{\sqrt{1-(2x+1)^2}} \underset{\substack{\uparrow\\ u\,=\,2x+1\\ du\,=\,2dx}}{=} \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(2x+1) + C$$

CHAPTER 7: TECHNIQUES OF INTEGRATION

CYU 7.1 $\int x\cos x\,dx$: $\begin{array}{rcl} u &=& x \\ du &=& dx \end{array}$ $\begin{array}{rcl} dv &=& \cos x\,dx \\ v &=& \sin x \end{array}$

$$\text{Then: } \int x\cos x\,dx = uv - \int v du = x\sin x - \int \sin x dx = x\sin x + \cos x + C$$

CYU 7.2 (a)

$$\int_0^1 x\ln(2x^2+1)dx \underset{\uparrow}{=} \frac{1}{4}\int_1^3 \ln u\,du \underset{\uparrow}{=} \frac{1}{4}[(u\ln u - u)|_1^3] = \frac{1}{4}[(3\ln 3 - 3) - (1\ln 1 - 1)]$$

$\begin{array}{rcl} u &=& 2x^2+1 \\ du &=& 4xdx \end{array}$ $\begin{array}{l} x = 0 \Rightarrow u = 1 \\ x = 1 \Rightarrow u = 3 \end{array}$ Theorem 7.1

$$= \frac{1}{4}[3\ln 3 - 3 - (0-1)]$$

$$= \frac{3}{4}\ln 3 - \frac{1}{2}$$

(b) $\int \tan^{-1} x dx$: $\begin{array}{rcl} u &=& \tan^{-1} x \\ du &=& \frac{1}{1+x^2}dx \end{array}$ $\begin{array}{rcl} dv &=& dx \\ v &=& x \end{array}$

$$\text{So: } \int \tan^{-1} x dx = uv - \int v du = x\tan^{-1} x - \int \frac{\boldsymbol{x}}{\boldsymbol{1+x^2}}\boldsymbol{dx}$$

$\begin{array}{rcl} u &=& 1+x^2 \\ du &=& 2xdx \end{array}$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{du}{u} = x\tan^{-1} x - \frac{1}{2}\ln|u| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

CYU 7.3 $\int e^x \cos x dx$: $\begin{array}{rcl} u &=& e^x \\ du &=& e^x dx \end{array}$ $\begin{array}{rcl} dv &=& \cos x\,dx \\ v &=& \sin x \end{array}$

$$\text{So: } \int e^x \cos x\,dx = uv - \int v du = e^x \sin x - \int \boldsymbol{e^x \sin x dx} \qquad (*)$$

Now: $\int \boldsymbol{e^x \sin x dx}$: $\begin{array}{rcl} u &=& e^x \\ du &=& e^x dx \end{array}$ $\begin{array}{rcl} dv &=& \sin x\,dx \\ v &=& -\cos x \end{array}$

$$\text{So: } \int \boldsymbol{e^x \sin x\,dx} = uv - \int v du = e^x(-\cos x) - \int e^x(-\cos x)dx = -e^x\cos x + \int e^x \cos x dx$$

Returning to (*):

$$\int e^x \cos x\,dx = e^x \sin x - [-e^x\cos x + \int e^x \cos x dx] = e^x(\sin x + \cos x) - \int e^x \cos x dx$$

$$\Rightarrow 2\int e^x \cos x dx = e^x(\sin x + \cos x) + C \Rightarrow \int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + C$$

CYU 7.4 Since $\int x\cos x\,dx = x\sin x + \cos x + C$ (See CYU 7.1):

$$\int_0^{\frac{\pi}{2}} x\cos x\,dx = (x\sin x + \cos x)\Big|_0^{\frac{\pi}{2}} = \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0\sin 0 + \cos 0) = \frac{\pi}{2} - 1$$

CYU 7.5 $$\int \sin^3 x\,dx \underset{\substack{\uparrow \\ \text{Theorem 7.2}}}{=} -\frac{\cos x}{3}\sin^2 x + \frac{2}{3}\int \sin x\,dx = -\frac{1}{3}\cos x\sin^2 x - \frac{2}{3}\cos x + C$$

CYU 7.6

$$\int \frac{dx}{\sqrt{3x - x^2}} = \int \frac{dx}{\sqrt{-\left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4}}} = \int \frac{dx}{\sqrt{\frac{9}{4} - \left(x - \frac{3}{2}\right)^2}}$$

since: $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}x + C$

we set our sights on turning the $\frac{9}{4}$ into a 1

$$= \int \frac{dx}{\sqrt{\frac{9}{4}\left[1 - \frac{\left(x - \frac{3}{2}\right)^2}{\frac{9}{4}}\right]}} = \frac{1}{\frac{3}{2}}\int \frac{dx}{\sqrt{1 - \left(\frac{x - \frac{3}{2}}{\frac{3}{2}}\right)^2}}$$

$$= \frac{2}{3}\int \frac{dx}{\sqrt{1 - \left(\frac{2}{3}x - 1\right)^2}}$$

$u = \frac{2}{3}x - 1$

$du = \frac{2}{3}dx$ $\longrightarrow$

$$= \frac{2}{3}\cdot\frac{3}{2}\int \frac{du}{\sqrt{1 - u^2}}$$

$$= \sin^{-1}u + C = \sin^{-1}\left(\frac{2}{3}x - 1\right) + C$$

CYU 7.7 Referring to Figure 7.1 we have:

$$\frac{x - 4}{(x - 3)(2x + 1)^2(x^2 + 5)^2} = \frac{A}{x - 3} + \boxed{\underbrace{\frac{B}{2x + 1} + \frac{C}{(2x + 1)^2}}_{\text{(ii) of Figure 7.1}}} + \boxed{\underbrace{\frac{Dx + E}{x^2 + 5} + \frac{Fx + G}{(x^2 + 5)^2}}_{\text{(v) of Figure 7.1}}}$$

CYU 7.8 $$\frac{1}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$$

$$1 = A(x^2 + x + 1) + (Bx + C)x \quad (*)$$

$1 = 0x^2 + 0x + 1$

Evaluate at $x = 0$ (only A survives:) $1 = A$

$\longrightarrow$ Equate the coefficients of x^2: $0 = A + B \Rightarrow 0 = 1 + B \Rightarrow B = -1$

While it is easy to spot the constant coefficient on the right side of (*) it will give us nothing new; namely: $1 = A$. So:

$\longrightarrow$ Equate the coefficients of x: $0 = A + C \Rightarrow 0 = 1 + C \Rightarrow C = -1$

Conclusion: $$\frac{1}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1} = \frac{1}{x} + \frac{-x - 1}{x^2 + x + 1} = \frac{1}{x} - \frac{x + 1}{x^2 + x + 1}$$

CYU 7.9 $\displaystyle\int\frac{dx}{x(x^2+x+1)} \overset{\substack{\text{See the above “Conclusion”}\\ \downarrow}}{=} \int\left(\frac{1}{x}-\frac{x+1}{x^2+x+1}\right)dx = \int\frac{dx}{x}-\int\frac{x+1}{x^2+x+1}dx = \ln|x|-\int\frac{x+1}{x^2+x+1}dx$

To evaluate $\displaystyle\int\frac{x+1}{x^2+x+1}dx$, we first set our sights on getting $2x+1$ in the numerator:

$$\int\frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int\frac{2x+2}{x^2+x+1}dx = \frac{1}{2}\int\frac{(2x+1)+1}{x^2+x+1}dx$$

$$= \underset{(*)}{\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx} + \underset{(**)}{\frac{1}{2}\int\frac{dx}{x^2+x+1}}$$

Continuing the good fight:

$$(*):\ \frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx \underset{\substack{\uparrow\\ u\,=\,x^2+x+1,\ du\,=\,(2x+1)dx}}{=} \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln(x^2\underset{\substack{\uparrow\\ \text{note that } x^2+x+1>0 \text{ for all } x}}{+}x+1)+C$$

$$(**):\ \frac{1}{2}\int\frac{dx}{x^2+x+1} = \frac{1}{2}\int\frac{dx}{\left(x^2+x+\frac{1}{4}\right)+1-\frac{1}{4}} = \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{2}\int\frac{dx}{\frac{3}{4}\left(\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2\right]+\boxed{1}\right)}$$

$\displaystyle\int\frac{dx}{1+x^2} = \tan^{-1}x+C$

want

so pull out $\frac{3}{4}$

$$= \frac{2}{3}\int\frac{dx}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}$$

$$u = \frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}},\ du = \frac{2}{\sqrt{3}}dx:\quad = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{du}{u^2+1} = \frac{1}{\sqrt{3}}\tan^{-1}u+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C$$

Putting it all together we have:

$$\int\frac{dx}{x(x^2+x+1)} = \ln|x|-\frac{1}{2}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C$$

CYU 7.10 $\displaystyle\int\frac{2x^2-4x+3}{x^2-4x+4}dx \overset{\downarrow}{=} \int\left(2+\frac{4x-5}{x^2-4x+4}\right)dx$

$$\begin{array}{r|l} & \ \ 2 \\ x^2-4x+4 & 2x^2-4x+3 \\ & \underline{2x^2-8x+8} \\ -: & \ \ 4x-5 \end{array}$$

$$= \int 2dx+\int\frac{4x-5}{(x-2)^2}dx = 2x+\int\frac{4x-5}{(x-2)^2}dx$$

Turning to $\int \frac{4x-5}{(x-2)^2}dx$:

$$\frac{4x-5}{(x-2)^2} = \frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$4x-5 = A(x-2)+B$$

$$x = 2:\quad 8-5 = B \Rightarrow \boxed{B = 3}$$

equating x-coefficient: $\boxed{A = 4}$

So: $\int \frac{4x-5}{(x-2)^2}dx = \int \frac{4}{x-2}dx + \int \frac{3}{(x-2)^2}dx = 4\ln|x-2| - \frac{3}{x-2} + C$

$$\boxed{\begin{aligned} u &= x-2 \\ du &= dx \end{aligned}} \longrightarrow \boxed{= 3\int \frac{du}{u^2} = 3\int u^{-2}du = -\frac{3}{u}+C}$$

Hence: $\int \frac{2x^2-4x+3}{x^2-4x+4}dx = 2x + \int \frac{4x-5}{(x-2)^2}dx = 2x + 4\ln|x-2| - \frac{3}{x-2} + C$

CYU 7.11 (a) $\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos x \cos^2 x dx = \int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)dx$

$$= \int_0^{\frac{\pi}{2}} \cos x dx - \int_0^{\frac{\pi}{2}} \cos x(\sin^2 x)dx$$

$$\boxed{\begin{aligned} &u = \sin x,\ du = \cos x dx \\ &\sin 0 = 0,\ \sin\tfrac{\pi}{2} = 1 \end{aligned}}$$

$$= \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^1 u^2 du \longleftarrow$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \frac{u^3}{3}\Big|_0^1 = (1-0) - \left(\frac{1}{3}-0\right) = \frac{2}{3}$$

(b) $\int \sin^3 x \cos^2 x\, dx = \int \sin x \sin^2 x \cos^2 x\, dx = \int \sin x(1-\cos^2 x)\cos^2 x\, dx$

$$= \int \sin x \cos^2 x dx - \int \sin x \cos^4 x dx$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x dx \end{aligned}} \longrightarrow = -\int u^2 du + \int u^4 du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

<u>CYU 7.12</u> (a)
$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\sin^4 x\,dx &= \int_0^{\frac{\pi}{2}}(\sin^2 x)^2\,dx = \int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)^2 dx\\
&= \int_0^{\frac{\pi}{2}}\left(\frac{1}{4}-\frac{2\cos 2x}{4}+\frac{\cos^2 2x}{4}\right)dx\\
&= \int_0^{\frac{\pi}{2}}\frac{dx}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos 2x\,dx+\frac{1}{4}\int_0^{\frac{\pi}{2}}\cos^2 2x\,dx\\
&= \left(\frac{x}{4}-\frac{\sin 2x}{4}\right)\Bigg|_0^{\frac{\pi}{2}}+\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1+\cos 4x}{2}dx\\
&= \left(\frac{\pi}{8}-0\right)+\frac{1}{8}\left(x+\frac{\sin 4x}{4}\right)\Bigg|_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{8}+\frac{1}{8}\left(\frac{\pi}{2}+0\right) = \frac{3\pi}{16}
\end{aligned}$$

(b)
$$\int x\sin^2 x^2\cos^2 x^2\,dx \underset{\substack{\uparrow\\ u\,=\,x^2\\ du\,=\,2xdx}}{=} \frac{1}{2}\int\sin^2 u\cos^2 u\,du \underset{\substack{\uparrow\\ \text{Example 7.11(b)}}}{=} \frac{1}{2}\left(\frac{1}{8}u-\frac{1}{32}\sin 4u+C\right) = \frac{x^2}{16}-\frac{\sin 4x^2}{64}+C$$

<u>CYU 7.13</u>
$$\begin{aligned}
\int(\tan x)^{-3/2}\sec^4 x\,dx &= \int(\tan x)^{-3/2}(1+\tan^2 x)\sec^2 x\,dx\\
&= \int((\tan x)^{-3/2}\sec^2 x+(\tan x)^{1/2}\sec^2 x)dx\\
&= \int(u^{-3/2}+u^{1/2})du = -2u^{-1/2}+\frac{2}{3}u^{3/2}+C\\
&= -\frac{2}{\sqrt{\tan x}}+\frac{2(\tan x)^{3/2}}{3}+C
\end{aligned}$$

$u = \tan x$
$du = \sec^2 x\,dx$

CYU 7.14

$1 - \sin^2\theta = \cos^2\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}3\cos\theta\,d\theta = \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta}3\cos\theta\,d\theta$$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

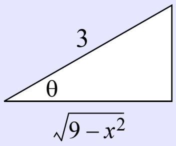
$$= \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cot^2\theta = \csc^2\theta - 1$:

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$



x and $\theta = \sin^{-1}\left(\frac{x}{3}\right)$:

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

CYU 7.15

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}}2\sec^2\theta\,d\theta = \int \frac{16\sec^2\theta\tan^3\theta}{\sqrt{4\sec^2\theta}}\,d\theta$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= 8\int \sec\theta\tan^3\theta\,d\theta$$

$$= 8\int \tan^2\theta\sec\theta\tan\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

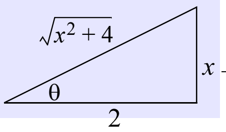
$$= 8\int (u^2 - 1)\,du$$

$$= 8\left(\frac{u^3}{3} - u\right) + C$$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$



$$= 8\left[\frac{\left(\frac{\sqrt{x^2+4}}{2}\right)^3}{3} - \frac{\sqrt{x^2+4}}{2}\right] + C$$

$$= \frac{\sqrt{x^2+4}(x^2+4)}{3} - 4\sqrt{x^2+4} + C$$

$$= \frac{(x^2-8)\sqrt{x^2+4}}{3} + C$$

CYU 7.16 $\displaystyle\int\frac{6x+11}{3x^2+2x+1}dx = \int\frac{(6x+2)+9}{3x^2+2x+1}dx = \int\frac{6x+2}{3x^2+2x+1}dx + 9\int\frac{dx}{3x^2+2x+1}$ (*)

noting that $(3x^2+2x+1)' = 6x+2$

Then: $\displaystyle\int\frac{6x+2}{3x^2+2x+1}dx = \int\frac{du}{u} = \ln|u| + C = \ln(3x^2+2x+1) + C$

never negative

And: $\displaystyle\int\frac{dx}{3x^2+2x+1} = \frac{1}{3}\int\frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}} = \frac{1}{3}\int\frac{dx}{(x^2+\frac{2}{3}x+\frac{1}{9})+\frac{1}{3}-\frac{1}{9}} = \frac{1}{3}\int\frac{dx}{(x+\frac{1}{3})^2+\frac{2}{9}}$

completing the square method

motivated by the 1 in the formula $\displaystyle\int\frac{dx}{1+x^2} = \tan^{-1}x + C$:

$$= \frac{1/3}{2/9}\int\frac{dx}{\left(\frac{3}{\sqrt{2}}x+\frac{1}{\sqrt{2}}\right)^2+\mathbf{1}}$$

$u = \frac{3}{\sqrt{2}}x + \frac{1}{\sqrt{2}}$

$du = \frac{3}{\sqrt{2}}dx$

$$= \frac{3}{2}\cdot\frac{\sqrt{2}}{3}\int\frac{du}{u^2+1}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}u + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

Returning to (*): $\displaystyle\int\frac{6x+11}{3x^2+2x+1}dx = \ln(3x^2+2x+1) + \frac{9}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

CHAPTER 8: L’Hôpital’s Rule AND IMPROPER INTEGRALS

CYU 8.1 (a) $\displaystyle\lim_{x\to0}\frac{(8+x)^{\frac{1}{3}}-2}{x} = \lim_{x\to0}\frac{\left[(8+x)^{\frac{1}{3}}-2\right]'}{x'} = \lim_{x\to0}\frac{\frac{1}{3}(8+x)^{-\frac{2}{3}}}{1} = \frac{1}{3}\cdot8^{-\frac{2}{3}} = \frac{1}{12}$

(b) $\displaystyle\lim_{x\to1}\frac{\tan(3x-3)}{\sin(2x-2)} = \lim_{x\to1}\frac{[\tan(3x-3)]'}{[\sin(2x-2)]'} = \lim_{x\to1}\frac{\sec^2(3x-3)\cdot3}{\cos(2x-2)\cdot2} = \frac{1(3)}{1(2)} = \frac{3}{2}$

(c) $\displaystyle\lim_{x\to\infty}\frac{x^{-1/2}}{\tan(x^{-1/2})} = \lim_{x\to\infty}\frac{(x^{-1/2})'}{[\tan(x^{-1/2})]'} = \lim_{x\to\infty}\frac{-\frac{1}{2}x^{-3/2}}{\sec^2(x^{-1/2})\left(-\frac{1}{2}x^{-3/2}\right)}$

$$= \lim_{x\to\infty}\frac{1}{\sec^2(x^{-1/2})} = 1$$

CYU 8.2 $$\lim_{x\to 0}\frac{1-\cos 2x}{5x^2} = \lim_{x\to 0}\frac{(1-\cos 2x)'}{(5x^2)'} = \lim_{x\to 0}\frac{\sin 2x\cdot 2}{10x} = \lim_{x\to 0}\frac{\sin 2x}{5x}$$
$$= \lim_{x\to 0}\frac{(\sin 2x)'}{(5x)'} = \lim_{x\to 0}\frac{\cos 2x\cdot 2}{5} = \frac{2}{5}$$

CYU 8.3 $$\lim_{x\to 0^-}\frac{\tan x}{x^2} = \lim_{x\to 0^-}\frac{(\tan x)'}{(x^2)'} = \lim_{x\to 0^-}\frac{\overset{\nearrow 1}{\sec^2 x}}{\underset{\searrow\, 0 \text{ (from the left)}}{2x}} = \underset{\nwarrow}{-\infty}$$

CYU 8.4 (a) $$\lim_{x\to\infty}\frac{5x^2+1}{x^2-3} = \lim_{x\to\infty}\frac{(5x^2+1)'}{(x^2-3)'} = \lim_{x\to\infty}\frac{10x}{2x} = \lim_{x\to\infty}5 = 5$$

(b) $$\lim_{x\to 0^-}\frac{1/x}{\ln(-x)} = \lim_{x\to 0^-}\frac{(x^{-1})'}{[\ln(-x)]'} = \lim_{x\to 0^-}\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x\to 0^-}-\frac{1}{x} = \infty$$

CYU 8.5 (a) $$\lim_{x\to -\frac{\pi}{4}}[(1+\tan x)\sec 2x] = \lim_{x\to -\frac{\pi}{4}}\left[\frac{1+\tan x}{\cos 2x}\right] = \lim_{x\to -\frac{\pi}{4}}\frac{(1+\tan x)'}{(\cos 2x)'}$$
$$= \lim_{x\to -\frac{\pi}{4}}\frac{\sec^2 x}{-2\sin 2x} = \frac{(\sqrt{2})^2}{-2(-1)} = 1$$

(b) $$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\tan x}\right) = \lim_{x\to 0}\frac{\tan x - x}{x\tan x} = \lim_{x\to 0}\frac{(\tan x - x)'}{(x\tan x)'}$$
$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{x\sec^2 x + \tan x}$$

multiply by $\frac{\cos^2 x}{\cos^2 x}$:
$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x+\sin x\cos x}$$
$$= \lim_{x\to 0}\frac{(1-\cos^2 x)'}{(x+\sin x\cos x)'}$$
$$= \lim_{x\to 0}\frac{2\cos x\sin x}{1+[-\sin^2 x+\cos^2 x]} = \frac{0}{2} = 0$$

CYU 8.6 (a) To determine $\lim\limits_{x\to\infty}(e^x+1)^{-\frac{2}{x}}$ we set our sights on finding $\lim\limits_{x\to\infty}\ln(e^x+1)^{-\frac{2}{x}}$:

$$\begin{aligned}\lim_{x\to\infty}\ln(e^x+1)^{-\frac{2}{x}} = \lim_{x\to\infty}-\frac{2}{x}\ln(e^x+1) &= -2\lim_{x\to\infty}\frac{\ln(e^x+1)}{x}\\ &= -2\lim_{x\to\infty}\frac{[\ln(e^x+1)]'}{x'}\\ &= -2\lim_{x\to\infty}\frac{e^x}{e^x+1}\\ &= -2\lim_{x\to\infty}\frac{(e^x)'}{(e^x+1)'} = -2\lim_{x\to\infty}\frac{e^x}{e^x} = -2\end{aligned}$$

Then: $\lim\limits_{x\to\infty}(e^x+1)^{-\frac{2}{x}} = e^{\lim\limits_{x\to\infty}\ln(e^x+1)^{-\frac{2}{x}}} = e^{-2} = \dfrac{1}{e^2}$

(b) To determine $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}$, we set our sights on finding $\lim\limits_{x\to 0}\ln(\cos x)^{1/x}$:

$$\lim_{x\to 0}\ln(\cos x)^{1/x} = \lim_{x\to 0}\frac{1}{x}\ln(\cos x) = \lim_{x\to 0}\frac{[\ln(\cos x)]'}{x'} = \lim_{x\to 0}\frac{-\tan x}{1} = 0$$

Then: $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0}\ln(\cos x)^{1/x}} = e^0 = 1$

CYU 8.7 Noting that: $\displaystyle\int xe^{-x}dx \underset{\substack{\uparrow\\ \begin{array}{ll} u=x & dv=e^{-x}dx\\ du=dx & v=-e^{-x}\end{array}}}{=} -xe^{-x}+\int e^{-x}dx = -xe^{-x}-e^{-x}+C$

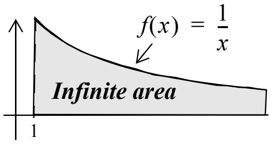
We have:
$$\begin{aligned}\int_0^\infty xe^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}(-xe^{-x}-e^{-x})\Big|_0^t &\\ = \lim_{t\to\infty}(-te^{-t}-e^{-t}+1) &\\ = -\lim_{t\to\infty}\frac{t}{e^t}-\cancelto{0}{\lim_{t\to\infty}\frac{1}{e^t}}+1 = -\lim_{t\to\infty}\frac{t}{e^t}+1 &\end{aligned}$$

Invoking l'Hôpital's rule: $\lim\limits_{x\to\infty}\dfrac{t}{e^t} = \lim\limits_{x\to\infty}\dfrac{t'}{(e^t)'} = \lim\limits_{x\to\infty}\dfrac{1}{e^t} = 0$

Conclusion: $\int_0^\infty xe^{-x}dx$ converges, with $\int_0^\infty xe^{-x}dx = 1$.

CYU 8.8 Revolving the indicated adjacent region about the x-axis we have:

$$V = \lim_{t\to\infty}\pi\int_1^t\left(\frac{1}{x}\right)^2dx = \pi\lim_{t\to\infty}(-x^{-1})\Big|_1^t = \pi\lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = \pi$$



CYU 8.9 By Theorem 8.3:

(a) In order for $\int_1^\infty\left(\frac{1}{x^p}+\frac{1}{x^q}\right)dx$ to converge, both p and q must be greater than 1.

(b) In order for $\int_1^\infty\left(\frac{1}{x^p}\cdot\frac{1}{x^q}\right)dx = \int_1^\infty\left(\frac{1}{x^{p+q}}\right)dx$ to converge, $p+q$ must be greater than 1.

(c) In order for $\int_1^\infty\frac{1/x^p}{1/x^q}dx = \int_1^\infty\left(\frac{1}{x^{p-q}}\right)dx$ to converge, $p-q$ must be greater than 1

CYU 8.10 (a) $\int_1^3\frac{dx}{(x-3)^2} = \lim_{t\to3^-}\int_1^t\frac{dx}{(x-3)^2} = \lim_{t\to3^-}-(x-3)^{-1}\Big|_1^t = -\lim_{t\to3^-}\left(\frac{1}{t-3}+\frac{1}{2}\right) = \infty.$

The integral diverges.

(b)
$$\begin{aligned}\int_{-3}^2\frac{dx}{(x+1)^{1/5}} &= \lim_{t\to-1^-}\int_{-3}^t\frac{dx}{(x+1)^{1/5}}+\lim_{t\to-1^+}\int_t^2\frac{dx}{(x+1)^{1/5}}\\ &= \lim_{t\to-1^-}\frac{5}{4}(x+1)^{4/5}\Big|_{-3}^t+\lim_{t\to-1^+}\frac{5}{4}(x+1)^{4/5}\Big|_t^2\\ &= \frac{5}{4}\lim_{t\to-1^-}[(t+1)^{4/5}-(-2)^{4/5}]+\frac{5}{4}\lim_{t\to-1^+}[3^{4/5}-(t+1)^{4/5}]\\ &= \frac{5}{4}(-2^{4/5}+3^{4/5})\end{aligned}$$

The integral converges.

CHAPTER 9: SEQUENCES AND SERIES

CYU 9.1 Let $(c_n) = (c, c, c, c, \dots)$. We show $\lim\limits_{n \to \infty} c_n = c$:

For given $\varepsilon > 0$, let $N = 1$. Then: $n > N \Rightarrow |c_n - c| = |c - c| = 0 < \varepsilon$.

CYU 9.2 (a-i) Let $\varepsilon > 0$ be given. We want to find N such that $n > N \Rightarrow \left|\left(7 - \frac{101}{n}\right) - 7\right| < \varepsilon$

$$\Leftrightarrow \frac{101}{n} < \varepsilon \Leftrightarrow n > \frac{101}{\varepsilon}$$

From the above, we see that if N is any integer greater than or equal to $\frac{101}{\varepsilon}$, then $n > N \Rightarrow \left|\left(7 - \frac{101}{n}\right) - 7\right| < \varepsilon$.

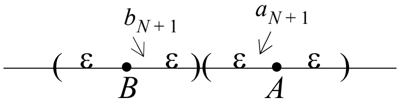
(a-ii) If $\varepsilon = \frac{1}{10}$, then $\frac{101}{\varepsilon} = \frac{101}{\frac{1}{10}} = 1010$. It follows, from (i), that $N = 1010$ is the smallest integer for which $n > N \Rightarrow \left|\left(7 - \frac{101}{n}\right) - 7\right| < \frac{1}{10}$.

(a-iii) If $\varepsilon = \frac{1}{100}$, then $\frac{101}{\varepsilon} = \frac{101}{\frac{1}{100}} = 10{,}100$. It follows, from (i) that $N = 10{,}100$ is the smallest integer for which $n > N \Rightarrow \left|\left(7 - \frac{101}{n}\right) - 7\right| < \frac{1}{100}$.

(b) For given $\varepsilon > 0$: $|a_n - 0| < \varepsilon \Leftrightarrow |a_n| < \varepsilon \Leftrightarrow \big||a_n| - 0\big| < \varepsilon$.

(c) (One possible answer) The sequence $(a_n) = (1, -1, 1, -1, 1, -1, \dots)$ diverges, while $(|a_n|) = (1, 1, 1, 1, 1, 1, \dots)$ converges to 1.

<u>CYU 9.3</u> (a) Suppose, to the contrary, that $A > B$. For $\varepsilon = \frac{B - A}{2}$, let N be such that $n > N \Rightarrow |a_n - A| < \varepsilon$ and $n > N \Rightarrow |b_n - B| < \varepsilon$.

It follows that $a_{N+1} > b_{N+1}$:  — a contradiction.

(b) (One possible answer) $(a_n) = (0, 0, 0, \dots)$ and $(b_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$: $a_n < b_n$ and $\lim a_n = \lim b_n = 0$.

CYU 9.4 Since $\lim\limits_{n\to\infty}\frac{n+1}{n} = 1$ (Example 9.1), and since both the square root and the logarithmic function are continuous on $(0, \infty)$, we have:

(a) $\lim\sqrt{\frac{n+1}{n}} = \sqrt{\lim\frac{n+1}{n}} = \sqrt{1} = 1$

(b) $\lim\left[\ln\left(\frac{n+1}{n}\right)\right] = \ln\left[\lim\left(\frac{n+1}{n}\right)\right] = \ln 1 = 0$

CYU 9.5 (a) For $a_n = n^{1/n}$: From Example 8.6(a), page 306: $\lim\limits_{x\to\infty} x^{1/x} = 1$. Consequently: $\lim\limits_{x\to\infty} n^{1/n} = 1$.

(b) For $a_n = \left(\frac{n+1}{n-1}\right)^n$: $\ln a_n = n\ln\left(\frac{n+1}{n-1}\right)$. Applying l'Hôpital's rule to the indeterminate form $x\ln\left(\frac{x+1}{x-1}\right)$ we have:

$$\lim_{\mathbf{x}\to\infty}\mathbf{x}\,\mathbf{ln}\left(\frac{\mathbf{x+1}}{\mathbf{x-1}}\right) = \lim_{x\to\infty}\frac{\left[\ln\left(\frac{x+1}{x-1}\right)\right]'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty}\frac{\frac{1}{\frac{x+1}{x-1}}\cdot\left(\frac{x+1}{x-1}\right)'}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\frac{x-1}{x+1}\cdot\frac{-2}{(x-1)^2}}{-x^{-2}}$$

$$= \lim_{x\to\infty}\frac{\frac{-2}{(x^2-1)}}{-\frac{1}{x^2}}$$

$$= 2\lim_{x\to\infty}\frac{x^2}{x^2-1} = \mathbf{2}$$

It follows that $\lim\limits_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = e^{\lim\limits_{n\to\infty}\ln\left(\frac{n+1}{n-1}\right)^n} = e^{\lim\limits_{n\to\infty}\mathbf{n}\,\mathbf{ln}\left(\frac{\mathbf{n+1}}{\mathbf{n-1}}\right)} = e^{\mathbf{2}}$.

CYU 9.6 (a-i) $\frac{a_{n+1}}{a_n} \geq 1 \Rightarrow a_{n+1} \geq a_n$ (a-ii) $\frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n$

(b) For $\left(\frac{e^n}{n!}\right)$: $\frac{a_{n+1}}{a_n} = \frac{(e^{n+1})/(n+1)!}{(e^n)/n!} = \frac{(e^{n+1})}{(n+1)!}\cdot\frac{n!}{e^n} = \frac{e}{n+1} < 1$ (for $n > 1$).

CYU 9.7 Assume that $\lim a_n = L$. Taking $\varepsilon = 1$ in Definition 9.1, we choose N such that $n > N \Rightarrow |a_n - L| < 1$. Then: $-1 + L < a_n < 1 + L$ for $n > N$; which, in turn, implies that $|a_n| < 1 + |L|$ for $n > N$. It follows that $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |L|\}$ is a bound for (a_n).

CYU 9.8 (One possible answer) $(a_n) = (0, 1, 0, 1, 0, 1, \ldots)$.

CYU 9.9 If $|r| > 1$ then $\lim\limits_{n \to \infty} r^n = \infty$ and the sequence (r^n) diverges. If $r = -1$, then $(r^n)_{n=1}^{\infty} = (-1, 1, -1, 1, \ldots)$ diverges (Exercise 63). If $r = 1$, then $(r^n) = (1, 1, 1, \ldots)$ converges (CYU 9.1).

The above facts, and Theorem 9.7, tell us that (r^n) converges if and only if $-1 < r \le 1$

CYU 9.10 For $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ we have:

(a) $s_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = 1 - \frac{1}{5} = \frac{4}{5}$.

(b) $s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.

(c) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty}\left(1 - \frac{1}{n+1}\right) = 1$. The series converges and its sum is 1.

CYU 9.11 (a) We first mold $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2}{3^n}$ into the form of Theorem 9.9 $\sum\limits_{n=1}^{\infty} ar^{n-1}$ and then go on from there:

$$\sum_{n=1}^{\infty}(-1)^n\frac{2}{3^n} = \sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)^{n-1} = \frac{-\frac{2}{3}}{1-\left(-\frac{1}{3}\right)} = -\frac{1}{2}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

(b)
$$\begin{aligned} 0.232323\ldots &= 0.23 + 0.0023 + 0.000023 + \cdots \\ &= 23\left(\frac{1}{100}\right) + 23\left(\frac{1}{100}\right)^2 + 23\left(\frac{1}{100}\right)^3 + \cdots \\ &= \sum_{n=1}^{\infty}\frac{23}{100}\left(\frac{1}{100}\right)^{n-1} = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{23}{99} \end{aligned}$$

CYU 9.12 From Example 9.7(c): $\sum\limits_{n=1}^{\infty}\frac{1}{3^n} = \frac{1}{2}$. Also $\sum\limits_{n=1}^{\infty}\frac{1}{2^n} = \sum\limits_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

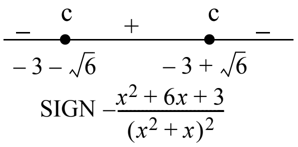
Thus: $\sum\limits_{n=1}^{\infty}\left[\frac{2}{3^n} + \frac{3}{2^n}\right] = 2\sum\limits_{n=1}^{\infty}\frac{1}{3^n} + 3\sum\limits_{n=1}^{\infty}\frac{1}{2^n} = 2\left(\frac{1}{2}\right) + 3(1) = 4$.

CYU 9.13 (a) Theorem 9.11 assures us that the alternating series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2}{3^n}$ converges: $\frac{2}{3^{n+1}} \le \frac{2}{3^n}$ since $3^n \le 3^{n+1}$, and $\lim\limits_{n \to \infty}\frac{2}{3^n} = 0$. Indeed, we were able to show in CYU 9.11(a) that $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2}{3^n} = -\frac{1}{2}$.

(b) For $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n+3}{n^2+n}$ we have:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty}\frac{n+3}{n^2+n} = \lim_{n \to \infty}\frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{1}{n}} \to \frac{0}{1} = 0.$$

To see that the condition $a_{n+1} \le a_n$ is satisfied we consider the derivative of the function $f(x) = \frac{x+3}{x^2+x}$: $f'(x) = \frac{(x^2+x)-(x+3)(2x+1)}{(x^2+x)^2} = -\frac{x^2+6x+3}{(x^2+x)^2}$. Noting that the denominator $(x^2+x)^2$ is never negative and that the numerator has zeros at $x = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-6 \pm \sqrt{36-12}}{2} = -3 \pm \sqrt{6}$, we find that f is **decreasing** to the right of $-3+\sqrt{6}$:



. It follows that $a_{n+1} \le a_n$ for all n. Thus the series converges by the Alternating Series Theorem.

CYU 9.14 (a) $\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{n!} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots$ satisfies the conditions of Theorem 9.11:

$$\frac{1}{(n+1)!} \le \frac{1}{n!}, \text{ and } \lim_{n \to \infty}\frac{1}{n!} = 0.$$

(b) You can easily verify that $\frac{1}{7!} < 0.0002$. It follows, from Theorem 9.12, that:

$$\left|\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!} - \left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)\right| < 0.0002$$

Noting that $1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!} \approx 0.368$, we conclude that, to within three decimal places: $\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{n!} \approx 0.368$.

CYU 9.15 As $f(x) = \frac{1}{x} > 0$ and decreasing for $x > 0$, the Integral Test applies:

Since $\int_1^{\infty} \frac{dx}{x} = \lim_{t \to \infty} \int_1^t \frac{dx}{x} = \lim_{t \to \infty} \ln|x| \Big|_1^t = \lim_{t \to \infty} (\ln t - \ln 1) = \lim_{t \to \infty} \ln t = \infty$,

$\sum \frac{1}{n}$ diverges.

CYU 9.16 (a) Since $\frac{1}{n^2 + n} < \frac{1}{n^2}$ and since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n^2 + n}$ converges, by the Comparison Test.

(b) Since $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ and since $\sum \frac{1}{\sqrt{n}}$ is a divergent p-series $(p = \frac{1}{2})$,

$\sum \frac{\sqrt{n}}{n-1}$ diverges.

CYU 9.17 (a) As $n \to \infty$: $\frac{1}{3^n - 100} \approx \frac{1}{3^n}$. Knowing that the geometric series $\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ converges we anticipate that $\sum \frac{1}{3^n - 100}$ will do the same. Let's make sure:

$$\lim_{n \to \infty} \frac{\frac{1}{3^n - 100}}{\frac{1}{3^n}} = \lim_{n \to \infty} \frac{3^n}{3^n - 100} = \lim_{n \to \infty} \frac{1}{1 - \frac{100}{3^n}} = 1 > 0$$

Conclusion: $\sum \frac{1}{3^n - 100}$ converges, by the Limit Comparison Test.

(b) As $n \to \infty$: $\frac{5\sqrt{n} + 100}{\sqrt{n^3 - 3n + 1}} \approx \frac{5\sqrt{n}}{\sqrt{n^3}} = \frac{5}{n^{\frac{3}{2} - \frac{1}{2}}} = \frac{5}{n} \approx \frac{1}{n}$. Knowing that the harmonic series $\sum \frac{1}{n}$ diverges we anticipate that $\sum \frac{5\sqrt{n} + 100}{\sqrt{n^3 - 3n + 1}}$ will do the same, and it does:

$$\lim_{n \to \infty} \frac{\frac{5\sqrt{n}+100}{\sqrt{n^3-3n+1}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n(5\sqrt{n}+100)}{\sqrt{n^3-3n+1}} = \lim_{n \to \infty} \frac{5n^{3/2}+100n}{\sqrt{n^3-3n+1}}$$

divide numertor and denominator by $n^{3/2}$: $$= \lim_{n \to \infty} \frac{5+\frac{100n}{n^{3/2}}}{\frac{\sqrt{n^3-3n+1}}{n^{3/2}}}$$

$n^{3/2} = \sqrt{n^3}$: $$= \lim_{n \to \infty} \frac{5+\frac{100n}{n^{3/2}}}{\sqrt{\frac{n^3-3n+1}{n^3}}}$$

$$= \lim_{n \to \infty} \frac{5+\frac{100n}{n^{3/2}}}{\sqrt{1-\frac{3}{n^2}+\frac{1}{n^3}}} = 5 > 0$$

Conclusion: $\sum \frac{5\sqrt{n}+100}{\sqrt{n^3-3n+1}}$ diverges, by the Limit Comparison Test.

CYU 9.18 (a) Assume that $\frac{a_{n+1}}{a_n} \to L > 1$ or $\frac{a_{n+1}}{a_n} \to \infty$. Let N be such that $\frac{a_{n+1}}{a_n} > 1$ for $n > N$. Since $a_{n+1} > a_n > 0$ for $n > N$, $\lim_{n \to \infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, by the Divergence Test.

(b) For the divergent series $\sum \frac{1}{n}$: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n+1} = 1$.

For the convergent series $\sum \frac{1}{n^2}$: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^2 = 1$.

CYU 9.19 (a) For $\sum \frac{n^3}{5^n}$: Since $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{(n+1)^3}{5^{n+1}}}{\frac{n^3}{5^n}} = \lim_{n \to \infty} \frac{1}{5}\left(\frac{n+1}{n}\right)^3 = \frac{1}{5} < 1$, $\sum \frac{n^3}{5^n}$ converges, by the Ratio Test.

(b) For $\sum \frac{(2n)!}{(n!)^2}$: Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{[2(n+1)]!}{[(n+1)!]^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n!)}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = 4 > 1$$

$\sum \frac{(2n)!}{(n!)^2}$ diverges, by the Ratio Test.

<u>CYU 9.20</u> (a) Assume that $L > 1$, and let $\varepsilon > 0$ be small enough so that $L - \varepsilon > 1$. Since $\sqrt[n]{a_n} \to L$, we can choose N such that $\sqrt[n]{a_n} > L - \varepsilon$ or $a_n > (L-\varepsilon)^n$ for $n > N$.

Since $L - \varepsilon > 1$, the geometric series $\sum\limits_{n=N+1}^{\infty} (L-\varepsilon)^n$ diverges. By the Comparison Test, so must $\sum a_n$, since eventually $a_n > (L-\varepsilon)^n$.

If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \infty$, then $a_n \to \infty$, and the series diverges by the Divergence Test.

(b) $$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{(n^{1/n})^p} = \frac{1}{\lim\limits_{n\to\infty} (n^{1/n})^p} \underset{\substack{\uparrow \\ \text{CYU 9.5(a), page 327}}}{=} \frac{1}{1^p} = 1$$

(c) Both $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = 1$ and $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1$ [see (b)]. $\sum \frac{1}{n^2}$ converges while $\sum \frac{1}{n}$ diverges.

<u>CYU 9.21</u> (a) Applying the Root Test we find that $\sum \left(\frac{3n+2}{2n+1}\right)^n$ diverges:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+2}{2n+1}\right)^n} = \lim_{n\to\infty} \left(\frac{3n+2}{2n+1}\right) = \frac{3}{2} > 1$$

(b) Applying the Root Test we find that $\sum \frac{1}{(\ln n)^n}$ converges:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$$

CYU 9.22 (a) Since $\left|\frac{n^2\cos n}{3^n}\right| \le \frac{n^2}{3^n}$, if we can show that $\sum \frac{n^2}{3^n}$ converges, it will then follow that $\sum \frac{n^2\cos n}{3^n}$ converges absolutely; and we can, via the Ratio Test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^2 = \frac{1}{3} < 1$$

(b) We could apply the Ratio Test to $\sum \frac{n^n}{n!}$ to show that $\sum (-1)^n \frac{n^n}{n!}$ fails to converge absolutely. That conclusion, however, follows directly from the fact that $\sum (-1)^n \frac{n^n}{n!}$ does not even converge conditionally by the Divergence Test:

$$\frac{n^n}{n!} = \overset{1}{\frac{n}{n}} \cdot \overbrace{\frac{n}{n-1} \cdot \frac{n}{n-2} \cdots \frac{n}{2} \cdot \frac{n}{1}}^{\text{all greater than 1}} > 1$$

CYU 9.23 (a) Observe that the absolute value of each element in the two series composed of the positive and the negative terms of the given series

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{16} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} - \frac{1}{2^{n+3}} + \ldots$$

are elements of the converging p-series $\sum \frac{1}{2^p}$. As such both of those positive series converge. Employing Theorem 9.20 we conclude that the given series converges absolutely, and therefore converges.

(b) The series of positive terms of $\frac{1}{3} - 1 + \frac{1}{3^2} - \frac{1}{2} + \frac{1}{3^3} - \frac{1}{3} + \ldots + \frac{1}{3^n} - \frac{1}{n} + \ldots$ is the convergent p-series $\sum \frac{1}{3^p}$. The series of negative terms $-\sum \frac{1}{n}$ diverges (negative of the harmonic series). That being the case, Theorem 9.20, is of no help to us; but Theorem 9.10, page 336, which tells us that if $\sum a_n$ and $\sum b_n$ converge then so must $\sum (a_n - b_n)$ converge can save the day. How? like this:

> **Assume** that the given series which we will now label $\sum a_n$ converges. Since the series $\sum b_n = \sum \frac{1}{3^p}$ converges, their difference $\sum (a_n - b_n)$ would have to converge. But it doesn't since $\sum (a_n - b_n) = -\sum \frac{1}{n}$.
>
> Conclusion: the given series diverges.

CYU 9.24 (a) Since, for $\sum a_n = \sum (-1)^n \frac{2^{2n-50}}{2n-100}$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{2^{2n+2-50}}{2n+2-100}}{\frac{2^{2n-50}}{2n-100}} = \lim_{n\to\infty}\frac{2^{2n-48}}{2n-98}\cdot\frac{2n-100}{2^{2n-50}}$$

$$= \lim_{n\to\infty} 2^2\cdot\frac{2n-100}{2n-98} = 4 > 1$$

the series $\sum (-1)^n \frac{2^{2n-50}}{2n-100}$ diverges, by the Ratio Test.

(b) Since, for $\sum a_n = \sum (-1)^n \frac{n^2}{2^n}$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$$

the series $\sum (-1)^n \frac{n^2}{2^n}$ converges absolutely by the Ratio Test, and therefore converges.

CYU 9.25 Consider: (i) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$ and (ii) $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \ldots$. Start with $1 - \frac{1}{2}$ and add enough of the terms of (i) to arrive at a sum $S_2 > 2$. Add $\left(-\frac{1}{2}\right)$ to S_2 along with enough of the remaining terms of (i) to arrive at a sum $S_3 > 3$. Add $\left(-\frac{1}{4}\right)$ to S_3 along with enough of the remaining terms of (i) to arrive at a sum $S_4 > 4$. Continuing in this manner one arrives at a rearrangement of the original series which diverges to ∞.

An initial impression might be that we are adding a lot more positive then negative terms of the original series. Not so. All terms of the original series will show up in the above rearrangement. Think about it.

CYU 9.26 (a) For $\sum a_n = \sum \frac{x^n}{n}$: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}|x|\left(\frac{n}{n+1}\right) = |x|$.

It follows, from the Ratio Test, that $\sum \frac{x^n}{n}$ converges absolutely for $-1 < x < 1$, so $R = 1$. As for the endpoints, we note that at $x = -1$ the series $\sum \frac{x^n}{n} = \sum \frac{(-1)^n}{n}$ converges (alternating harmonic series), and that it diverges at $x = 1$: $\sum \frac{x^n}{n} = \sum \frac{1}{n}$ (harmonic series). Conclusion: $\sum \frac{x^n}{n}$ has a radius of convergence of 1 and its interval of convergence is $[-1, 1)$.

(b) For $\sum a_n = \sum n!x^n$:

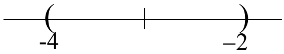
$$\lim_{n \to \infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n \to \infty} |x|(n+1) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0 \end{cases}$$

Conclusion: $\sum n!x^n$ has a radius of convergence of 0 and its interval of convergence is $\{0\}$.

(c) For $\sum \frac{(x+3)^n}{n-2}$:

$$\lim_{n \to \infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty}\left|\frac{\frac{(x+3)^{n+1}}{(n+1)-2}}{\frac{(x+3)^n}{n-2}}\right| = \lim_{n \to \infty}\left|\frac{(x+3)^{n+1}}{n-1} \cdot \frac{n-2}{(x+3)^n}\right| = \lim_{n \to \infty} |x+3|$$

The Ratio Test assures us that $\sum \frac{(x+3)^n}{n-2}$ converges (absolutely) when $|x+3| < 1$ and diverges when $|x+3| > 1$. It follows that the series has radius of convergence $R = 1$.

Here is the interval $|x+3| < 1$: 

. Challenging the endpoints for convergence we find that $\sum \frac{(x+3)^n}{n-2}$ converges at $x = -4$ and diverges at $x = -2$:

$$\sum \frac{(-4+3)^n}{n-2} = \sum \frac{(-1)^n}{n-2} \text{ — converging alternating harmonic series.}$$

$$\sum \frac{(-2+3)^n}{n-2} = \sum \frac{1}{n-2} \text{ — diverging harmonic series.}$$

Conclusion: $\sum \frac{(x+3)^n}{n-2}$ has a radius of convergence of 1 and its interval of convergence is $[-4, -2)$.

CYU 9.27 $f''(x) = \left[\left(\frac{1}{1-x}\right)'\right]' = ([(1-x)^{-1}]')' = \underset{(*)}{[(1-x)^{-2}]'} = 2(1-x)^{-3} = \frac{2}{(1-x)^3}$.

Applying Theorem 9.26 to (*) we have:

$$[(1-x)^{-2}]' = \sum_{n=1}^{\infty} (nx^{n-1})' = \sum_{n=2}^{\infty} (n-1)nx^{n-2}$$

So, for $|x| < 1$: $\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \dots$.

$$= 2 + 6x + 12x^2 + 20x^3 + \dots$$

CYU 9.28 (a) We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ [Example 9.20(a)]. Replacing x with $1-x$ we have:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}[(-1)(x-1)]^n$$

$$= \sum_{n=0}^{\infty}(-1)^n(x-1)^n, \quad \text{for } |x-1| < 1$$

(b) Starting with $\frac{1}{x^2} = -\left(\frac{1}{x}\right)'$, we turn to (a) and Theorem 9.26 to arrive at:

$$\frac{1}{x^2} = -\left(\sum_{n=0}^{\infty}(-1)^n(x-1)^n\right)' = -\sum_{n=0}^{\infty}(-1)^n n(x-1)^{n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+2}(n+1)(x-1)^n$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \ldots$$

CYU 9.29 For $f(x) = \ln(1-x)$ we have:

$$f(x) = \ln(1-x) \qquad c_0 = \frac{f(0)}{0!} = \frac{0}{0!} = 0$$

$$f'(x) = \frac{-1}{1-x} = -(1-x)^{-1} \qquad c_1 = \frac{f'(0)}{1!} = \frac{-1}{1!} = -1$$

$$f''(x) = -(1-x)^{-2} \qquad c_2 = \frac{f''(0)}{2!} = \frac{-1}{2!} = -\frac{1}{2}$$

$$f^{(3)} = -2(1-x)^{-3} \qquad c_3 = \frac{f^{(3)}(0)}{3!} = \frac{-2}{3!} = -\frac{1}{3}$$

Pattern: $c_n = -\frac{1}{n}$ for $n \geq 1$, and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ becomes:

$$f(x) = \ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n} \text{ (note that } c_0 = 0\text{)}.$$

Moreover, since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}|x| \to |x|$ as $n \to \infty$, the Ratio Test tells us that the above power series representation holds for $|x| < 1$.

CYU 9.30 (a) For $f(x) = \cos x$ we have

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos(0) = \mathbf{1} \\ f'(x) &= -\sin x & f'(0) &= -\sin(0) = \mathbf{0} \\ f''(x) &= -\cos x & f''(0) &= -\cos(0) = \mathbf{-1} \\ f^{(3)}(x) &= \sin x & f^{(3)}(0) &= \sin(0) = \mathbf{0} \end{aligned}$$

Since $f^{(4)}(x) = \cos x = f(x)$, the above value-pattern of **1**, **0**, **−1**, **0** will keep repeating; bringing us to the Mclaurin series of the cosine function:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{\mathbf{1}}{0!}x^0 + \frac{\mathbf{0}}{1!}x^1 + \frac{\mathbf{-1}}{2!}x^2 + \frac{\mathbf{0}}{3!}x^3 + \frac{\mathbf{1}}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Which is seen to converge (absolutely) for all x:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2(n+1)}}{[2(n+1)]!}}{\frac{x^{2n}}{(2n)!}}\right| = \frac{(2n)!}{(2n+2)!}\left|\frac{x^{2n+2}}{x^{2n}}\right|$$

$$= \frac{1}{(2n+1)(2n+2)}|x^2| \to 0 < 1 \text{ as } n \to \infty$$

(for all x)

Since, for all x and n, $\underset{\text{the } M \text{ in Theorem 9.31}}{|f^{(n)}(x)| \le \underset{\uparrow}{1}}$: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(b) For $f(x) = \sin x$ we have

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = \mathbf{1} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) = \mathbf{0} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) = \mathbf{-1} \\ f^{(3)}(x) &= -\cos x & f^{(3)}\left(\frac{\pi}{2}\right) &= -\cos\left(\frac{\pi}{2}\right) = \mathbf{0} \end{aligned}$$

Since $f^{(4)}(x) = \sin x = f(x)$, the above value-pattern of **1**, **0**, **−1**, **0** will keep repeating. Bringing us to the Taylor series of the sine function about $\frac{\pi}{2}$:

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/2)}{n!}\left(x-\frac{\pi}{2}\right)^n = \frac{1}{0!}\left(x-\frac{\pi}{2}\right)^0+\frac{0}{1!}\left(x-\frac{\pi}{2}\right)^1+\frac{-1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{0}{3!}\left(x-\frac{\pi}{2}\right)^3+\cdots$$

$$= 1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}$$

Incidently, it follows that: $\sin x \underset{\substack{\uparrow\\ \text{Theorem 9.32(ii)}}}{=} \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \underset{\downarrow}{=} \sum_{n=0}^{\infty}(-1)^n\frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}$

follows also from Theorem 9.32(iii) and the fact that $\sin x = \cos\left(x-\frac{\pi}{2}\right)$

(c)

$$(\sin x)' = \left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right]' = \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)'$$

Theorem 9.26, page 367: $= 1-\frac{3x^2}{3!}+\frac{5x^4}{5!}-\frac{7x^6}{7!}+\ldots$

$$= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots = \cos x$$

CYU 9.31 Starting with $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, we have $e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^nx^n}{n!}$. Employing Theorem 9.10, page 336: $f(x) = e^x+2e^{2x} = \sum_{n=0}^{\infty}\frac{x^n}{n!}+2\sum_{n=0}^{\infty}\frac{2^nx^n}{n!} = \sum_{n=0}^{\infty}\frac{(2^{n+1}+1)x^n}{n!}$.

CYU 9.32 (The binomial Theorem):

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n\sum_{k=0}^{\infty}\binom{n}{k}\left(\frac{b}{a}\right)^k = \sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k \underset{\uparrow}{=} \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots\overset{\mathbf{0}}{(\boldsymbol{n-n})}\ldots(n-k+1)}{k!} = \mathbf{0}\text{, for } k>n$$

CYU 9.33 Noting that for $0\le x\le 4$: $0\le e^x\le e^4$ and that $|x-2|\le 2$, we invoke Taylor's Inequality and set our sights on finding the smallest N for which:

$$|E_N(x)|\le\frac{e^4}{(N+1)!}2^{N+1}\le 0.0001$$

Turning to a calculator we found that while $\frac{e^4}{(N+1)!}2^{N+1} > 0.0001$ for $N \le 11$; at $N = 12: \frac{e^4}{(12+1)!}2^{12+1} \approx 0.00007 < 0.0001$.

Conclusion: Thirteen terms are needed. Then:

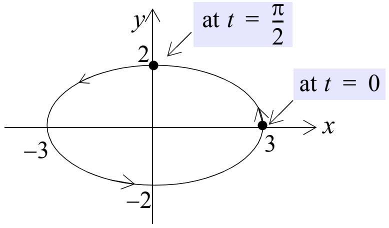
$$\left| e^x - e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots + \frac{(x-2)^{12}}{12!}\right)\right| < 0.0001 \text{ for } 0 \le x \le 4.$$

CHAPTER 10:
PARAMETRIZATION OF CURVES AND POLAR COORDINATES

CYU 10.1 From $x = 3\cos t,\ y = 2\sin t$: $\cos t = \frac{x}{3},\ \sin t = \frac{y}{2} \Rightarrow \cos^2 t = \frac{x^2}{9},\ \sin^2 t = \frac{y^2}{4}$

Employing the Pythagorean Identity: $\cos^2 t + \sin^2 t = 1$ (Theorem 1.5(i), page 37), we arrive at the rectangular equation:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1:$$



CYU 10.2 Setting $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t} = \frac{3(t+1)(t-1)}{2t}$ to 0, we conclude that a horizontal tangent line occurs when $t = \pm 1$. Turning to $x(t) = t^2,\ y(t) = t^3 - 3t$, we find the corresponding points on the curve namely:

$$[(-1)^2, (-1)^3 - 3(-1)] = (1, 2) \text{ and } [(1)^2, (1)^3 - 3(1)] = (1, -2)$$

Noting that the denominator in the above expression for $\frac{dy}{dx}$ is zero at $t = 0$, and the numerator is not zero at $t = 0$, we conclude that a vertical tangent line occurs at the point $(x(0), y(0)) = (0, 0)$.

As for concavity:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{3t^2-3}{2t}\right)'}{(t^2)'} = \frac{\frac{3}{2}(t-t^{-1})'}{2t} = \frac{3\left(1+\frac{1}{t^2}\right)}{4t} = \frac{3(t^2+1)}{4t^3}$$

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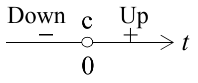
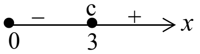


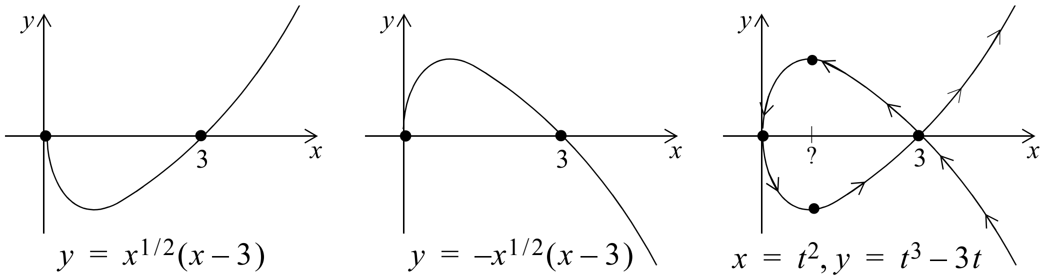
CYU 10.3 From $x = t^2$: $t = \pm\sqrt{x}$. Substituting in $y = t^3 - 3t$, one option at a time:

(1) For $t = x^{1/2}$: $y = (x^{1/2})^3 - 3x^{1/2} = x^{3/2} - 3x^{1/2} = x^{1/2}(x-3)$.

Domain of: $f(x) = x^{1/2}(x-3)$: $[0, \infty)$. SIGN f: 

. As $x \to \infty$ $f(x)$ will resemble, in shape, that of $y = x^{3/2}$ — all of which brings us to the anticipated graph in (a) below.

(2) For $t = -x^{1/2}$: $y = (-x^{1/2})^3 - 3(-x^{1/2}) = -x^{3/2} + 3x^{1/2}$. Its graph, which is simply the "negative" of the one in (a) appears in (b) below.

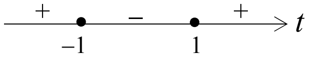


$y = x^{1/2}(x-3)$ $y = -x^{1/2}(x-3)$ $x = t^2, y = t^3 - 3t$

(a) (b) (c)

We merged (a) and (b) to arrive at the curve in (c). As for its indicated direction:

$$\frac{dy}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$$

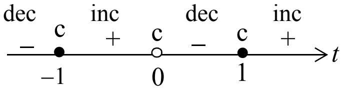
SIGN: 

↓

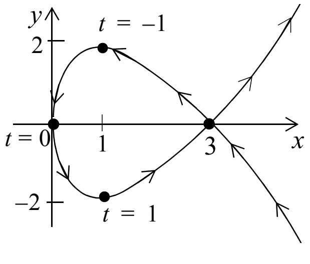
as t increases from $-\infty$ to -1, y increases to 2
for $-1 < t < 1$, y decreases, and then increases again.

We could use the calculus to challenge the functions in (a) and (b), but choose, instead, to analyze the directed curve in (c).

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3(t+1)(t-1)}{2t}$$

SIGN: 

$x = t^2$: $x = 1$ $x = 1$



Something "special" happens at $t = \pm 1$; which is to say, at $x = (\pm 1)^2 = 1$. Tracing the curve in (c) we see that a maximum occurs when $t = -1$, and a minimum at $t = 1$. Indeed, from the above SIGN-chart we see that the slope is positive for t between -1 and 0, and negative for $t < -1$, indicating that a maximum occurs when $t = -1$. Similarly, the SIGN-chart tells us that between $t = 0$ and $t = 1$ the slope is negative, and that it is positive for $t > 1$, indicating that a minimum occurs at $t = 1$.

CYU 10.4 For $x = t^3 - t^2,\ y = t^2e^t + 3$: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{t^2e^t + 2te^t}{3t^2 - 2t} = \dfrac{te^t(t+2)}{t(3t-2)} = \dfrac{e^t(t+2)}{3t-2}$.

At this point we know that a horizontal tangent line occurs when $t = -2$; which is to say, at $(x, y) = \left(-12, \dfrac{4}{e^2} + 3\right)$. Turning to the second derivative we have:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{te^t + 2e^t}{3t-2}\right]}{\frac{d}{dt}(t^3 - t^2)} = \frac{\frac{(3t-2)(te^t + e^t + 2e^t) - 3(te^t + 2e^t)}{(3t-2)^2}}{3t^2 - 2t}$$

$$= \frac{e^t(3t^2 + 4t - 12)}{t(3t-2)^3} \longleftarrow \boxed{\text{using the quadratic formula: } t = \frac{-2 \pm 2\sqrt{10}}{3}}$$

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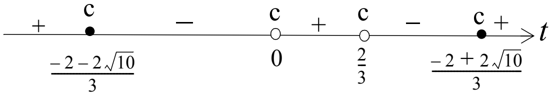


From the above sign information we conclude that a local maximum occurs when $t = -2$ [at $(x, y) = \left(-12, \dfrac{4}{e^2} + 3\right) \approx (-12, 3.54)$], and that inflection points occur when:

$t = 0$ $(0, 3)$

$t = \dfrac{2}{3}$ $\approx(-0.15, 3.9)$

$t = \dfrac{-2-2\sqrt{10}}{3}$ $\approx(-29.1, 3.5)$

$t = \dfrac{-2+2\sqrt{10}}{3}$ $\approx(0.9, 11.8)$

CYU 10.5 In integral form we have $L = \displaystyle\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

From $x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$ we have:

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (-3\sin t + 3\sin 3t)^2 + (3\cos t - 3\cos 3t)^2 \\ &= 9[(-\sin t + \sin 3t)^2 + (\cos t - \cos 3t)^2] \\ &= 9(\sin^2 t - 2\sin t\sin 3t + \sin^2 3t + \cos^2 t - 2\cos t\cos 3t + \cos^2 3t) \\ &= 9[(\sin^2 t + \cos^2 t) + (\sin^2 3t + \cos^2 3t) - 2(\mathbf{\cos 3t\cos t + \sin 3t\sin t})] \\ &= 9[1 + 1 - 2\mathbf{\cos(3t - t)}] = 9(2 - 2\cos 2t) = 18(1 - \cos 2t)\end{aligned}$$

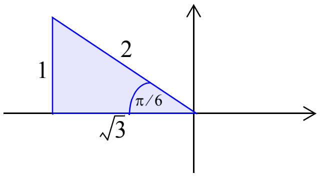
Bringing us to:

$$\begin{aligned}L = \int_0^{\pi}\sqrt{18(1-\cos 2t)}\,dt &= \sqrt{18}\int_0^{\pi}\sqrt{(1 - (\cos^2 t - \sin^2 t))}\,dt \\ &= 3\sqrt{2}\int_0^{\pi}\sqrt{(1 - \cos^2 t) + \sin^2 t}\,dt \\ &= 3\sqrt{2}\int_0^{\pi}\sqrt{2\sin^2 t}\,dt = 6\int_0^{\pi}\sin t\,dt = 6(-\cos t|_0^{\pi}) \\ &= -6(\cos\pi - \cos 0) = 12\end{aligned}$$

CYU 10.6 (a) Let $P = \left(-2, -\frac{\pi}{6}\right)$. Turning to the equations $x = r\cos\theta,\ y = r\sin\theta$:

$$x = -2\cos\left(-\frac{\pi}{6}\right) = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

$$y = -2\sin\left(-\frac{\pi}{6}\right) = -2\left(-\frac{1}{2}\right) = 1$$

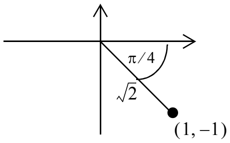


we find that P has rectangular coordinates $(-\sqrt{3}, 1)$.

(b) Let $P = (1, -1)$. From $r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$, we have:

$$r^2 = (1)^2 + (1)^2 = 2 \text{ or } r = \sqrt{2}, \quad \text{and} \quad \tan\theta = \frac{-1}{1} = -1$$

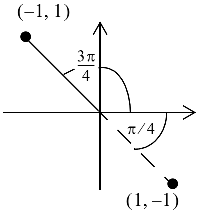
Since $\tan\left(-\frac{\pi}{4} + 2k\pi\right) = -1$ for any integer k, all of the following are polar representations of the point P:



$$P = \left(\sqrt{2}, -\frac{\pi}{4} + 2k\pi\right) \text{ for any integer } k.$$

In addition (see comments directly below Figure 10.4):

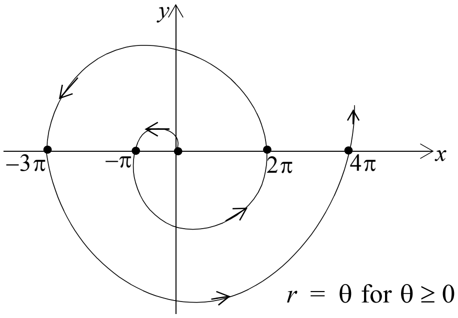
$$P = \left(-\sqrt{2}, \frac{3\pi}{4} + 2k\pi\right) \text{ for any integer } k.$$



CYU 10.7 The adjacent spiral figure pretty much speaks for itself. As the angle θ gets larger and larger, so does $r = \theta$. In particular:

When $\theta = 0$, $r = 0$, and $(0, 0)$ is on the curve.

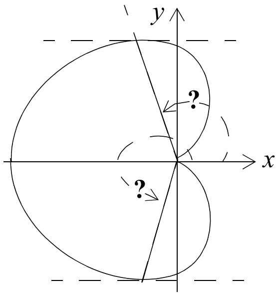
When $\theta = \pi$, $r = \pi$, and $(0, -\pi)$ lies on the curve (mark off π units on the terminal side of the angle $\theta = \pi$ in standard position).



$r = \theta$ for $\theta \geq 0$

Note that the curve intersects the y-axis when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, with corresponding $y = r$-values: $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

CYU 10.8 The adjacent curve of $r = f(\theta) = 1 - \cos\theta$ displays two local extreme points. Let's find them:

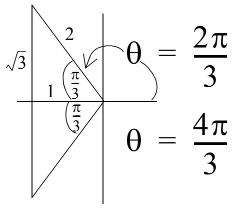


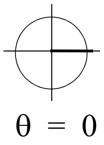
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\sin\theta\sin\theta + (1 - \cos\theta)\cos\theta}{\sin\theta\cos\theta - (1 - \cos\theta)\sin\theta}$$

$$= \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{2\sin\theta\cos\theta - 1}$$

Intent on finding where $\frac{dy}{dx} = 0$, we determine where the numerator is 0:

$$\begin{aligned}
\sin^2\theta + \cos\theta - \cos^2\theta &= 0\\
(1-\cos^2\theta) + \cos\theta - \cos^2\theta &= 0\\
2\cos^2\theta - \cos\theta - 1 &= 0\\
(2\cos\theta + 1)(\cos\theta - 1) &= 0 \Rightarrow \cos\theta = -\frac{1}{2} \quad \text{and} \quad \cos\theta = 1
\end{aligned}$$





As for the top question mark in the above figure:

$$\theta = \frac{2\pi}{3} \text{ with}$$

$$r = 1 - \cos\frac{2\pi}{3} = \frac{3}{2}$$

The bottom question mark:

$$\theta = \frac{4\pi}{3} \text{ with}$$

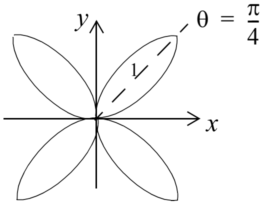
$$r = 1 - \cos\frac{4\pi}{3} = \frac{3}{2}$$

You can use the bridges $x = r\cos\theta$, $y = r\sin\theta$ to find the rectangular coordinates of those points. The rectangular coordinates of the local maximum $\left(\frac{3}{2}, \frac{2\pi}{3}\right)$:

$$(x, y) = \left(\frac{3}{2}\cos\frac{2\pi}{3}, \frac{3}{2}\sin\frac{2\pi}{3}\right) = \left[\frac{3}{2}\left(-\frac{1}{2}\right), \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = \left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$$

By symmetry, we conclude that $\left(-\frac{3}{4}, -\frac{3\sqrt{3}}{4}\right)$ are the rectangular coordinates of the local minimum $\left(\frac{3}{2}, \frac{4\pi}{3}\right)$.

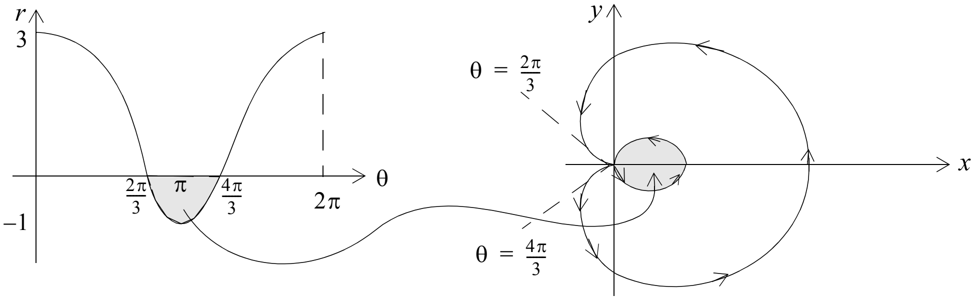
CYU 10.9 (a) Taking advantage of symmetry, we quadruple the area of the leaf in the first quadrant:



$$A = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 2\theta)^2 d\theta = 2\int_0^{\frac{\pi}{2}} \sin^2(2\theta) d\theta$$

$$\begin{aligned}
\begin{array}{r}\phi = 2\theta\\ d\phi = 2d\theta\end{array} \quad \theta = 0 \Rightarrow \phi = 0, \theta = \frac{\pi}{2} \Rightarrow \phi = \pi: \quad &= \int_0^{\pi} \sin^2(\phi) d\phi\\
\text{Theorem 1.5(iiiv), page 37:} \quad &= \frac{1}{2}\int_0^{\pi} (1 - \cos 2\phi) d\varphi\\
&= \frac{1}{2}\left(\phi - \frac{\sin 2\phi}{2}\right)\Bigg|_0^{\pi} = \frac{\pi}{2}
\end{aligned}$$

(b) As you can see from the construction of the curve below, the loop in question is traced out as θ runs from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$.



Taking advantage of symmetry, we choose to double the area of the inner loop lying below the x-axis (as θ runs from $\frac{2\pi}{3}$ to π):

$$A = 2\int_{\frac{2\pi}{3}}^{\pi}\left[\frac{1}{2}(2\cos\theta+1)^2\right]d\theta = \int_{\frac{2\pi}{3}}^{\pi}(4\cos^2\theta+4\cos\theta+1)d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\pi}\left(4\cdot\frac{1+\cos 2\theta}{2}+4\cos\theta+1\right)d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\pi}(2+2\cos 2\theta+4\cos\theta+1)d\theta$$

$$= (3\theta+\sin 2\theta+4\sin\theta)\Big|_{\frac{2\pi}{3}}^{\pi}$$

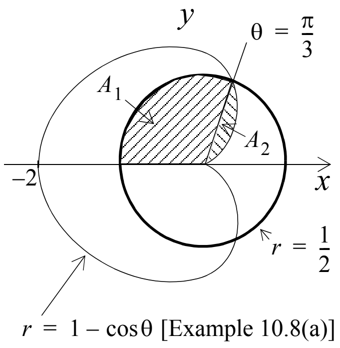
$$= 3\pi-\left(2\pi-\frac{\sqrt{3}}{2}+2\sqrt{3}\right) = \pi-\frac{3\sqrt{3}}{2}$$

CYU 10.10 By symmetry, the area in question is two times the shaded region in the adjacent figure:

$$r = 1-\cos\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$



$$A = 2\left[\overset{A_2}{\int_0^{\frac{\pi}{3}}\frac{1}{2}(1-\cos\theta)^2d\theta} + \overset{A_1}{\int_{\frac{\pi}{3}}^{\pi}\frac{1}{2}\left(\frac{1}{2}\right)^2d\theta}\right]$$

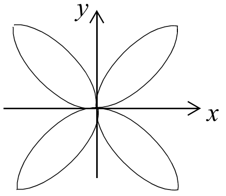
$$= \int_0^{\frac{\pi}{3}}(1-2\cos\theta+\cos^2\theta)d\theta + \int_{\frac{\pi}{3}}^{\pi}\frac{1}{4}d\theta$$

$$= \left((\theta-2\sin\theta)\Big|_0^{\frac{\pi}{3}}\right) + \int_0^{\frac{\pi}{3}}\frac{1+\cos 2\theta}{2}d\theta + \left(\frac{\theta}{4}\Big|_{\frac{\pi}{3}}^{\pi}\right)$$

$$= \frac{\pi}{3}-2\sin\frac{\pi}{3}+\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{3}}+\frac{\pi}{6}$$

$$= \frac{\pi}{3}-\sqrt{3}+\left(\frac{\pi}{6}+\frac{\sqrt{3}}{8}\right)+\frac{\pi}{6} = \frac{2\pi}{3}-\frac{7\sqrt{3}}{8}$$

CYU 10.11 Taking advantage of symmetry, we quadruple the length of the leaf in the first quadrant:



$$L = 4\int_0^{\frac{\pi}{2}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta = 4\int_0^{\frac{\pi}{2}}\sqrt{\sin^2 2\theta+(2\cos 2\theta)^2}d\theta \approx 9.69$$

ADDITIONAL THEORETICAL DEVELOPMENT

THEOREM 8.1, PAGE 301

Theorem 8.1 L'Hôpital's Rule: "$0/0$" type	Let c be a real number, or $\pm\infty$. Assume that, apart from c, f and g are differentiable on an open interval (a, b) containing c with $g'(x) \neq 0$. If $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = 0$ and if $\lim\limits_{x \to c} \frac{f'(x)}{g'(x)} = L$ Then: $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$

The following result will be used in the proof of the above theorem:

GENERALIZED MEAN VALUE THEOREM	If f and g are continuous on $[a, b]$ and differentiable on (a, b) and if $g'(x) \neq 0$ for $x \in (a, b)$, then there exists $d \in (a, b)$ for which: $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$ (Reduces to the Mean Value Theorem of page 121 if $g(x) = x$)

PROOF: The Mean Value Theorem assures us that since $g'(x) \neq 0$ on (a, b), $g(b) - g(a) \neq 0$. That being the case, we turn our attention to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

Noting that F satisfies the conditions of Rolle's Theorem (page 121) we conclude that $F'(c) = 0$ for some $c \in (a, b)$. Turning to $F'(x)$ we have:

$$F'(x) = \left(f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]\right)' = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(x)$$

Consequently, for some $c \in (a, b)$:

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) = 0; \text{ or: } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

PROOF (of Theorem 8.1): Let us first consider the case where c is a real number. Since the statement of the theorem does not assure us that either f or g is defined at c, we introduce functions F and G which agree with f and g away from c, and are continuous on (a, b):

$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$$

We now show that $\lim\limits_{x \to c^+} \frac{f(x)}{g(x)} = L$ (a similar argument can be used to show that $\lim\limits_{x \to c^-} \frac{f(x)}{g(x)} = L$):

Noting that F and G are continuous on $[a, x]$ for $a < x < c$, and that they are differentiable on $[c, x]$, we apply the Generalized Mean Value Theorem to find y with $a < y < x$ for which:

$$\frac{F'(y)}{G'(y)} = \frac{F(b)-F(a)}{G(b)-(Ga)}$$

Taking into account the fact that for $x \neq c$: $F(x) = f(x)$, $G(x) = g(x)$, and that as x approaches c from the right so must y (since $a < y < x$) we have:

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{F(x)}{G(x)} = \lim_{y \to c^+} \frac{F(x)}{G(x)} = \lim_{y \to c^+} \frac{f'(x)}{g'(x)} = L$$

Turning to the case $c = \infty$, we introduce a new variable t such that $x = \frac{1}{t}$. Then:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0^+} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}$$

by previous argument: $$= \lim_{t \to 0^+} \frac{\left[f\left(\frac{1}{t}\right)\right]'}{\left[g\left(\frac{1}{t}\right)\right]'} = \lim_{t \to 0^+} \frac{f'\left(\frac{1}{t}\right) \cdot \left(\frac{1}{t}\right)'}{g'\left(\frac{1}{t}\right) \cdot \left(\frac{1}{t}\right)'} = \lim_{t \to 0^+} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}$$

as $t \to 0^+, x \to \infty$ and $x = \frac{1}{t}$ $$= \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$$

A similar argument can be used for the case $c = -\infty$.

THEOREM 9.2 (C) AND (D), PAGE 323

Theorem 9.2 (c) and (d)

If $\lim a_n = A$ and $\lim b_n = B$, then:

(c) $\lim (a_n b_n) = AB$

(d) $\lim \frac{a_n}{b_n} = \frac{A}{B}$, providing no $b_n = 0$ and $B \neq 0$.

PROOF: (c) Let $\varepsilon > 0$. We are to find N such that:

$$n > N \Rightarrow |a_n b_n - \alpha\beta| < \varepsilon$$

In order to get $|a_n - \alpha|$ and $|b_n - \beta|$ into the picture (for we have control over those two expressions), we insert the clever zero $-\boldsymbol{a_n}\beta + \boldsymbol{a_n}\beta$ in the expression $|a_n b_n - \alpha\beta|$:

$$\begin{aligned} |a_n b_n - \alpha\beta| &= |a_n b_n - \boldsymbol{a_n}\beta + \boldsymbol{a_n}\beta - \alpha\beta| \\ &= |(a_n b_n - a_n\beta) + (a_n\beta - \alpha\beta)| \\ &\leq |a_n b_n - a_n\beta| + |a_n\beta - \alpha\beta| = \underbrace{|a_n||b_n - \beta|}_{\textbf{(i)}} + \underbrace{|\beta||a_n - \alpha|}_{\textbf{(ii)}} \end{aligned}$$

The next step is to find an N such that **both** (i) and (ii) are less than $\frac{\varepsilon}{2}$.

Focusing on (i): $|a_n||b_n-\beta|$:

The temptation is to let N_β be such that $n > N_\beta \Rightarrow |b_n-\beta| < \frac{\varepsilon}{2|a_n|}$ (yielding $|a_n||b_n-\beta| < |a_n|\frac{\varepsilon}{2|a_n|} = \frac{\varepsilon}{2}$). No can do. For one thing, if $a_n = 0$, then the expression $\frac{\varepsilon}{2|a_n|}$ is undefined. More importantly: $\frac{\varepsilon}{2|a_n|}$ is **NOT A CONSTANT**! We can, however, take advantage of the fact that there exists an $M > 0$ such that $|a_n| < M$ for every n (CYU 9.7, page 328), and choose N_β such that $n > N_\beta \Rightarrow |b_n-\beta| < \frac{\varepsilon}{2M}$. Then: $n \ge N_\beta \Rightarrow |a_n||b_n-\beta| < M\frac{\varepsilon}{2M} = \frac{\varepsilon}{2}$.

Focusing on (ii): $|\beta||a_n-\alpha|$.

Wanting $|\beta||a_n-\alpha|$ to be less than $\frac{\varepsilon}{2}$, one might be tempted to choose N_α such that $n > N_\alpha \Rightarrow |a_n-\alpha| < \frac{\varepsilon}{2|\beta|}$. But what if $\beta = 0$? To get around this potential problem we choose N_α such that $n > N_\alpha \Rightarrow |a_n-\alpha| < \frac{\varepsilon}{2|\beta|+1}$. No problem now:

$$n \ge N_\alpha \Rightarrow |\beta||a_n-\alpha| < |\beta|\frac{\varepsilon}{2|\beta|+1} \underset{\uparrow \text{ since } \frac{|\beta|}{2|\beta|+1} < \frac{|\beta|}{2|\beta|}}{<} \frac{\varepsilon}{2}$$

Letting $N = \max\{N_\alpha, N_\beta\}$, we see that, for $n > N$:

$$|a_nb_n-\alpha\beta| \le |a_n||b_n-\beta| + |\beta||a_n-\alpha| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

(d) Appealing to (c), we establish the fact that $\lim \frac{a_n}{b_n} = \frac{\alpha}{\beta}$, by showing that $\lim \frac{1}{b_n} = \frac{1}{\beta}$:

Let $\varepsilon > 0$ be given. We are to find N such that:

$$n > N \Rightarrow \left|\frac{1}{b_n} - \frac{1}{\beta}\right| = \frac{|\beta - b_n|}{|b_n||\beta|} < \varepsilon \qquad (*)$$

Since $b_n \to \beta \ne 0$, we can choose N_1 such that $n > N_1 \Rightarrow |b_n-\beta| < \frac{|\beta|}{2}$. Noting that for $n > N_1$: $|b_n| > \frac{|\beta|}{2}$ we see that, for $n > N_1$:

$$\frac{|\beta-b_n|}{|b_n||\beta|} = \frac{1}{|b_n|}\cdot\frac{|\beta-b_n|}{|\beta|} < \frac{1}{\frac{|\beta|}{2}}\cdot\frac{|\beta-b_n|}{|\beta|} = \frac{2}{|\beta|^2}|b_n-\beta| \quad (**)$$

Since $b_n \to \beta$, we can choose N_2 such that: $n \ge N_2 \Rightarrow |b_n-\beta| < \frac{|\beta|^2}{2}\varepsilon \qquad (***)$

Letting $N = \max\{N_1, N_2\}$, we find that, for $n \geq N$:

$$\frac{|\beta - \boldsymbol{b_n}|}{|\boldsymbol{b_n}||\beta|} \underset{\text{By (**)}}{\underset{\uparrow}{<}} \frac{2}{|\beta|^2}|b_n - \beta| \underset{\text{By (***)}}{\underset{\uparrow}{<}} \frac{2}{|\beta|^2} \cdot \frac{|\beta|^2}{2}\varepsilon = \varepsilon$$

thereby establishing (*).

THEOREM 9.5, PAGE 328

Theorem 9.5 Every bounded monotone sequence converges.

As previously noted axioms are "dictated truths" upon which, with the cement of logic, mathematical theories are constructed. One such axiom, called the Principle of Mathematical Induction, was introduced on page 83. Here, we will need another axiom, the so called *Completion Axiom*:

Every nonempty subset *S* of $\Re$ that is bounded from above has a least upper bound.

You may be able to anticipate the meaning of the terminology appearing in the above axiom on your own; but just in case:

a is an **upper bound** of *S* if $a \geq s$ for every $s \in S$

and: α is the **least upper bound** of *S*, denoted by $\text{lub } S$, if it satisfies the following two properties:

(i) α is an upper bound of *S*, and:

(ii) For any given $\varepsilon > 0$ there exists some $s \in S$ (which depends on ε) such that $s > \alpha - \varepsilon$. (That is: $\alpha - \varepsilon$ is not itself an upper bound.)

For example: $\text{lub}\{1, 2, 5\} = 5$, and $\text{lub}(-\infty, 7) = \text{lub}(-\infty, 7] = 7$

Proof (of Theorem 9.5): Let $(a_n)_{n=1}^{\infty}$ be increasing and bounded from above. The completion axiom assures us that the set $\{a_n\}$ has a least upper bound: α. We show that $(a_n)_{n=1}^{\infty}$ converges to α:

Let $\varepsilon > 0$ be given. Since α is the least upper bounded of $\{a_n\}$, and since $\alpha - \varepsilon$ lies to the left of α, there must exists a term a_N such that $a_N > \alpha - \varepsilon$. Since $(a_n)_{n=1}^{\infty}$ is an increasing sequence: $a_n > \alpha - \varepsilon$ for every $n > N$, and since $\alpha = \text{lub } \{a_n\}$: $a_n \leq \alpha$ for every *n*. It follows that $|a_n - \alpha| < \varepsilon$ for every $n > N$.

A similar argument can be used to show that every decreasing sequence bounded from below converges.

THEOREM 9.23, PAGE 360

Theorem 9.23

(a) If $\sum a_n$ converges absolutely to L, then any series $\sum b_n$ obtained by rearranging the terms of $\sum a_n$ also converges to L.

(b) If $\sum a_n$ converges conditionally then, for **any** given L, the terms of the series can be rearranged so that the resulting series converges to L. The terms can also be rearranged so that the resulting series diverges.

PROOF (a): Let $s_n = \sum_{i=1}^{n} a_i$ and $t_m = \sum_{i=1}^{m} b_i$. We show $\lim_{m \to \infty} t_m = L$ by showing that for any given $\varepsilon > 0$ there exists N such that $m > N \Rightarrow |t_m - L| < \varepsilon$:

Let N_1 be such that $n > N_1 \Rightarrow |s_n - L| < \frac{\varepsilon}{2}$. Since $\sum a_n$ converges absolutely, we can also choose N_2 such that for all $n > m > N_2$: $\sum_{k=m}^{n} |a_k| < \frac{\varepsilon}{2}$. Consider $M = \max\{N_1, N_2\}$. Next, we choose N such that each term $\{a_1, a_2, \ldots, a_N\}$ is contained in $\{b_1, b_2, \ldots, b_M\}$ (note that $N \geq M$). Putting all of this together we have:

$$n > N \Rightarrow |t_m - L| = |t_m - s_n + s_n - L| \leq |t_m - s_n| + |s_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(b) We first show that:

If $\sum a_n$ is a conditionally convergent series, and if $\sum c_n$ and $\sum d_n$ denote the series composed of the positive and negative terms of $\sum a_n$, respectively, then:

$\sum c_n$ diverges to ∞ and $\sum d_n$ diverges to $-\infty$.

PROOF: Can both $\sum c_n$ and $\sum d_n$ converge? No, for if $\sum c_n = L$ and $\sum d_n = -K$, then the sequence of partial sums of $\sum |a_n|$ would be bounded by $L + K$, indicating that $\sum a_n$ converges absolutely (a contradiction).

Can exactly one of the series $\sum c_n$ and $\sum d_n$ diverge? No for if, say, $\sum c_n$ diverges to ∞ and that $\sum d_n$ converges to $-K$, then surely $\sum a_n$ would diverge to ∞ (a contradiction).

For any $L > 0$, we now construct a rearrangement $\sum b_n$ of the terms in the series $\sum a_n$ converging to L:

Let c_k denote the k^{th} positive term, and d_k the k^{th} negative term of the series $\sum a_n$, respectively. Since $\sum a_n$ converges we know there exists N such that $n > N \Rightarrow |a_n| < \varepsilon$. It follows that $|c_k| < \varepsilon$ and $|d_k| < \varepsilon$ for $k > N$.

Since $\sum c_n$ diverges to ∞ we can choose n_1 such that:

$$c_1 + c_2 + \ldots + c_{n_1} > L \text{ but } \mathrm{c}_1 + c_2 + \ldots + c_{n_1 - 1} \le L$$

(add just enough of the positive terms to end up to the right of L)

Since $\sum d_n$ diverges to $-\infty$ we can choose n_2 such that:

$$(c_1 + c_2 + \ldots + c_{n_1}) + (d_1 + d_2 + \ldots + d_{n_2}) < L \text{ but } (c_1 + c_2 + \ldots + c_{n_1}) + (d_1 + d_2 + \ldots + d_{n_2}) \ge L$$

(add just enough of the negative terms to end up to the left of L)

The next step is to add just enough of the remaining positive terms to end up to the right of L, and so on.

Let s_n denote the n^{th} partial sum of the above constructed rearranged sequence. For n large enough so that at least N of the c_i's and N of the d_i's appear in s_n we have $|s_n - L| < \varepsilon$.

A similar arguments can be used for $L < 0$ or $L = 0$.

THEOREM 9.25, PAGE 365

Theorem 9.25

Convergence Theorem for Power Series

For a given power series $\sum c_n(x-a)^n$ there are only three possibilities:

(i)The series converges absolutely for all x.

(ii)The series converges only at $x = a$.

(iii)There exists $R > 0$ such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.

PROOF: We show that if neither (i) nor (ii) is satisfied, then (iii) must hold:

$$\text{Consider the set } S = \left\{ x \ne a \,\middle|\, \sum c_n(x-a)^n \text{ converges absolutely} \right\}.$$

Theorem 9.24, and the assumption that (ii) does not hold, assures us that $S \ne \emptyset$.

Theorem 9.24, and the assumption that (i) does not hold, assures us that S is bounded from above.

The Completion Axiom, introduced in the proof of Theorem 9.5 (page B-4), assures us that S has a least upper bound R.

Can $\sum c_n(x_0-a)^n$ fail to converge absolutely if $|x_0 - a| < R$? No, for R is an upper bound of S.

Can $\sum c_n(x_0-a)^n$ converge if $|x_0 - a| > R$? No, for R is the least upper bound of S.

THEOREM 9.29, PAGE 377

Theorem 9.29
Lagrange's
Remainder
Theorem

If f has derivatives of all orders in an open interval I containing a, then for each positive integer N and for each $x \in I$ there exists $\boldsymbol{c}$ between a and x such that

$$E_N(x) = \frac{f^{(N+1)}(\boldsymbol{c})}{(N+1)!}(x-a)^{N+1}$$

PROOF: We want to find an expression for $E_N(x)$ in

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(N)}(a)}{N!}(x-a)^N + E_N(x)$$

With that in mind we note that for any **fixed** $x \in I$, there is a number L (which depends on x) for which:

$$f(x) = f(a) + f'(a)(x-4) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(N)}(a)}{N!}(x-a)^N + \frac{L}{(N+1)!}(x-a)^{N+1} \quad (*)$$

We now consider the function F, given by:

$$F(t) = f(x) - \left[f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \ldots + \frac{f^{(N)}(t)}{N!}(x-t)^N + \frac{L}{(N+1)!}(x-t)^{N+1}\right] \quad (**)$$

with domain $a \le t \le x$ if $x > a$ and $x \le t \le a$ $x < a$. F is differentiable, with:

$$\begin{aligned} F'(t) &= -f'(t) + \left[f'(t) - \frac{f''(t)}{1!}(x-t)\right] + \left[\frac{f''(t)}{1!}(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots \\ &\quad + \left[\frac{f^{(N)}(t)}{(n-1)!}(x-t)^{N-1} - \frac{f^{(N+1)}(t)}{N!}(x-t)^N\right] + \frac{L}{N!}(x-t)^N \\ &= -\frac{f^{(N+1)}(t)}{N!}(x-t)^N + \frac{L}{N!}(x-t)^N \quad \text{for all } t \text{ between } a \text{ and } x \end{aligned}$$

From (*) and (**) we have: $F(a) = F(x) = 0$. Rolle's Theorem (page 121) tells us that there exists c between a and x for which $F'(c) = 0$:

$$-\frac{f^{(N+1)}(c)}{N!}(x-c)^N + \frac{L}{N!}(x-c)^N = 0$$

$$L = f^{(N+1)}(c)$$

Setting $t = a$ and $L = f^{(N+1)}(c)$ in (**), we have (recall that $F(a) = 0$):

$$f(x) = \left[f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(N)}(a)}{N!}(x-a)^N + \underbrace{\frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}}_{E_N(x)}\right]$$

APPENDIX C

ANSWERS TO ODD EXERCISES

1.1 SETS AND FUNCTIONS (PAGE 9)

1. $(-\infty, \infty)$ **3.** $(-\infty, -100) \cup (-100, 1) \cup (1, \infty)$ **5.** $[-7, \infty)$

7. $(f+g)(2) = 6, (f-g)(2) = 8, (fg)(2) = -7, \left(\frac{f}{g}\right)(2) = -7, (2f)(2) = 14, (f\circ g)(2) = 1,$
$(g\circ f)(2) = -6$

9. $(f+g)(2) = 6, (f-g)(2) = 6, (fg)(2) = 0, \left(\frac{f}{g}\right)(2)$ undefined,$(2f)(2) = 12, (f\circ g)(2) = 0,$
$(g\circ f)(2) = -4$

11. $(f+g)(2) = \frac{36}{7}, (f-g)(2) = -\frac{34}{7}, (fg)(2) = \frac{5}{7}, \left(\frac{f}{g}\right)(2) = \frac{1}{35}, (2f)(2) = \frac{2}{7},$
$(f\circ g)(2) = \frac{1}{10}, (g\circ f)(2) = \frac{22}{7}$

13. $(f+g)(x) = -x+4, (f-g)(x) = 3x+2, (fg)(x) = -2x^2-5x+3$
$\left(\frac{f}{g}\right)(x) = \frac{x+3}{-2x+1}, (2f)(x) = 2x+6, (f\circ g)(x) = -2x+4, (g\circ f)(x) = -2x-5$

15. $(f+g)(x) = x^2+2x+2, (f-g)(x) = x^2-4, (fg)(x) = x^3+4x^2+2x-3$
$\left(\frac{f}{g}\right)(x) = \frac{x^2+x-1}{x+3}, (2f)(x) = 2x^2+2x-2, (f\circ g)(x) = x^2+7x+11$
$(g\circ f)(x) = x^2+x+2$

17. $(f+g)(x) = \frac{-x^3+2x^2-3x+7}{-x+2}, (f-g)(x) = \frac{x^3-2x^2+3x-5}{-x+2}$
$(fg)(x) = \frac{x^2+3}{-x+2}, \left(\frac{f}{g}\right)(x) = \frac{1}{-x^3+2x^2-3x+6}$
$(2f)(x) = \frac{2}{-x+2}, (f\circ g)(x) = \frac{1}{-x^2-1}, (g\circ f)(x) = \frac{3x^2-12x+13}{(-x+2)^2}$

19. **(a)** $-6x+3$ **(b)** -9 **21.** **(a)** x^2+3x+2 **(b)** 12

23. $f(2) = 15, f(2+h) = 3h+15, f(x+h) = 3x+3h+9$

25. $f(2) = -1, f(2+h) = -h^2-3h-1, f(x+h) = -x^2+(1-2h)x-(h^2-x-1)$

27. $f(2) = \frac{4}{7}, f(2+h) = \frac{h^2+4h+4}{2h+7}, f(x+h) = \frac{x^2+2xh+h^2}{2x+2h+3}$ **29.** 3 **31.** $-2x+1-h$

33. $\frac{2x^2+2xh+6x+3h}{(2x+2h+3)(2x+3)}$ **35.** $f(0) = 0, f(1) = 2$ **37.** $f(-1) = -8, f(1) = 1, f(7) = -14$
10 is not in the domain of f

39. Since $ab = |a||b|$ or $ab = -|a||b|$:

$$|ab| = ||a||b|| = |a||b| \text{ or } |ab| = |-|a||b|| = ||a||b|| = |a||b|$$

In either case we have: $|ab| = |a||b|$.

41. $|a| = |(a-b)+b| \underset{\uparrow\text{ triangle inequality}}{\leq} |a-b| + |b| \Rightarrow |a-b| \geq |a| - |b|$ **43.** Odd **45.** Even **47.** Neither

1.2 ONE-TO-ONE FUNCTIONS AND THEIR INVERSES (PAGE 17)

1.
$$\begin{aligned} -5a-1 &= -5b-1 \\ 5a &= 5b \\ a &= b \end{aligned}$$

3.
$$\begin{aligned} a^3+1 &= b^3+1 \\ a^3 &= b^3 \\ a &= b \end{aligned}$$

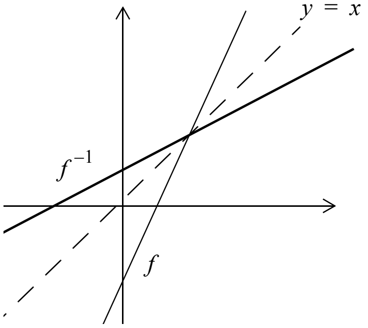
5.
$$\begin{aligned} \frac{4}{2a-3} &= \frac{4}{2b-3} \\ 8b-12 &= 8a-12 \\ 8b &= 8a \\ b &= a \end{aligned}$$

7.
$$\begin{aligned} \sqrt{a+1}+2 &= \sqrt{b+1}+2 \\ \sqrt{a+1} &= \sqrt{b+1} \\ a+1 &= b+1 \\ a &= b \end{aligned}$$

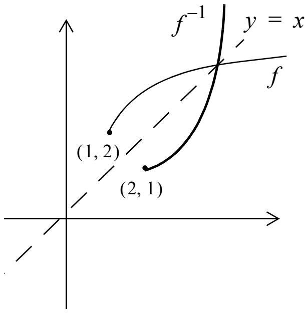
9.
$$\begin{aligned} \sqrt{\frac{3a}{2a+1}} &= \sqrt{\frac{3b}{2b+1}} \\ \frac{3a}{2a+1} &= \frac{3b}{2b+1} \\ 6ab+3a &= 6ba+3b \\ 3a &= 3b \\ a &= b \end{aligned}$$

11. No unique answers **13.** No unique answer **15.** $f^{-1}(x) = 1-x$ **17.** $f^{-1}(x) = \frac{5x+3}{2x}$

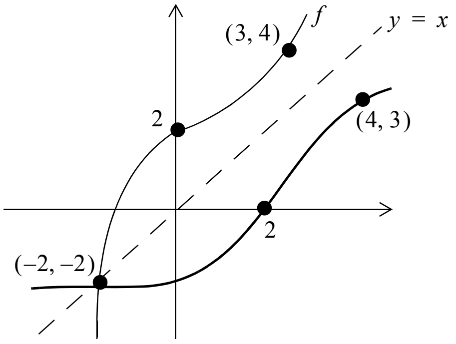
19. $f^{-1}(x) = \frac{3x+2}{x-5}$ **21.** $f^{-1}(x) = \frac{1}{4}x^2 - 3$ **23.**



25.



27.



1.3 EQUATIONS AND INEQUALITIES (PAGE 27)

1. (a) 12 (b) $(-\infty, 12)$ (c) $[12, \infty)$ **3.** (a) $-\frac{4}{5}$ (b) $\left(-\infty, -\frac{4}{5}\right]$

5. (a) 0, 3, –2 (b) $[-2, 3]$ (c) $(-\infty, -2) \cup (3, \infty)$

7. (a) $-1, \frac{1 \pm \sqrt{5}}{2}$ (b) $\left(-\infty, -1\right] \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$ (c) $\left(-1, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right)$

9. (a) $-2, -\frac{3}{2}, 1$ (b) $\left(-2, -\frac{3}{2}\right) \cup (1, \infty)$ **11.** (a) $-2, -\frac{3}{2}, 1$ (b) $[1, \infty) \cup \left\{-2, -\frac{3}{2}\right\}$

13. (a) $-\frac{3}{2}, 0, 1$ (b) $\left[-\frac{3}{2}, 1\right]$ **15.** (a) $\pm\frac{\sqrt{3}}{3}, \pm\sqrt{5}$ (b) $\left[-\sqrt{5}, -\frac{\sqrt{3}}{3}\right] \cup \left[\frac{\sqrt{3}}{3}, \sqrt{5}\right]$

17. (a) 1 (b) $(-1, 0) \cup (1, \infty)$ (c) $(-\infty, -1) \cup (0, 1]$

19. (a) –1, 2 (b) $[-1, 0) \cup [2, \infty)$ (c) $(-\infty, -1] \cup (0, 2]$ **21.** No unique answer

23. No unique answer **25.** No unique answer **27.** No unique answer

1.4 TRIGONOMETRY (PAGE 38)

1. $\frac{\pi}{6}$ **3.** $\frac{\pi}{3}$ **5.** $\frac{2}{3}\pi$ **7.** $-\frac{5}{6}\pi$ **9.** 45° **11.** –30° **13.** 90° **15.** 210°

17. 1 **19.** 0 **21.** –1 **23.** 0 **25.** $\frac{1}{\sqrt{2}}$ **27.** $-\sqrt{3}$ **29.** $\frac{2}{\sqrt{3}}$ **31.** $\frac{1}{\sqrt{3}}$

33. $\tan x$ **35.** 1 **37.** $\frac{1}{8}\tan^2 x$ **39.** 2 **41.** $\sec x + \tan x$

2.1 LIMITS: AN INTUITIVE INTRODUCTION (PAGE 51)

1. $\frac{2}{3}$ **3.** 2 **5.** $\frac{10}{7}$ **7.** Does Not Exist **9.** $\frac{1}{5}$ **11.** 0 **13.** $\frac{2}{3}$ **15.** –2

17. $\frac{1}{4}$ **19.** Does Not Exist **21.** $\frac{1}{2}$ **23.** $-\frac{1}{3}$ **25.** $3x^2$ **27.** 0 **29.** 0

31. $\lim_{x \to 2} f(x) = 4$ Removable discontinuity **33.** Jump discontinuity

35. $\lim_{x \to 3} f(x) = 5$ Removable discontinuity **37.** Jump discontinuity

39. No unique answer **41.** No unique answer

2.2 THE DEFINITION OF A LIMIT (PAGE 61)

1. $\frac{3}{2}$ **3.** 1 **5.** $\sqrt{5}-2$ **7.** (Largest) $\delta = \frac{\varepsilon}{5}$ 9. (Largest) $\delta = \varepsilon$

11. (Largest) $\delta = 2\varepsilon$ **13.** (One possible answer) $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$

15. (One possible answer) $\delta = \min\left(1, \frac{\varepsilon}{7}\right)$ **17.** Let $\varepsilon > 0$ be given. If $\delta = 1$ (or any positive number): $0 < |x-c| < \delta \Rightarrow |f(x)-d| = |d-d| = 0 < \varepsilon$

19. $\lim\limits_{x\to 2} 3x^2 = 3\lim\limits_{x\to 2} x^2 = 3\lim\limits_{x\to 2} x \lim\limits_{x\to 2} x = 3\cdot 2\cdot 2 = 12$

21. $\lim\limits_{x\to -2}(2x+1)^3 = [\lim\limits_{x\to -2}(2x+1)]^3 = [2\lim\limits_{x\to -2} x + \lim\limits_{x\to -2} 1]^3 = [2(-2)+1]^3 = -27$

23. $\lim\limits_{x\to 3}(x^3-25)^3 = [\lim\limits_{x\to 3}(x^3-25)]^3 = [(\lim\limits_{x\to 3} x)^3 - \lim\limits_{x\to 3} 25]^3 = [27-25]^3 = 8$

25. Any a, $b = -\frac{3}{2}$ **27.** No unique answer **29.** No unique answer

31. Suppose that $L \neq M$. Let $\varepsilon = \frac{|L-M|}{4} > 0$. Choose $\delta_1 > 0$ and $\delta_1 > 0$ such that:

$$0 < |x-c| < \delta_1 \Rightarrow |f(x)-L| < \varepsilon \text{ and } 0 < |x-c| < \delta_2 \Rightarrow |f(x)-M| < \varepsilon$$

For $\delta = \min(\delta_1, \delta_2)$ we have:

$$0 < |x-c| < \delta \Rightarrow |\boldsymbol{L}-\boldsymbol{M}| = |L-M+f(x)-f(x)|$$

Tiangle Inequality: $\leq |f((x)-M)| + |f(x)-L| < \varepsilon + \varepsilon = 2\frac{|L-M|}{4} = \frac{|\boldsymbol{L}-\boldsymbol{M}|}{\boldsymbol{2}}$

A contradiction, since $|\boldsymbol{L}-\boldsymbol{M}| \not\leq \frac{|\boldsymbol{L}-\boldsymbol{M}|}{\boldsymbol{2}}$.

33. We first show that $\lim\limits_{x\to c}\frac{1}{g(x)} = \frac{1}{M}$:

For given $\varepsilon > 0$ we are to find $\delta > 0$ such that $0 < |x-c| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M-g(x)}{Mg(x)}\right| < \varepsilon$.

Since $\lim\limits_{x\to c} g(x) = M$, there exist $\delta_1 > 0$ such that:

$$0 < |x-c| < \delta_1 \Rightarrow |g(x)-M| < \frac{|M|}{2}$$

$$\Rightarrow |\boldsymbol{M}| = |M-g(x)+g(x)| \leq |M-g(x)| + |g(x)| < \frac{|\boldsymbol{M}|}{\boldsymbol{2}} + |\boldsymbol{g(x)}|$$

$$\Rightarrow |\boldsymbol{g(x)}| > \frac{|\boldsymbol{M}|}{\boldsymbol{2}} \Rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|}$$

$$\Rightarrow \frac{1}{|Mg(x)|} = \frac{1}{|M||g(x)|} < \frac{1}{|M|}\cdot\frac{2}{|M|} = \frac{2}{M^2}$$

Since $\lim\limits_{x \to c} g(x) = M$, there exist $\delta_2 > 0$ such that:

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min(\delta_1, \delta_2)$. Then, for $0 < |x - c| < \delta$:

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{Mg(x)}\right| < \frac{2}{M^2} \cdot \frac{M^2}{2}\varepsilon = \varepsilon$$

Applying Theorem 2.3(c) we then have:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = [\lim_{x \to c} f(x)] \cdot \left[\lim_{x \to c} \frac{1}{g(x)}\right] = L \cdot \frac{1}{M} = \frac{L}{M}$$

35. If $\lim\limits_{x \to c} f(x) = f(c)$ and $\lim\limits_{x \to c} g(x) = g(c) \neq 0$, then, by Theorem 2.3(d):

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} = \frac{f(c)}{g(c)} = \left(\frac{f}{g}\right)(c)$$

37. A consequence of the fact that:

$$|f(x) - 0| < \varepsilon \Leftrightarrow |f(x)| < \varepsilon \Leftrightarrow ||f(x)|| < \varepsilon \Leftrightarrow ||f(x)| - 0| < \varepsilon$$

39. A consequence of Exercise 38 and Theorem 2.4(d).

3.1 TANGENT LINES AND THE DERIVATIVE (PAGE 75)

1. 16 **3.** –1 **5.** 0 **7.** 1 **9.** $\frac{3}{2\sqrt{7}}$ **11.** $\frac{1}{3}$ **13.** 1 **15.** $6x$

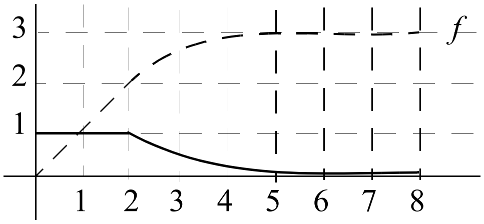
17. $-4x + 1$ **19.** $-\frac{1}{(x+1)^2}$ **21.** $\frac{1}{2\sqrt{x+3}}$ **23.** $\frac{-x^2+1}{(x^2+1)^2}$ **25.** $-2x - 1$

27. $-\frac{1}{(2x+3)^{3/2}}$ **29.** $y = 4x - 1$ **31.** $f'(2) \approx 1$, $f'(4) = f'(7) = 0$

33. **(a)** No limit at 3 and 4 **(b)** Not continuous at 1, 2, 3, and 4

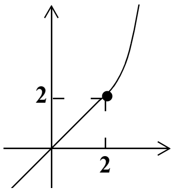
(c) Not differentiable at 1, 2, 3, and 4.

35.



37. No unique answer **39.** No unique answer

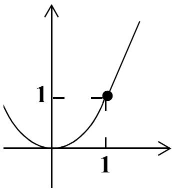
41.



$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = 1$$

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = 4$$

43.



$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = 2$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = 2$$

3.2 DIFFERENTIATION FORMULAS (PAGE 86)

1. $15x^4 + 12x^2$ **3.** $21x^2 + 10x - 4 - \frac{4}{x^5}$ **5.** $-7x^6 + 4x + \frac{1}{x^2} + \frac{2}{x^3}$ **7.** $3x^2 + 6x$

9. $\frac{1}{2\sqrt{x}}$ **11.** $\frac{3x^2 + 24x + 13}{(x+4)^2}$ **13.** $\frac{-30x}{(3x^2+1)^2}$ **15.** $28x^6 + 12x^5 + 25x^4 + 24x^3 + 9x^2 + 2x$

17. $-\frac{1}{\sqrt{x}(\sqrt{x}-1)^2}$ **19.** 154 **21.** 96 **23.** 8 **25.** $\frac{3}{2}$ **27.** 3 **29.** 13

31. $\frac{17}{6}$ **33.** $\frac{7}{4}$ **35.** $y = 5x - 4$ **37.** $y = -5x - 8$ **39.** $\left(-\frac{1}{2}, \frac{31}{24}\right), \left(1, \frac{1}{6}\right)$

41. $\left(-\frac{1}{3}, \frac{37}{54}\right), (2, -1)$ **43.** The equation $3x^2 + 2x = -4$: $3x^2 + 2x + 4 = 0$ has no solution.

45. Slope of tangent line at $x = c$: $\frac{1}{2\sqrt{c}}$. Equation of tangent line at $x = c$ with $b = -4$: $y = \frac{1}{2\sqrt{c}}x - 4$. Since the point $(c, \sqrt{c} + 2)$ on the curve must also be a point on the tangent line, we have $\sqrt{c} + 2 = \frac{1}{2\sqrt{c}}c - 4$. But this equation has no solution:

$$\sqrt{c} + 2 = \frac{1}{2\sqrt{c}}c - 4 \Rightarrow \sqrt{c} + 2 = \frac{1}{2}\sqrt{c} - 4 \Rightarrow \sqrt{c} = -6$$

No!

47. $(-3, -3)$ **49.** $p(x) = 3x^2 + 5x - 12$ **51.** $p(x) = 3x^3 - 2x^2 - 2x + 1$ **53.** $y = -\frac{1}{7}x + \frac{44}{7}$

55.
$$\left[\frac{f(x)}{g(x)}\right]' = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \to 0} \frac{g(x)f(x+h) - f(x)g(x+h)}{h[g(x)g(x+h)]}$$
$$= \lim_{h \to 0} \frac{g(x)f(x+h) - \mathbf{g(x)f(x)} + \mathbf{g(x)f(x)} - f(x)g(x+h)}{h[g(x)g(x+h)]}$$
$$= \lim_{h \to 0} \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{h[g(x)g(x+h)]}$$
$$= \lim_{h \to 0} \frac{1}{[g(x)g(x+h)]}\left[g(x) \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h} - f(x) \lim_{h \to 0} \frac{[g(x+h) - g(x)]}{h}\right]$$
$$= \frac{1}{[g(x)]^2}[g(x)f'(x) - f(x)g'(x)] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

57.
$$(fgh)'(x) = ([f(x)g(x)]h(x))' = [f(x)g(x)]h'(x) + h(x)[f(x)g(x)]'$$
$$= [f(x)g(x)]h'(x) + h(x)[f(x)g'(x) + g(x)f'(x)]$$
$$= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

59. Let $P(n)$ be the proposition that the sum of the first n integers equals $\frac{n(n+1)}{2}$.

I. Since the sum of the first integer is 1, $P(1)$ is true.

II. Assume $P(k)$ is true; that is: $\boldsymbol{1+2+3+\cdots+k} = \boldsymbol{\frac{k(k+1)}{2}}$

III. We show that $P(k+1) = \frac{(k+1)(k+2)}{2}$, thereby completing the proof:

$$\begin{aligned}(\boldsymbol{1+2+3+\cdots+k})+(k+1) &= \boldsymbol{\frac{k(k+1)}{2}}+(k+1) = \frac{(k^2+k)+2(k+1)}{2}\\ &= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}\end{aligned}$$

61. Let $P(n)$ be the proposition that if the functions $f_1, f_2, \ldots, f_n$ are differentiable, then so is their sum, and that $[f_1(x)+f_2(x)+\ldots+f_n(x)]' = f_1'(x)+f_2'(x)+\ldots+f_n'(x)$

I. Theorem 3.2(d) assures us that the proposition is true for $n = 2$.

II. Assume $P(k)$ is true: $[f_1(x)+f_2(x)+\ldots+f_k(x)]' = f_1'(x)+f_2'(x)+\ldots+f_k'(x)$

III. We show that $P(k+1)$, thereby completing the proof:

where $h(x) = f_1(x)+f_2(x)+\ldots+f_k(x)$

$$\begin{aligned}[f_1(x)+f_2(x)+\ldots+f_k(x)+f_{k+1}(x)]' &= [\overset{\uparrow}{h}(x)+f_{k+1}(x)]'\\ \text{by I:}\quad &= h'(x)+f_{k+1}'(x)\\ &= [f_1(x)+f_2(x)+\ldots+f_k(x)]'+f_{k+1}'(x)\\ \text{by II:}\quad &= f_1'(x)+f_2'(x)+\ldots+f_k'(x)+f_{k+1}'(x)\end{aligned}$$

63. Let $P(n)$ be the proposition that the n^{th} derivative of x^n is $n!$, for any positive integer Integer n.

I. $P(1)$ is the proposition that the derivative of x is 1, which follows from Theorem 3.2(b).

II. Assume that $P(k)$ is true: i.e. $\frac{d^k}{dx^k}(x^k) = k!$

III. We show $P(k+1)$ is true: i.e. $\frac{d^{k+1}}{dx^{k+1}}(x^{k+1}) = (k+1)!$

$$\frac{d^{k+1}}{dx^{k+1}}(x^{k+1}) = \frac{d^k}{dx^k}(x^{k+1})' = \frac{d^k}{dx^k}[(k+1)x^k] \underset{\substack{\uparrow\\ \text{Theorem 3.2(c)}}}{=} (k+1)\frac{d^k}{dx^k}(x^k) \underset{\substack{\uparrow\\ \text{II}}}{=} (k+1)k! = (k+1)!$$

3.3 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS AND THE CHAIN RULE (PAGE 100)

1. $15(x^2+3x-10)^{14}(2x+3)$ **3.** $\dfrac{3x^2+2}{2\sqrt{x^3+2x}}$ **5.** $\dfrac{3x+6}{2(x+1)^{3/2}}$ **7.** $4x\cos(2x^2+1)$

9. $-\sin x\cos(\cos x)$ **11.** $-x\cos x\sin(\sin x)+\cos(\sin x)$ **13.** $\dfrac{2\sin x\cos^2 x+\sin^3 x}{\cos^2 x}$

15. $(2x+1)\cos(x^2+x-1)\sec^2[\sin(x^2+x-1)]$ **17.** $\sqrt{\sec(2x+3)}\tan(2x+3)$

19. $-2\cos(\cos^2 x)\cos x\sin x$ **21.** $4x\sin x^2\cot(\cos x^2)\csc^2(\cos x^2)$

23. $\frac{4}{3}x\sin x^2\cot(\cos x^2)[\csc(\cos x^2)]^{\frac{2}{3}}$ **25.** -9 **27.** $-\dfrac{6x+1}{(3x^2+x+1)^2}$ **29.** 4 **31.** 2 **33.** 18

35. $y = -20x-36$ **37.** $y = 1$ **39.** $\frac{\pi}{6}, \frac{7\pi}{6}$ **41.** $(2, 1)$ **43.** $y = -\frac{1}{\pi}x+\frac{\pi^2+2}{4}$ **45.** 0

47. For $x>0$**:** $-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -x \le x\sin\frac{1}{x} \le x$. It follows that $\lim\limits_{x\to 0^+} x\sin\frac{1}{x} = 0$.

For $x<0$**:** $-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -x \ge x\sin\frac{1}{x} \ge x$. It follows that $\lim\limits_{x\to 0^-} x\sin\frac{1}{x} = 0$.

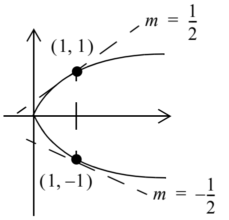
49. For any $x \ne 1$**:** $-1 \le \sin\dfrac{100}{(x-1)} \le 1 \Rightarrow -(x-1)^2 \le (x-1)^2\sin\dfrac{100}{(x-1)} \le (x-1)^2$. Noting that $\lim\limits_{x\to 1}[-(x-1)^2] = \lim\limits_{x\to 1}(x-1)^2 = 0$, we apply the Pinching Theorem and conclude that $\lim\limits_{x\to 1}\left[(x-1)^2\sin\dfrac{100}{(x-1)}\right] = 0$.

51. 3 **53.** 3 **55.** $\frac{7}{3}$ **57.** 4 **59.** 1 **61.** 0 **63.** $\left(1+\frac{r}{12}\right)^{11}$

65.
$$\begin{aligned}(h\circ g\circ f)'(x) &= [(h\circ g)\circ f]'(x) = (h\circ g)'[f(x)]\cdot f'(x) = h'(g[f(x)])g'[f(x)]\cdot f'(x)\\ &= h'[(g\circ f)(x)]\cdot g'[f(x)]\cdot f'(x)\end{aligned}$$

3.4 IMPLICIT DIFFERENTIATION (PAGE 108)

1.



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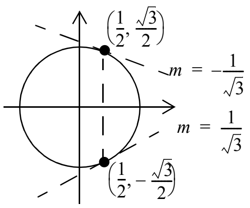

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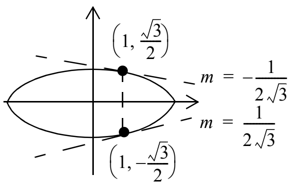

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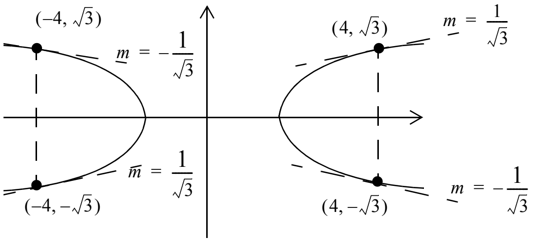


9. $y = -2x + 4$ **11.** $y = 4x - 8$ **13.** $y = -x + 6$

15. $y = -2x + 2$ **17.** $y = -4x + 9$ **19.** $y = \frac{3\pi - 2}{2}x + \frac{3\pi}{2}$ **21.** $y = x$ **23.** $y = \frac{4}{2 - \pi}x + \frac{\pi^2}{\pi - 2}$

25. $(0, 2)$ **27.** $y = -\frac{23}{30}x + \frac{113}{30}$ **29.** $\frac{-y^2 + 2xy + 3}{2xy - x^2 + 2}$ **31.** $\frac{y}{2x^2y + 4xy + 2y - x^2 - x}$

33. $\frac{1}{y^2\cos y + 2y\sin y - 1}$ **35.** $\frac{25}{64}$ **37.** 4 **39.** $\frac{2xy^3 - 2x^4}{y^5}$ **41.** $\frac{4y}{9x^2}$

43. Finding the points of intersection of $xy = 2$ and $x^2 - y^2 = 3$: $xy = 2 \Rightarrow y = \frac{2}{x}$.
Substituting in $x^2 - y^2 = 3$:

$$x^2 - \left(\frac{2}{x}\right)^2 = 3 \Rightarrow x^4 - 3x^2 = 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) - 0 \Rightarrow x = \pm 2$$

When $x = 2$, $y = 1$ and when $x = -2$, $y = -1$. Points of intersection: $(2, 1), (-2, -1)$.
Differentiating $xy = 2$ and $x^2 - y^2 = 3$ we have:

$$xy' + y = 0 \Rightarrow y' = -\frac{y}{x} \text{ and } 2x - 2yy' = 0 \Rightarrow y' = \frac{x}{y}$$

It follows that the slopes of the tangent lines at the points $(2, 1), (-2, -1)$ are negative reciprocals of each other.

45. Finding the points of intersection of $x^2 + y^2 = 4$ and $2x + 3y = 0$:

$$2x + 3y = 0 \Rightarrow y = -\frac{2}{3}x. \text{ Substituting in } x^2 + y^2 = 4:$$

$$x^2 + \left(-\frac{2}{3}x\right)^2 = 4 \Rightarrow \frac{13}{9}x^2 = 4 \Rightarrow x = \pm\frac{6}{\sqrt{13}}$$

When $x = \frac{6}{\sqrt{13}}$, $y = -\frac{2}{3} \cdot \frac{6}{\sqrt{13}} = -\frac{4}{\sqrt{13}}$ and when $x = -\frac{6}{\sqrt{13}}$, $y = -\frac{2}{3}\left(-\frac{6}{\sqrt{13}}\right) = \frac{4}{\sqrt{13}}$.
Points of intersection: $\left(\frac{6}{\sqrt{13}}, -\frac{4}{\sqrt{13}}\right), \left(-\frac{6}{\sqrt{13}}, \frac{4}{\sqrt{13}}\right)$.
Differentiating $x^2 + y^2 = 4$ and $2x + 3y = 0$ we have:

$$2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y} \text{ and } 2 + 3y' = 0 \Rightarrow y' = -\frac{2}{3}$$

At $\left(\frac{6}{\sqrt{13}}, -\frac{4}{\sqrt{13}}\right)$: $y' = -\frac{x}{y} = -\frac{\frac{6}{\sqrt{13}}}{-\frac{4}{\sqrt{13}}} = \frac{3}{2}$ which is the negative reciprocal of $-\frac{2}{3}$.

At $\left(-\frac{6}{\sqrt{13}}, \frac{4}{\sqrt{13}}\right)$: $y' = -\frac{x}{y} = -\frac{-\frac{6}{\sqrt{13}}}{\frac{4}{\sqrt{13}}} = \frac{3}{2}$ which is again the negative reciprocal of $-\frac{2}{3}$.

47. Differentiating both sides of the equation $(x-x_0)^2+(y-y_0)^2 = r^2$ we have $2(x-x_0)+2(y-y_0)y' = 0 \Rightarrow y' = -\frac{x-x_0}{y-y_0}$. So, the slope of the tangent line to any point (x, y) on the circle is $m = -\frac{x-x_0}{y-y_0}$. Moreover, the slope of the line passing through the center of the circle (x_0, y_0) and a point (x, y) on the circle (the direction of the radius) is given by $m_r = \frac{y-y_0}{x-x_0}$. Since $m_r = -\frac{1}{m}$, the tangent line is perpendicular to the radius.

3.5 RELATED RATES (PAGE 115)

1. (a) $7500\frac{\text{cm}^3}{\text{min}}$ (b) $600\frac{\text{cm}^2}{\text{min}}$ (c) $1200\frac{\text{cm}^3}{\text{min}}$ **3.** (a) $-10{,}000\pi\frac{\text{cm}^3}{\text{min}}$ (b) $-400\pi\frac{\text{cm}^2}{\text{min}}$ (c) $-1600\pi\frac{\text{cm}^3}{\text{min}}$

5. $-40\pi\frac{\text{ft}^3}{\text{min}}$ **7.** (a) $12\pi\frac{\text{ft}^2}{\text{sec}}$ (b) $\frac{20}{\pi}\frac{\text{ft}}{\text{sec}}$ **9.** $c = 10$ **11.** $0\frac{\text{in}^2}{\text{sec}}$ **13.** (a) $\frac{\sqrt{3}}{2}\frac{\text{in}^2}{\text{min}}$ (b) $3\frac{\text{in.}}{\text{min}}$

(c) $0\frac{\text{radians}}{\text{min}}$ **15.** $-\frac{1}{9}\frac{\text{radians}}{\text{sec}}$ **17.** (a) $-\frac{3}{2}\frac{\text{in}^2}{\text{min}}$ (b) $\frac{3}{2}\frac{\text{in}^2}{\text{min}}$ **19.** $-\frac{1}{12}\frac{\text{radians}}{\text{min}}$

21. We are told that $\frac{dV}{dt} = -kS$ for some positive number k. So: $4\pi r^2\frac{dr}{dt} = -k4\pi r^2$ or $\frac{dr}{dt} = -k$

23. (a) $\frac{2}{\pi}\frac{\text{ft}}{\text{min}}$ (b) $\frac{1}{\pi}\frac{\text{ft}}{\text{min}}$ (c) $\frac{dh}{dt} = \frac{8}{\pi^{1/3}(420)^{2/3}}\frac{\text{ft}}{\text{min}}$ and $\frac{dr}{dt} = \frac{4}{\pi^{1/3}(420)^{2/3}}\frac{\text{ft}}{\text{min}}$

25. $\frac{1}{160}\frac{\text{lb/in.}^2}{\text{min}}$ **27. (a)** $\frac{657{,}5000}{\sqrt{5{,}822{,}500}} \approx 272\frac{\text{ft}}{\text{min}}$ **(b)** $\frac{962{,}500}{\sqrt{9{,}062{,}500}} \approx 320\frac{\text{ft}}{\text{min}}$

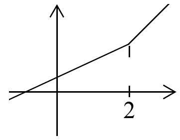
29. (a) $\left(\frac{200}{3}\pi+3\right)\frac{\text{in}^3}{\text{sec}}$ (b) $2\frac{\text{in}^3}{\text{sec}}$ **31.** (a) $5\frac{\text{ft}}{\text{min}}$ (b) $\frac{5}{2}\frac{\text{ft}}{\text{min}}$

4.1 THE MEAN VALUE THEOREM (PAGE 129)

1. $c = 0$ **3.** $c = -\frac{1}{2}$ **5.** $c = -\frac{1}{\sqrt{3}}$ **7.** $c = \frac{5}{4}$ **9.** $c = -1+\sqrt{2}$

11. $f(x) = \frac{1}{x^2}$ is not defined throughout the interval $[-2, 2]$.

13. (a)



(b) No: not differentiable at 2. **(c)** No

15. If there exist $a < x_1 < x_2 < b$ with $f(x_1) = f(x_2) = 0$, then, by Rolle's Theorem there exists $x_1 < c < x_2$ such that $f'(c) = 0$.

17. Let $f(x) = x^3 + 6x^2 + 15x - 23$. Since $f(-1) < 0$ and $f(2) > 0$, by the Intermediate Value Theorem we know that there exists $-1 < c < 2$ such that $f(c) = 0$ (c is a solution of the equation $x^3 + 6x^2 + 15x - 23 = 0$). Assume, now, that there are two solutions a and b with $a < b$ (we will arrive at a contradiction). Since f is differentiable everywhere and since $f(a) = f(b) = 0$ there must exist $c \in (a, b)$ such that $f'(c) = 0$ (Rolle's Theorem). However: $f'(x) = 3x^2 + 12x + 15$ is never 0, since the discriminant of $3x^2 + 12x + 15$ is negative.

19. Let $f(x) = 2x - 1 - \sin x$. Since $f(0) < 0$ and $f(2) > 0$, by the Intermediate Vale Theorem we know that there exists $0 < c < 2$ such that $f(c) = 0$ (c is a solution of the equation $2x - 1 - \sin x = 0$).

Assume, now, that there are two solutions a and b with $a < b$ (we will arrive at a contradiction). Since f is differentiable everywhere and since $f(a) = f(b) = 0$ there must exist $c \in (a, b)$ such that $f'(c) = 0$ (Rolle's Theorem). However: $f'(x) = 2 - \cos x$ is never 0, since $|\cos x| \le 1$ for all x.

21. Assume that $x^4 + 50x^2 - 300$ has three solutions x_1, x_2, x_3 with $x_1 < x_2 < x_3$ (we will arrive at a contradiction). Consider the function $f(x) = x^4 + 50x^2 - 300$.

Since $f(x_1) = f(x_2) = f(x_3) = 0$ we can apply Rolle's Theorem twice to arrive at c_1, c_2 with $x_1 < c_1 < x_2$, $x_2 < c_2 < x_3$, such that $f'(c_1) = f'(c_2) = 0$.

But $f'(x) = 4x^3 + 100x = x(4x^2 + 100) = 0$ has but one solution; namely: $x = 0$

23. If $a = b$ then surely $|\sin b - \sin a| \le |b - a|$; indeed both $\sin b - \sin a$ and $b - a$ are 0. Assume, therefore that $a < b$. Applying the Mean Value Theorem to the function $f(x) = \sin x$ we conclude that there exists $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$; or: $\cos c = \frac{\sin b - \sin a}{b - a}$. Taking the absolute value of both sides brings us to $|\cos c| = \left|\frac{\sin b - \sin a}{b - a}\right|$. Noting that $|\cos x| \le 1$ for all x, and that $\left|\frac{\sin b - \sin a}{b - a}\right| = \frac{|\sin b - \sin a|}{|b - a|}$ we have: $1 \ge \frac{|\sin b - \sin a|}{|b - a|}$, or: $|\sin b - \sin a| \le |b - a|$.

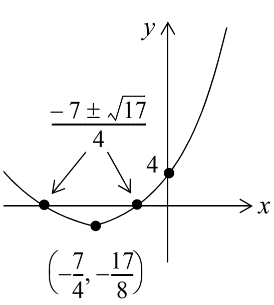
25. Let $s_1(t)$ and $s_2(t)$ denote the distances of runner 1 and runner 2 from the starting line t seconds after the start of the race, and suppose the finish line is reached at time t_0. Consider the function $S(t) = s_1(t) - s_2(t)$. Since $S(0) = S(t_0) = 0$ there must be a $0 < t_c < t_0$ such that $S'(t_c) = 0$, or that $s_1'(t_c) - s_2'(t_c) = 0$. The desired conclusion now follows from the fact that the derivative of displacement with respect to time is velocity.

27. The Mean Value Theorem tells us that there exists $0 \le c \le 2$ for which $f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) - 6}{2}$. The condition that $f'(x) \ge 1$ for $0 \le x \le 2$ leads us to the inequality: $\frac{f(2) - 6}{2} \ge 1$; or: $f(2) \ge 8$.

29. Consider the function $h(x) = f(x) - g(x)$. Being the difference of two differentiable functions, h is differentiable on $[a, b]$. Since $h(a) = h(b) = 0$, there exists $c \in (a, b)$ such that $h'(c)' = f'(c) - g'(c) = 0$ (Rolle's Theorem); which is to say: $f'(c) = g'(c)$.

4.2 GRAPHING FUNCTIONS (PAGE 146)

1.



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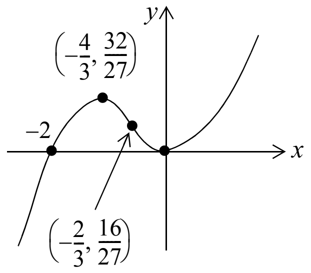

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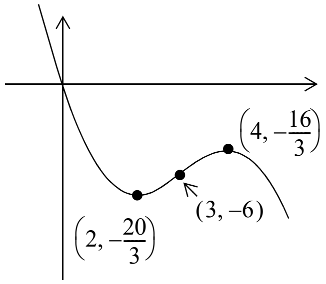

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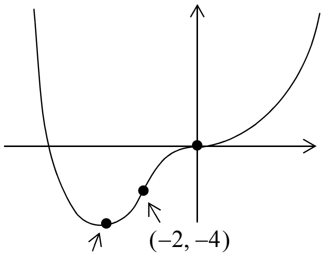

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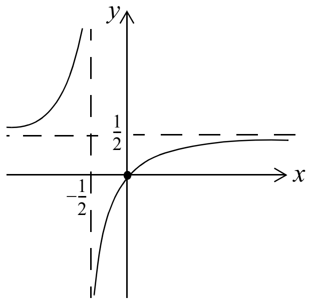

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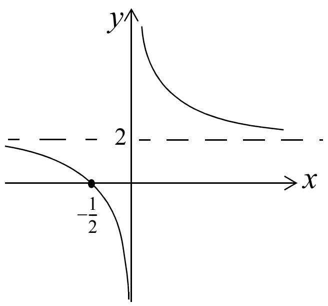

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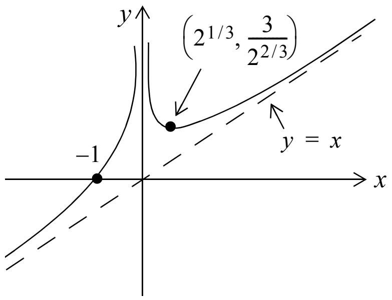

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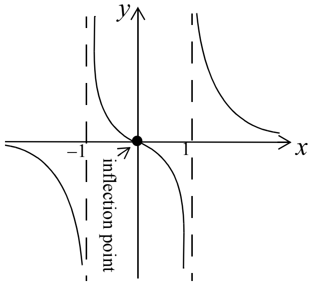

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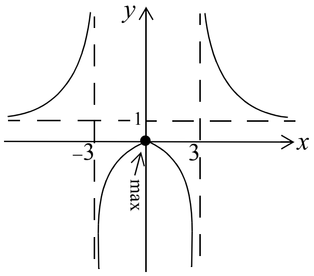

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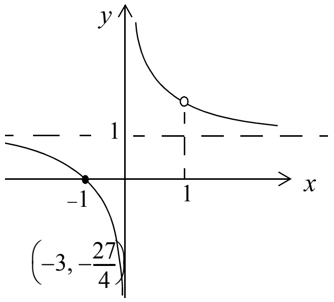

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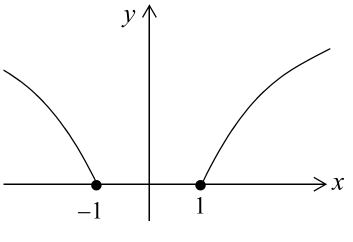

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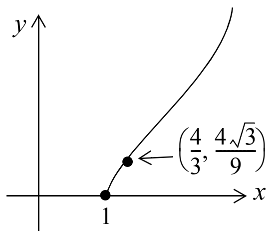

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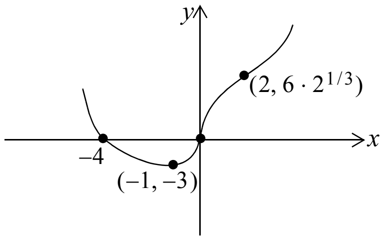

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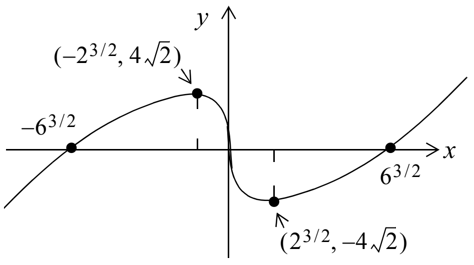

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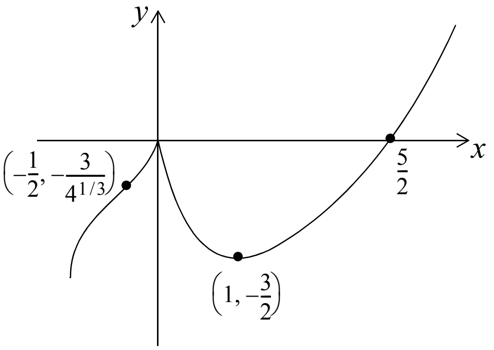

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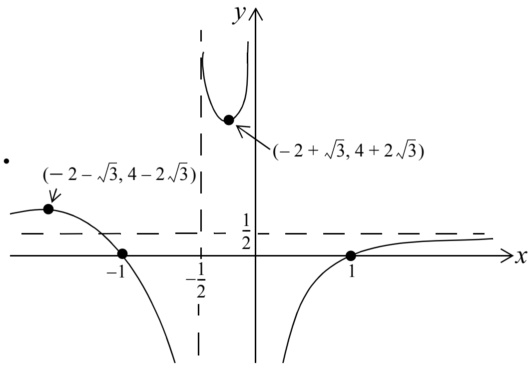


33. Absolute Minimum at -1, Absolute Maximum at 0.

35. Absolute Minimum at -3, Absolute Maximum at 3.

37. Absolute Minimum at 2, Absolute Maximum at 3.

39 through **50:** No unique answers. **51.** 5 weeks

53. Since $\frac{x^2-1}{x^2+x-2} = \frac{(x+1)(x-1)}{(x+2)(x-1)} = \frac{(x+1)}{(x+2)}$ for $x \neq 1$, the function has a removable discontinuity at 1. Since the denominator is 0 at -2 and the numerator is not, a vertical asymptote occurs at that point.

55. (a) The derivative of a cubic polynomial is a quadratic polynomial which cannot have more than two zeros.

(b) No unique answer.

(c) Suppose that $f(x) = ax^3+bx^2+cx+d$ assumes a (local) minimum at x_1 and at x_2, with $x_1 < x_2$. It follows that the SIGN of $f'(x)$ must be positive immediately to the right of both points, and negative immediately to their left: $\underset{x_1}{\overset{-\quad+}{\bullet}}\overline{\qquad\qquad\qquad}\underset{x_2}{\overset{-\quad+}{\bullet}}$. But for this to happen, there must be a sign change between x_1 and at x_2; which is to say, an additional zero x_3 of $f'(x)$ with $x_1 < x_3 < x_2$. This is not possible since $f'(x) = 3ax^2+2bx+c$ can have at most two zeros.

57. (a) The second derivative $f''(x) = (ax^3+bx^2+cx+d)'' = (3ax^2+2bx+c)' = 6ax+2b$ has but one zero: $x = -\frac{b}{3a}$. Concavity will change about that point. (b) No unique answer.

4.3 OPTIMIZATION (PAGE 160)

1. 200 units **3.** $\sqrt{\frac{10}{3}}$ hours **5.** minimum: $475\frac{\text{bacteria}}{\text{cm}^3}$, maximum: $875\frac{\text{bacteria}}{\text{cm}^3}$

7. $x = \frac{1}{2}a$ **9.** $x = 10$ **11.** $725 **13.** 1 **15.** $\frac{32\sqrt{3}}{9}$ **17.** 4in.^{2}

19. 36 inches wide by 48 inches high. **21.** 64 in. by 128 in. by 96 in.

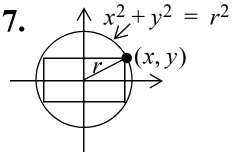
23. 150 ft by 100 ft, with the inner fence parallel to the 100 ft side.

25. 3.38 miles north of A. **27.** One hour and 32 minutes. **29.** $2r = \frac{30}{4+\pi}$ft, $h = \frac{15}{4+\pi}$ft

31. 4 machines. **33.** The strongest beam of depth d and width w is realized when $d = \sqrt{2}w$.

35. The perimeter $P = 2l + 2w$ of a rectangle of length l and width w, subject to the condition that $l \times w = A$, is to be minimized. From the given condition we have $l = \frac{A}{w}$, and therefore $P = 2\frac{A}{w} + 2w$. Setting the derivative of P to zero and solving we have: $(2Aw^{-1} + 2w)' = -2Aw^{-2} + 2 = 0 \Rightarrow w^2 = A \Rightarrow w = \sqrt{A}$. Then: $l = \frac{A}{w} = \frac{A}{\sqrt{A}} = \sqrt{A}$.

37.



We want to maximize $A = 4xy$ (note that xy is the area of the rectangle lying in the first quadrant). From $x^2 + y^2 = r^2$ we have: $y = \sqrt{r^2 - x^2}$; or $A = 4x\sqrt{r^2 - x^2}$.

To simplify calculations we find x for which $A^2 = 16x^2(r^2 - x^2) = 16(r^2x^2 - x^4)$ is greatest:

$$[16(r^2x^2 - x^4)]' = 0 \Rightarrow 16[2r^2x - 4x^3] = 0 \Rightarrow x = 0 \text{ or } x = \pm\frac{r}{\sqrt{2}}$$

Since x has to be a positive number ($x = 0$ would yield a minimum area), we conclude that maximum area occurs when $x = \frac{r}{\sqrt{2}}$, and that, consequently: $y = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \frac{r}{\sqrt{2}}$.

39. 0.43 **41.** 0.75 **43.** 9.3 miles from plant A.

5.1 THE INDEFINITE INTEGRAL (PAGE 175)

1. $3x + C$ **3.** $x^6 + x^5 + C$ **5.** $\frac{x^5}{25} + \frac{3}{4x^4} + C$ **7.** $\frac{3}{5}x^5 + \frac{4}{3x^3} - \frac{1}{2x^4} + C$

9. $\frac{x^4}{2} - \frac{5x^3}{3} + C$ **11.** $\frac{x^4}{4} - \frac{x^2}{2} + C$ **13.** $\frac{1}{6}x^3 - \frac{1}{2x} + \frac{1}{10x^5} + C$ **15.** $\frac{1}{2}x^2 + x - \frac{1}{x} - \frac{1}{2x^2} + C$

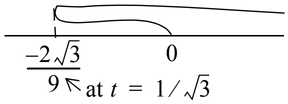
17. $\frac{5}{2}x^{\frac{2}{5}} + C$ **19.** $\frac{2}{7}x^{\frac{7}{2}} + \frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{3}{2}} + C$ **21.** $\frac{3}{13}x^{\frac{13}{3}} + \frac{6}{5}x^{\frac{5}{3}} + C$ **23.** $\sec x - \tan x + C$

25. $\sec x - \tan x + x + C$ **27.** $f(x) = \frac{3}{2}x^2 + 5x - \frac{123}{2}$ **29.** $f(x) = x^3 + \frac{5}{2}x^2 - \frac{5}{2}$

31. $f(x) = \frac{1}{4}x^4 + \frac{5}{2}x^2 - 2x + 1$ **33.** $f(x) = -\frac{3}{x} - \frac{5}{2x^2} + \frac{15}{2}$ **35.** $f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 3x + \frac{11}{6}$

37. $f(x) = \frac{1}{2}x^3 + \frac{5}{2}x^2 + x - 3$ **43.** $f(x) = \frac{x^3}{3} + \frac{14}{3}$ **45** $320\sqrt{2}\frac{\text{ft}}{\text{sec}}$ **47** $\frac{225}{4}$ft

49. (a)



(b) Right: $t > \frac{1}{\sqrt{3}}$; left: $0 \le t < \frac{1}{\sqrt{3}}$ (c) $t > \frac{1}{\sqrt{3}}$

(d) $\frac{4\sqrt{3}}{9} + 120$ m **51.** $\frac{1936}{15}$ft

53. (a) $\frac{v_0^2}{64}$ft (b) Time when object reached a height h:

$$\begin{aligned} -16t^2 + v_0 t &= h \\ 16t^2 - v_0 t + h &= 0 \\ t &= \frac{v_0 \pm \sqrt{v_0^2 - 64h}}{32} \end{aligned}$$

Velocity at height h when object is going up:

$$v\left(\frac{v_0 - \sqrt{v_0^2 - 64h}}{32}\right) = -32\left(\frac{v_0 - \sqrt{v_0^2 - 64h}}{32}\right) + v_0 = \sqrt{v_0^2 - 64h}$$

Velocity at height h when object is coming down:

$$v\left(\frac{v_0 + \sqrt{v_0^2 - 64h}}{32}\right) = -32\left(\frac{v_0 + \sqrt{v_0^2 - 64h}}{32}\right) + v_0 = -\sqrt{v_0^2 - 64h}$$

5.2 THE DEFINITE INTEGRAL (PAGE 186)

1. 3 **3.** 6 **5.** $\frac{35}{6}$ **7.** 4 **9.** –4 **11.** $\frac{27}{8}$ **13.** $\frac{4\sqrt{2}-2}{3}$ **15.** $-\frac{46}{35}$

17. $1 + \frac{1}{\sqrt{2}}$ **19.** $2 - \sqrt{2}$ **21. 0** **23.** $\frac{1}{4}$ **25.** $\frac{4\sqrt{2}-2}{3}$ **27.** $\frac{5}{2}$

29. (a) \$6,381 **(b)** \$2,765 **31.** 200 days **33.** The \$3,000 machine **35.** 27

37. 6 **39.** 0 **41.** $\sqrt{3x^4+1}$ **43.** $\frac{\sin x}{x^2+1}$ **45.** $x^2 + x$

47. $H(x) = T[g(x)]$ where $T(x) = \int_a^x f(t)dt$. So: $H'(x) = T'[g(x)]g'(x) = f[g(x)] \cdot g'(x)$.

49. $2\sqrt{48x^4+1}$ **51.** $\frac{\cos x}{\sin^2 x + 1} - \frac{2x}{x^4+1}$ **53.** $2x^5 - 10x^3$ **55.** $x\cos x + \sin x$

57. $4x^2\sec^2 x^2 + 2\tan x^2$

59. $\int_0^2 (f(x) - g(x))dx$

$$\begin{aligned} &= \lim_{n\to\infty}\left[\int_0^{1-\frac{1}{n}} [f(x)-g(x)]dx + \int_{1-\frac{1}{n}}^{1+\frac{1}{n}} [f(x)-g(x)]dx + \int_{1+\frac{1}{n}}^{2} [f(x)-g(x)]dx\right] \\ &= \lim_{n\to\infty}\left[0 + \int_{1-\frac{1}{n}}^{1+\frac{1}{n}} [f(x)-g(x)]dx + 0\right] \\ &\le \lim_{n\to\infty}\left\{\max|f(x)-g(x)|\left|\left[\left(1+\frac{1}{n}\right) - \left(1-\frac{1}{n}\right)\right]\right|\right\} < \lim_{n\to\infty}\frac{2}{n} = 0 \end{aligned}$$

5.3 THE SUBSTITUTION METHOD (PAGE 194)

1. $\frac{(x-5)^{16}}{16}+C$ **3.** $\frac{-1}{28(2x-5)^{14}}+C$ **5.** $\frac{(x^2+5)^{16}}{32}+C$ **7.** $\frac{\sqrt{5x^2-4}}{5}+C$

9. $\frac{1}{2}\left(3x^2-\frac{1}{(x^2-3)}\right)+C$ **11.** $\frac{1}{2}\tan^2 x+C$ **13.** $-\frac{3x^2+3x+1}{3(x+1)^3}+C$

15. $\frac{2}{5}(x-3)^{5/2}+\frac{14}{3}(x-3)^{3/2}+22(x-3)^{1/2}+C$ **17.** $\frac{63}{18}$ **19.** $\frac{2}{5}(\sqrt{10}-\sqrt{5})$

21. $-\frac{1}{36}$ **23.** 0 **25.** $\frac{1}{2}$ **27.** $\frac{4}{15}(\sqrt{2}+1)$ **29.** $\frac{2}{9}(11\sqrt{11}-27)$

5.4 AREA AND VOLUME (PAGE 205)

1. 9 **3.** $\frac{1}{4}$ **5.** $\frac{9}{2}$ **7.** $\frac{9}{8}$ **9.** $\frac{4}{3}$ **11.** 16 **13.** 2 **15.** $\frac{7}{6}$ **17.** 3

19. 1 **21.** $\frac{51}{4}$ **23.** $\frac{2}{3}$ **25.** $\frac{4+\sqrt{2}}{3}$ **27.** $-1+\sqrt{2}$ **29.** $\frac{26\pi}{3}$ **31.** $\frac{28\pi}{15}$ **33.** $\frac{\pi}{30}$

35. 8π **37.** $\frac{4}{3}\pi r^3$ **39.** $\frac{2\pi}{9}$ **41.** $\frac{512\pi}{45}$ **43.** $\frac{\pi}{3}$ **45.** $\frac{72\pi}{5}$ **47.** $\frac{64\pi}{15}$

49. 16π **51.** $\frac{\pi}{96}(96-\pi^3)$ **53.** $\frac{\pi}{3}$ **55.** $\frac{\pi}{3}$ **57.** $\frac{32\pi}{5}$ **59.** $\frac{16\pi}{3}$ **61.** π

63. 8π **65.** $\frac{35\pi}{6}$ **67.** $\frac{212\pi}{15}$ **69.** $\frac{1}{3}l^2h$ **71.** $\frac{16}{3}r^3$ **73.** $\frac{4}{3}$

5.5 ADDITIONAL APPLICATIONS (PAGE 214)

1. $\int_1^5\sqrt{1+4x^2}\,dx$; 24.40 **3.** $\int_4^9\sqrt{1+\frac{1}{2x-5}}\,dx$; 5.35 **5.** $\int_{-2}^{2}\sqrt{36x^2+48x+17}\,dx$; 27.27

7. $\frac{8}{27}(19\sqrt{19}-1)$ **9.** $\frac{59}{24}$ **11.** $\frac{3}{2}$ **13.** $4\int_0^a\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\,dx$

15. The unit circle has perimeter 2π. The graph of the function $f(x) = \sqrt{1-x^2}$ over the interval $[-1, 1]$ is the upper half of the unit circle $x^2+y^2 = 1$. It follows, that: $\int_{-1}^{1}\sqrt{1+[(\sqrt{1-x^2})']^2}\,dx = \pi$. The claim that $\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = \pi$ now follows from the observation that:

$$\sqrt{1+[(\sqrt{1-x^2})']^2} = \sqrt{1+\left[\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\right]^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \sqrt{\frac{1}{1-x^2}}.$$

17. (a) $\frac{25}{4}$J (b) $\frac{25}{16}$J (c) 80 cm **19.** $\frac{3}{2}$ feet **21.** $\frac{103}{72}$m **23.** 3W

25. (a) $15{,}552\pi$ ft-lbs (b) $36{,}288\pi$ ft-lbs (c) $2{,}963.45\pi$ ft-lbs

27. $\frac{875}{8}$ ft-lb **29.** 1400 ft-lb **31.** $\bar{x} = -\frac{1}{5}$ **33.** $\frac{5}{7}$ pounds **35.** $\frac{50}{9}$ft **37.** $(-1, 3)$

39. $\left(\frac{34}{5}, -\frac{102}{5}\right)$ **41.** $\left(\frac{3}{4}, \frac{3}{10}\right)$ **43.** $\left(\frac{1}{2}, \frac{2}{5}\right)$ **45.** $\left(\frac{8}{5}, 2\right)$ **47.** $22w$ lb

49. $(27 + 18\sqrt{3})w$ lb **51.** $576w$ lb **53.** 28,912 lb

6.1 THE NATURAL LOGARITHMIC FUNCTION (PAGE 229)

1. $3x^2\ln x + x^2$ **3.** $\frac{2 + 2\ln x}{x}$ **5.** $\ln x\cos x + \frac{\sin x}{x}$ **7.** $-\tan x$ **9.** $\frac{x\ln x\cos x - \sin x}{x(\ln x)^2}$

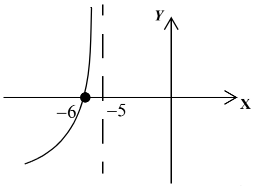
11. $\frac{4x\ln(x^2+1)}{x^2+1}$ **13.** $\frac{1 + \ln x}{2\sqrt{x\ln x}}$ **15.** $\tan x$ **17.** $\frac{2\cos(\ln x^2)}{x}$ **19.** $-\frac{1}{x^2}$ **21.** $\frac{2 - \ln x}{x(\ln x)^3}$

23. $-\frac{1 + (\ln x)^2}{4(x\ln x)^{3/2}}$ **25.** (a) $\frac{6x+1}{3x^2+x}$ (b) $\frac{1 + 6\ln x}{x}$ (c) $108x^3 + 54x^2 + 12x + 1$ (d) $\frac{1}{x\ln x}$

27. $y = 2x - 2$ **29.** $y = 0$ **31.** $(e, 1)$ **33.** $\frac{1}{3}\ln|x^3 + 2| + C$ **35.** $\frac{1}{2}\ln|\ln x| + C$

37. $\sin(\ln x) + C$ **39.** $-2\ln|\csc\sqrt{x}| + C$ **41.** $\frac{1}{2}\ln 2$ **43.** $\frac{\ln(x^2+1)}{2} + 2$ **45.** $-\frac{1}{2}\left(\ln\frac{1}{x}\right)^2 + \frac{3}{2}$

47.



49. $\frac{1 + e^2}{2}$ **51.** $\frac{e^5 - 6}{5}\pi$ **53.** $\ln\frac{9}{2}$ **55.** e

57. Since $(\ln x^r)' = \frac{1}{x^r} \cdot rx^{r-1} = \frac{r}{x}$, and $(r\ln x)' = r \cdot \frac{1}{x} = \frac{r}{x}$, the two functions can differ only by a constant (Theorem 5.1): $\ln x^r = r\ln x + C$. Evaluating this equation at $x = 1$:

$$\ln 1 = r\ln 1 + C \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow \ln x^r = r\ln x$$

6.2 THE NATURAL EXPONENTIAL FUNCTION (PAGE 239)

1. $2e^{2x}$ **3.** $x^2e^x(x + 3)$ **5.** $2e^x(e^x - 1)$ **7.** $\frac{(2x-1)e^{2x}}{2x^2}$ **9.** $5(x + e^x)^4(1 + e^x)$

11. $10x(x^2 + e^2)^4$ **13.** $\cos x e^{\sin x}$ **15.** $e^x(\sin x^2 + 2x\cos x^2)$ **17.** $\frac{2x(\sec^2 x^2 - \tan x^2)}{e^{x^2}}$

19. $\frac{2x(1 + e^{x^2})}{x^2 + e^{x^2}}$ **21.** $e^{e^x + x}$ **23.** $\frac{y(2x - e^y)}{xye^y + 1}$ **25.** $e^x(x + 2)$ **27.** $-\frac{1}{x^2} + 2$

29. $-e^{2x}(5\cos 3x + 12\sin 3x)$ **31.** (a) $(6x+1)e^{3x^2+x}$ (b) $e^x(6e^x+1)$

(c) $108x^3 + 54x^2 + 12x + 1$ (d) $e^{e^x + x}$

33. $y = 4e^4x - 7e^4$ **35.** $y = e^{\pi/2}x + e^{\pi/2}\left(1 - \frac{\pi}{2}\right)$ **37.** $\left(\frac{1}{2}, e^{\frac{1}{4}}\right)$ **39.** $\frac{e^{x^3}}{3} + C$

41. $-e^{1/x} + C$ **43.** $x - e^{-x} + C$ **45.** $\sin e^x + C$ **47.** $\ln|e^x - e^{-x}| + C$ **49.** $e^{\frac{1}{\sqrt{2}}} - 1$

51. $f(x) = \frac{1}{2}e^{x^2} + \frac{3}{2}$ **53.** $f(x) = \frac{e^{2x}}{4} + \frac{x}{2} + \frac{3}{4}$

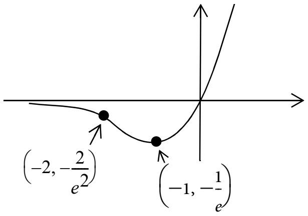
55. $(Ae^{-x} + Bxe^{-x})'' = (-Ae^{-x} + Be^{-x} - Bxe^{-x})' = Ae^{-x} - Be^{-x} + Bxe^{-x} - Be^{x} = Ae^{-x} + Bxe^{-x} - 2Be^{x}$

and $2(Ae^{-x} + Bxe^{-x})' = 2(-Ae^{-x} + Be^{-x} - Bxe^{-x}) = -2Ae^{-x} + 2Be^{-x} - 2Bxe^{-x}$

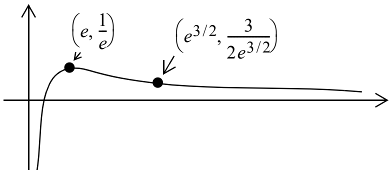
So: $(Ae^{-x} + Bxe^{-x})'' + 2(Ae^{-x} + Bxe^{-x})' + (Ae^{-x} + Bxe^{-x})$

$= Ae^{-x} + Bxe^{-x} - 2Be^{x} - 2Ae^{-x} + 2Be^{-x} - 2Bxe^{-x} + Ae^{-x} + Bxe^{-x} = 0$

57. 2, 3 **59.**



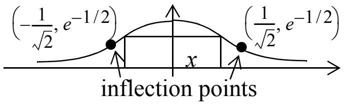
61.



63. $\frac{9}{8}$

65. $\frac{e^2+1}{2}\pi$ **67.** $f(x) = e^{-x^2} \Rightarrow f'(x) = -2xe^{-x^2}$

$f''(x) = -2e^{-x^2}(1 + \sqrt{2}x)(1 - \sqrt{2}x)$ So:



$A = 2xe^{-x^2} \Rightarrow A' = \frac{2(1-2x^2)}{e^{x^2}}$

Then: $A' = 0$ at $x = \pm\frac{1}{\sqrt{2}}$

69. \$8,187.31 **71.** (a) 6.91 billion (b) 2030

73. Since $\ln(e^a)^b = b\ln e^a = ba$ and $\ln e^{ab} = ab$, and since the natural logarithmic function is one-to-one: $(e^a)^b = e^{ab}$.

75. For $g(x) = e^{-x}f(x)$: $g'(x) = e^{-x}f'(x) - e^{-x}f(x) = 0$. Applying Theorem 4.3(c), page 124, we have: $e^{-x}f(x) = c$ for some constant c, or: $f(x) = \frac{c}{e^{-x}} = ce^{x}$.

6.3 a^x AND $\log_a x$ (PAGE 247)

1. $2\ln 5 \cdot 5^{2x}$ **3.** $x^2 3^x(x\ln 3 + 3)$ **5.** $\frac{2^{2x+1}[(2\ln 2)\ln 16 - 1]}{x^2}$ **7.** $5^{\sin x}\cos x \ln 5$

9. $5^x(2x\cos x^2 + \ln 5 \sin x^2)$ **11.** $\frac{1 - (\ln 3)x\ln x}{x3^x}$ **13.** $2^x\left(\frac{1}{x\ln 2} + \ln 2 \log_2 x\right)$

15. $\frac{1}{\ln 2 \cdot x(\log_2 x)}$ **17.** $\frac{1}{(\ln 2)^2 x \log_2 x}$ **19.** $(3^x)^x(1 + \ln 3x)$ **21.** $\frac{5^{x^2}}{\ln 25} + C$

23. $-\dfrac{5^{1/x}}{\ln 5} + C$ **25.** $\dfrac{1}{\ln 2}$

27. Turning to the formula $A(t) = A_0 e^{kt}$**(*)** of Theorem 6.8, page 236, we substitute H for t to arrive at: $\dfrac{A_0}{2} = A_0 e^{kH}$, or: $e^{kH} = \dfrac{1}{2}$. Applying the natural logarithmic functions to both sides we have: $kH = \ln\left(\dfrac{1}{2}\right)$, or: $k = -\dfrac{\ln 2}{H}$. Substituting in (*):

$$A(t) = A_0 e^{-\frac{\ln 2}{H}t} = A_0(e^{\ln 2})^{-\frac{t}{H}} = A_0 2^{-\frac{t}{H}}.$$

29. 29.3 days **31.** $b^{\log_b x} = x \Rightarrow \log_a b^{\log_b x} = \log_a x \Rightarrow (\log_b x)(\log_a b) = \log_a x \Rightarrow \log_b x = \dfrac{\log_a x}{\log_a b}$

33. (a) 10^{-3} Watts/m^2 (b) 10^{-9} Watts/m^2 (c) 10^{-11} Watts/m^2

35. 10,000 **37.** 251 times more intense. **39.** 8.2

6.4 INVERSE TRIGONOMETRIC FUNCTIONS (PAGE 256)

1. $\left[\dfrac{1}{e}, e\right]$ **3.** $(-\infty, \infty)$ **5.** $[-1, 1]$ **7.** $\dfrac{2x}{\sqrt{1-x^4}}$ **9.** $-\dfrac{2x}{\sqrt{1-x^4}}$

11. $-\dfrac{\sin x}{1+\cos^2 x}$ **13.** $\dfrac{1}{\sqrt{e^{2x}-1}}$ **15.** $\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$ **17.** $\dfrac{x-(1+x^2)\tan^{-1}x}{x^2(1+x^2)}$

19. $\dfrac{1}{(1+4x^2)\sqrt{\tan^{-1}2x}}$ **21.** $\dfrac{1}{|x+1|\sqrt{2x+1}}$ **23.** $-\dfrac{2x}{|x|(x^2+1)\sqrt{x^2+2}}$ **25.** $y = x$

27. $y = \dfrac{1}{\sqrt{3}}x + \dfrac{\pi-\sqrt{3}}{3}$ **29.** $\sin^{-1}\left(\dfrac{x}{3}\right) + C$ **31.** $\dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{\sqrt{3}}\right) + C$ **33.** $\sin^{-1}(e^x) + C$

35. $\ln|\sin^{-1}x| + C$ **37.** $2\tan^{-1}\sqrt{x} + C$ **39.** $\dfrac{1}{3}\tan^{-1}\left(\dfrac{\sin x}{3}\right) + C$ **41.** $\dfrac{\pi}{12}$ **43.** $\dfrac{4\pi}{3}$

45. $\dfrac{\pi^2}{72}$ **47.** $f(x) = \dfrac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + \dfrac{4-\pi}{4\sqrt{3}}$ **49.** $\dfrac{1}{6}\tan^{-1}9$ **51.** $\dfrac{3}{80}$ radian/sec

53. $\tan(\tan^{-1}x) = x \Rightarrow [\tan(\tan^{-1}x)]' = 1 \Rightarrow \sec^2(\tan^{-1}x)\cdot(\tan^{-1}x)' = 1$

$$\Rightarrow (\tan^{-1}x)' = \frac{1}{\sec^2(\tan^{-1}x)} \underset{\substack{\uparrow \\ \text{since } \sec^2 x \,=\, 1+\tan^2 x}}{=} \frac{1}{1+[\tan(\tan^{-1}x)]^2} = \frac{1}{1+x^2}$$

55. $\sec(\sec^{-1}x) = x \Rightarrow [\sec(\sec^{-1}x)]' = 1 \Rightarrow \sec(\sec^{-1}x)\cdot\tan(\sec^{-1}x)\cdot(\sec^{-1}x)' = 1$

$$\Rightarrow (\sec^{-1}x)' = \frac{1}{\sec(\sec^{-1}x)\cdot\tan(\sec^{-1}x)} = \frac{1}{x}\tan(\sec^{-1}x) \underset{\substack{\uparrow \\ \text{since } 1+\tan^2 x \,=\, \sec^2 x}}{=} \frac{1}{x\sqrt{x^2-1}}$$

57. (a) Let a be the positive square root of a^2. Then:

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{dx}{\sqrt{a^2\left[1-\left(\frac{x}{a}\right)^2\right]}} = \int\frac{dx}{a\sqrt{\left[1-\left(\frac{x}{a}\right)^2\right]}} \underset{\substack{\uparrow \\ u \,=\, \frac{x}{a},\ du \,=\, \frac{dx}{a}}}{=} \int\frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}\frac{x}{a} + C$$

(b) $$\left(\sin^{-1}\frac{x}{a}\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\frac{x}{a}\right)' = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$$

59. (a) Let a be the positive square root of a^2. Then:

$$\int\frac{dx}{a^2+(x+b)^2} = \int\frac{dx}{a^2\left[1+\left(\frac{x+b}{a}\right)^2\right]} \underset{\substack{\uparrow \\ u \,=\, \frac{x+b}{a},\ du \,=\, \frac{dx}{a}}}{=} \frac{1}{a}\int\frac{du}{1+u^2} = \frac{1}{a}\tan^{-1}u + C = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$$

(b) $$\left[\frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right)\right]' = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x+b}{a}\right)^2}\cdot\left(\frac{x+b}{a}\right)' = \frac{1}{a^2}\cdot\frac{1}{1+\frac{(x+b)^2}{a^2}} = \frac{1}{a^2+(x+b)^2}$$

61. $\dfrac{\sqrt{\pi^2-4}}{\pi}$ **63.** $\sqrt{\dfrac{4-\pi}{\pi}}$

7.1 INTEGRATION BY PARTS (PAGE 268)

1. $-\dfrac{x+1}{e^x}+C$ **3.** $\dfrac{\sin 3x - 3x\cos 3x}{9}+C$ **5.** $\dfrac{\sin ax - ax\cos ax}{a^2}+C$ **7.** $\dfrac{x^3(3\ln x-1)}{9}+C$

9. $\dfrac{(9x^2-6x+2)e^{3x}}{27}+C$ **11.** $\dfrac{\sqrt{1+x^2}(1+x^2-3)}{3}+C$ **13.** $-\dfrac{x^2+2x+2}{e^x}+C$

15. $\dfrac{2x^{3/2}}{9}(3\ln x-2)+C$ **17.** $x\left(\ln\dfrac{1}{x}+1\right)+C$ **19.** $\dfrac{1}{2}(x^2-c^2)\ln(x+c)-\dfrac{1}{4}x^2+\dfrac{1}{2}cx+C$

21. $\dfrac{1}{2}x[\cos(\ln x)+\sin(\ln x)]+C$ **23.** $\sin x[\ln(\sin x)-1]+C$ **25.** $(x^2-7x+7)e^x+C$

27. $x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$ **29.** $\sin x - \frac{1}{3}\sin^3 x + C$ **31.** $\frac{1}{2}\left[\tan^{-1}x - \frac{x}{x^2+1}\right] + C$

33. $\frac{1}{6}[\ln(x^2+1) + 2x^3\tan^{-1}x - x^2] + C$ **35.** $-\frac{2}{5}\sin 3x\sin 2x - \frac{3}{5}\cos 3x\cos 2x + C$ **37.** $e-2$

39. $\frac{1}{27}\left(2 - \frac{17}{e^3}\right)$ **41.** $\frac{1}{4}(\pi - 2\ln 2)$ **43.** $5\ln 5 - 4$ **45.** $-\frac{\pi}{8}$ **47.** $\frac{1}{3}(2-\sqrt{2})$

49. 9π **51.** $\pi(e-2)$ **53.** $2e^3 + 1$ ft. **55.** $\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

$$\boxed{\begin{array}{ll} u = x^n & dv = e^x dx \\ du = nx^{n-1}dx & v = e^x \end{array}}$$

57.

$$\boxed{\begin{array}{ll} u = \sec^{n-2}x & dv = \sec^2 x dx \\ du = (n-2)(\sec^{n-2}x)\sec x\tan x dx & v = \tan x \end{array}}$$

$$\int \sec^n x dx = \int(\sec^{n-2}x)\sec^2 x dx \overset{\downarrow}{=} (\sec^{n-2}x)\tan x - (n-2)\int(\sec^{n-2}x)\tan^2 x dx$$

$$\boxed{\tan^2 x = \sec^2 x - 1:} = (\sec^{n-2}x)\tan x - (n-2)\int(\sec^n x - \sec^{n-2}x)dx$$

$$= (\sec^{n-2}x)\tan x - (n-2)\int\sec^n x dx + (n-2)\int(\sec^{n-2}x)dx$$

Thus: $\int\sec^n x dx = (\sec^{n-2}x)\tan x - (n-2)\int\sec^n x dx + (n-2)\int(\sec^{n-2}x)dx$

$$\int\sec^n x dx + (n-2)\int\sec^n x dx = (\sec^{n-2}x)\tan x + (n-2)\int(\sec^{n-2}x)dx$$

$$\int\sec^n x dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int(\sec^{n-2}x)dx$$

59.

$$\boxed{\begin{array}{ll} u = (\ln x)^n & dv = x^m dx \\ du = n(\ln x)^{n-1}\cdot\frac{1}{x}dx & v = \frac{x^{m+1}}{m+1} \end{array}}$$

$$\int x^m(\ln x)^n dx \overset{\downarrow}{=} \frac{x^{m+1}(\ln x)^n}{m+1} - \int\frac{x^{m\cancel{+1}}}{m+1}\cdot\frac{n(\ln x)^{n-1}}{\cancel{x}}dx$$

$$= \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx$$

61.

$$\boxed{\begin{array}{ll} u = \ln(x+\sqrt{x^2+a^2}) & dv = dx \\ du = \dfrac{dx}{\sqrt{x^2+a^2}} & v = x \end{array}}$$

$$\ln(x+\sqrt{x^2+a^2})dx \overset{\downarrow}{=} x\ln(x+\sqrt{x^2+a^2}) - \int\frac{x}{\sqrt{x^2+a^2}}dx \quad \boxed{\begin{array}{l} w = x^2+a^2 \\ dw = 2xdx \end{array}}$$

$$= x\ln(x+\sqrt{x^2+a^2}) - \frac{1}{2}\int w^{-\frac{1}{2}}dw$$

$$= x\ln(x+\sqrt{x^2+a^2}) - w^{\frac{1}{2}} + C = x\ln(x+\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + C$$

63.

$$\begin{array}{ll} u = \sin bx & dv = e^{ax} \\ du = b\cos bx dx & v = \frac{1}{a}e^{ax} \end{array}$$

$$\begin{array}{ll} u = \cos bx & dv = e^{ax}dx \\ du = -b\sin bx dx & v = \frac{1}{a}e^{ax} \end{array}$$

$$\begin{aligned} \int e^{ax}\sin bx dx &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx dx \\ &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left[\frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx dx\right] \\ &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx dx \end{aligned}$$

Thus: $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin bx dx = \frac{e^{ax}}{a}\left(\sin bx - \frac{b}{a}\cos bx\right) + C$

$$\Rightarrow \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$$

65. $u(v + C) - \int(v + C)du = uv + uC - \int vdu - \int Cdu = uv + uC - \int vdu - Cu = uv - \int vdu$

7.2 COMPLETING THE SQUARE AND PARTIAL FRACTIONS (PAGE 278)

1. $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$ **3.** $\frac{1}{4}\tan^{-1}\left(\frac{x+3}{4}\right) + C$ **5.** $\sin^{-1}\left(\frac{x-3}{3}\right) + C$ **7.** $\frac{1}{10}\tan^{-1}\left(\frac{2x+2}{5}\right) + C$

9. $\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right) + C$ **11.** $-\sqrt{3 + 2x - x^2} + \sin^{-1}\left(\frac{x-1}{2}\right) + C$ **13.** $-\frac{\pi}{24}$ **15.** $\frac{\pi}{6}$

17. $\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C$ **19.** $\ln\frac{(x-2)^2}{|x-1|} + C$ **21.** $\frac{1}{4}\ln|(x+1)(x-1)^3| - \frac{1}{2x-2} + C$

23. $\frac{2}{21}\ln|3x-2| + \frac{1}{14}\ln|2x+1| + C$ **25.** $\ln\left|\frac{(x-3)^2}{x^2+x}\right| + C$

27. $\ln|(x+4)^3(x-3)^2| + \frac{1}{x+4} + C$ **29.** $\ln\left[\frac{x^2}{(x+2)^2}\right] - \frac{1}{2x^2} + C$ **31.** $\frac{1}{x^2+1} + \frac{1}{2}\ln(x^2+1) + C$

33. $\frac{1}{2}\ln(x^2+1) - \ln|x| + \tan^{-1}x + C$ **35.** $\frac{5}{2}\ln(x^2+2) + \ln(x^2+1) - 3\tan^{-1}x + C$

37. $x + \ln\left|\frac{x}{(x+2)^3}\right| + C$ **39.** $-\frac{x^2}{2} - 3x + \frac{1}{2}\ln(x^2+1) + C$ **41.** $\frac{1}{5}\ln\left|\frac{\sin x - 2}{\sin x + 3}\right| + C$

43. $\frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$ **45.** $\frac{-\ln 2}{6}$ **47.** $\ln(4.41)$ **49.** $3 + \sqrt{3} - \ln 3 + \frac{49\pi}{12}$

51. $\ln 4 - \tan^{-1}2 + \frac{\pi}{4}$ **53.** $\ln\left|\frac{2(x-2)}{x-1}\right|$ **55.** $\frac{3}{4}\ln\frac{21}{13} + \ln 3$ **57.** $\pi\left(\frac{19}{105} + \frac{1}{8}\ln\frac{15}{7}\right)$

59. $\frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \Rightarrow 1 = A(x+a) + B(x-a) = a \Rightarrow x = -a: B = -\frac{1}{2a};\ x = a: A = \frac{1}{2a}$

$$\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\int\frac{dx}{x-a} - \frac{1}{2a}\int\frac{dx}{x+a} = \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C = \frac{1}{2a}\left|\frac{x-a}{x+a}\right| + C$$

7.3 POWERS OF TRIGONOMETRIC FUNCTIONS AND TRIGONOMETRIC SUBSTITUTION (PAGE 288)

1. $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ **3.** $\frac{1}{9}\cos^3 3x - \frac{1}{3}\cos 3x + C$ **5.** $\frac{3}{8}x + \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$

7. $\frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$ **9.** $\frac{1}{2}\tan^2 x - \ln|\sec x| + C$ **11.** $-\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

13. $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$ **15.** $\frac{1}{10}\tan^2 5x + \frac{1}{20}\tan^4 5x + C$ **17.** $x\sec x - \ln|\sec x + \tan x| + C$

19. $\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$ **21.** $\frac{1}{3}\csc^3 x - \frac{1}{5}\csc^5 x + C$ **23.** $\frac{1}{2}\sin 2x + C$

25. $\frac{2}{3}(\tan x)^{\frac{3}{2}} + \frac{2}{7}(\tan x)^{\frac{7}{2}} + C$ **27.** $\frac{5\sqrt{2}}{12}$ **29.** 0 **31.** $\sqrt{3} - \frac{\pi}{3}$ **33.** $\frac{3\pi}{4}$

35. $48\sqrt{3}$ **37.** $\frac{1}{2}(\sin^{-1}x - x\sqrt{1-x^2}) + C$ **39.** $-\sqrt{9-x^2} + C$ **41.** $-\frac{\sqrt{25-x^2}}{25x} + C$

43. $\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C$ **45.** $\frac{\sqrt{4x^2-9}}{9x} + C$ **47.** $-\frac{x}{\sqrt{9x^2-1}} + C$

49. $-2\sqrt{2-x^2} + \frac{1}{3}(2-x^2)^{\frac{3}{2}} + C$ **51.** $-\frac{(x^2+16)^{\frac{3}{2}}}{48x^3} + C$ **53.** $\frac{1}{4}\sin^{-1}2x + \frac{1}{2}x\sqrt{1-4x^2} + C$

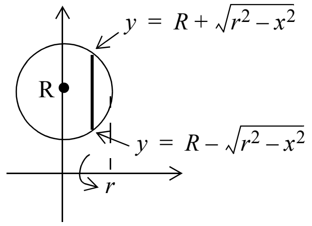
55. $-\frac{1}{3}(1-e^{2x})^{\frac{3}{2}} + C$ **57.** $\frac{25}{2}\sin^{-1}\left(\frac{x-2}{5}\right) + \left(\frac{x-2}{2}\right)\sqrt{21+4x-x^2} + C$

59. $\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1} + 2x + 1\right| + C$

61. $x^2 + 2\ln(x^2-4x+8) + \frac{3}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$ **63.** $4\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)$ **65.** $\ln 2$

67. $\frac{5}{96}$ **69.** $\frac{3\pi}{2}$ **71.** 4 **73.** $\frac{\pi^2}{4} + \frac{\pi}{2}$

75.



$x = r\sin\theta$
$dx = r\cos\theta d\theta$

$$\begin{aligned} V &= 2\pi\int_0^r [(R+\sqrt{r^2-x^2})^2 - (R-\sqrt{r^2-x^2})^2]dx \\ &= 2\pi\int_0^{\frac{\pi}{2}} [(R+r\cos\theta)^2 - (R-r\cos\theta)^2]r\cos\theta d\theta \\ &= 2\pi\int_0^{\frac{\pi}{2}} (4Rr\cos\theta)r\cos\theta d\theta = 8\pi Rr^2\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta \\ &= 4\pi Rr^2\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta \\ &= 4\pi Rr^2\left[\theta + \frac{1}{2}\sin 2\theta\Big|_0^{\pi/2}\right] = 2\pi^2 Rr^2 \end{aligned}$$

77. $\int \tan^n x \sec^m x dx = \int \tan x \tan^{n-1} x \sec x \sec^{m-1} x dx = \int \sec^{m-1} x \tan^{n-1} x \sec x \tan x dx$. Since n is odd, $n-1$ is even, say $n-1 = 2k$. Then:

$$\tan^{n-1} x = \tan^{2k} x = (\tan^2 x)^k = (\sec^2 x - 1)^k = (\sec^2 x - 1)^{\frac{n-1}{2}}$$

Consequently: $\int \tan^n x \sec^m x dx = \int \sec^{m-1} x (\sec^2 x - 1)^{\frac{n-1}{2}} \sec x \tan x dx$

7.4 A HODGEPODGE OF INTEGRALS (PAGE 296)

1. $-3\ln|x| + x - \frac{2}{x} + C$ **3.** $-\ln|x^2 + 2x + 5| - \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + x + C$ **5.** $\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C$

7. $-\frac{1}{3}\frac{(x^3+1)}{e^{x^3}} + C$ **9.** $\sin^{-1}(e^x) + C$ **11.** $\frac{x}{2}[\sin(\ln 2x) - \cos(\ln 2x)] + C$

13. $\frac{1}{3}\ln\frac{|e^x - 1|}{e^x + 2} + C$ **15.** $\frac{2}{15}(x-5)^{\frac{3}{2}}(3x+20) + C$

17. $-\frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} - 6x^{\frac{1}{6}} + 3\ln\left|\frac{1+x^{1/6}}{1-x^{1/6}}\right| - 2\sqrt{x} + C$ **19.** $\sin^{-1}\left(\frac{2x-3}{3}\right) + C$

21. $\sqrt{x^2 - 1} - \tan^{-1}\sqrt{x^2 - 1} + C$ **23.** $\ln\left|\sqrt{x^2 + 9} + x\right| - \frac{x}{\sqrt{x^2 + 9}} + C$

25. $\frac{x-1}{9\sqrt{x^2 - 2x + 10}} + C$ **27.** $\frac{1}{10}\tan^{-1}\left(\frac{2x+2}{5}\right) + C$ **29.** $-\frac{\tan^{-1} x}{x} + \ln\frac{|x|}{\sqrt{1+x^2}} + C$

31. $\frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C$ **33.** $2\tan x + 2\sec x - x + C$

35. $\sin x \cos x - \ln|\sin x \cos x + 1| + C$ **37.** $\frac{1}{2}\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2} + 2}\right| + C$ **39.** $\ln(\sqrt{2} + 1)$

41. $\frac{1}{4}\ln 2 + \frac{\pi}{8}$ **43.** $\frac{\sqrt{3}\pi}{9}$ **45.** $\frac{1}{4}(3e^4 + 1)$ **47.** $-\frac{\pi}{8}$ **49.** $\frac{1}{6}(3\sqrt{3} - \pi)$

51. $\frac{1}{2}(\sqrt{3} - \sqrt{2})$ **53.** $2 - \sqrt{2}$ **55.** $\frac{2}{5} - \frac{1}{2}\cos\frac{\pi}{10}$ **57.** $\frac{4}{9}$ **59.** $-\frac{9}{28}$ **61.** $\frac{\pi}{18}(8\sqrt{3} - 9)$

63. (a) $\cos(A-B) - \cos(A+B) = (\cos A \cos B + \sin A \sin B) - (\cos A \cos B - \sin A \sin B) = 2\sin A \sin B$

(b) $\sin(A+B) + \sin(A-B) = (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B) = 2\sin A \cos B$

(c) $\cos(A-B) + \cos(A+B) = (\cos A \cos B + \sin A \sin B) + (\cos A \cos B - \sin A \sin B) = 2\cos A \cos B$

8.1 L'HOPITAL'S RULE (PAGE 308)

1. $\frac{1}{8}$ **3.** 0 **5.** 0 **7.** 1 **9.** 0 **11.** 1 **13.** 0 **15.** 1 **17.** 4 **19.** $\frac{1}{3}$

21. 2 **23.** $\frac{1}{2}$ **25.** 2 **27.** ∞ **29.** $\ln a - \ln b$ **31.** $-\frac{1}{6}$ **33.** 2 **35.** 0

37. 1 **39.** 0 **41.** 0 **43.** $\frac{1}{2}$ **45.** $\ln 2$ **47.** 1 **49.** e **51.** $\frac{1}{\sqrt{e}}$

53. ∞ **55.** 0 and ∞ **57.** $\frac{3}{2}$ **59.** $\frac{1}{2}$ **61.** $\frac{1}{6}$ **63.** $a = \pm 4$

65. Using the facts that $(e^x)' = e^x$ and that the n^{th} derivative of x^n equals $n!$ (Exercise 64, page 88), we apply L'Hopital's rule n times to go from $\lim\limits_{x \to \infty} \frac{e^x}{x^n}$ to: $\lim\limits_{x \to \infty} \frac{e^x}{n!} = \infty$.

8.2 IMPROPER INTEGRALS (PAGE 316)

1. 1 **3.** -1 **5.** $-\frac{1}{4}$ **7.** $\frac{\pi}{4}$ **9.** Diverges **11.** $\frac{1}{2}$ **13.** 0 **15.** $\pi(1 + \sqrt{2})$

17. Diverges **19.** 6 **21.** 4π **23.** Diverges **25.** $-10 \cdot 2^{\frac{4}{5}}$ **27.** $10 \cdot 2^{3/5}$

29. $\tan^{-1} 2 + \ln 5 - \frac{\pi}{2}$ **31.** 2 **33.** $-\frac{3}{2}$ **35.** Diverges **37.** $\frac{\pi}{2}$ **39.** $-\frac{1}{4}$

41. $3(1 - 2^{2/3})$ **43.** $a > -1$ **45.** 3 **47.** $n > -1$

49. $\int_0^{\infty} \sin x dx = \lim\limits_{t \to \infty} \int_0^t \sin x dx = \lim\limits_{t \to \infty} (-\cos x)\Big|_1^t = \lim\limits_{t \to \infty} -[\cos t - \cos 0] = \lim\limits_{t \to \infty} [-\cos t + 1]$
which diverges since the cosine continues to vary from -1 to 1.

$$\int_{-\infty}^0 \sin x dx = \lim_{t \to -\infty} \int_t^0 \sin x dx = \lim_{t \to -\infty} (-\cos x)\Big|_t^0 = \lim_{t \to -\infty} [-\cos 0 - \cos t] = \lim_{t \to -\infty} [-1 + \cos t]$$

which diverges since the cosine continues to vary from -1 to 1.

$\lim\limits_{t \to \infty} \int_{-t}^t \sin x dx = \lim\limits_{t \to \infty} (-\cos x)\Big|_{-t}^t = \lim\limits_{t \to \infty} -[\cos t - \cos(-t)] = 0$, since $\cos t = \cos(-t)$.

51. 1 **53.** $\frac{4}{3}$ **55.** 2π **57.** $\frac{\pi}{7}$

9.1 SEQUENCES (PAGE 330)

1. $\frac{n}{n+1}$ **3.** $(-1)^{n+1}\frac{n^2}{3n-1}$ **5.** (a) 1, (b) 50, (c) 5,000, (d) smallest integer $\geq \frac{1}{2\varepsilon^2}$

7. (a) $\frac{2}{5}$, (b) 2, (c) 20, (d) smallest integer $\geq \frac{1}{5\varepsilon}$ **9.** As $n \to \infty$, $\frac{n}{10^{100}} \to \infty$

11. For $n > 100$, $a_n = \frac{n}{\sqrt{n+100}} > \frac{n}{\sqrt{2n}} = \frac{\sqrt{n}}{\sqrt{2}}$, which tends to ∞ as $n \to \infty$.

13. Converges to 0 **15.** Converges to 5 **17.** Diverges **19.** Diverges **21.** Converges to 0

23. Diverges **25.** Diverges **27.** Diverges **29.** Converges to 5 **31.** 1 **33.** 2

35. $\frac{3}{5}$ **37.** $\frac{1}{\sqrt{2}}$ **39.** 1 **41.** $\ln\frac{1}{5}$ **43.** 2 **45.** Diverges **47.** Converges to 0

49. Converges to $\frac{1}{2}$ **51.** Converges to $\frac{1}{2}$ **53.** Increasing **55.** Decreasing **57.** Decreasing

59. Decreasing **61.** Sequence is decreasing and bounded. **65.** No unique answer.

67. $a_n = \dfrac{n}{rn+1} = \dfrac{1}{r+\frac{1}{n}} \to \dfrac{1}{r}$ as $n \to \infty$, since $\frac{1}{n} \to 0$(as $n \to \infty$).

69. For given $\varepsilon > 0$ choose N such that $n > N \Rightarrow |a_n - L| < \varepsilon$. Since $|a_n - L| \geq |a_n| - |L|$: $n > N \Rightarrow ||a_n| - |L|| < \varepsilon$.

9.2 SERIES (PAGE 342)

1. $\sum_{n=1}^{4} \frac{1}{2n+1}, \sum_{n=0}^{3} \frac{1}{2n+3}$ **3.** $\sum_{n=1}^{5} 10^{n-1}, \sum_{n=0}^{4} 10^{n}$ **5.** $\sum_{1}^{\infty}(-1)^n\frac{5n}{2^n}, \sum_{0}^{\infty}(-1)^{n+1}\frac{5n+5}{2^{n+1}}$

7. (a) $\frac{5}{4}$ (b) $\frac{1}{4}$ (c) $\frac{5}{4}$ **9.** Converges to $\frac{7}{2}$ **11.** Converges to $\frac{500}{99}$ **13.** Converges to 3

15. Diverges **17.** Diverges **19.** Converges to $\frac{1}{6}$ **21.** Converges to $\frac{10}{3}$

23. Converges to $-\frac{5}{2}$ **25.** Converges **27.** Converges **29.** Diverges **31.** Diverges

33. 10,000 terms **35.** 3 terms **37.** 10^8 terms **39.** $s_n = 1 - \frac{1}{\sqrt{n+1}}$, sum $= 1$

41. $s_n = -\frac{1}{\ln 2} + \frac{1}{\ln(n+2)}$, sum $= -\frac{1}{\ln 2}$ **43.** $s_n = \frac{3}{2}\left(1 - \frac{1}{2n+1}\right)$, sum $= \frac{3}{2}$

45. (a) $$\sum_{n=1}^{\infty}(a_n - b_n) = \lim_{N\to\infty}\sum_{n=1}^{N}(a_n - b_n) \underset{\substack{\uparrow \\ \text{Theorem 9.2(b), page 322}}}{=} \lim_{N\to\infty}\sum_{n=1}^{N} a_n - \lim_{N\to\infty}\sum_{n=1}^{N} b_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

(b) $$\sum_{n=1}^{\infty} ca_n = \lim_{N\to\infty}\sum_{n=1}^{N} ca_n \underset{\substack{\uparrow \\ \text{Theorem 9.2(b), page 322}}}{=} c\left(\lim_{N\to\infty}\sum_{n=1}^{N} a_n\right) = c\sum_{n=1}^{\infty} a_n$$

47. (a) Suppose that $\sum(a_n + b_n)$ converges. By Exercise 45(a), $\sum[(a_n + b_n) - a_n] = \sum b_n$ would also converge — contradicting a given condition.

(b) No unique answer.

49. 300 ft **51.** 32 ft^2

9.3 SERIES OF POSITIVE TERMS (PAGE 354)

1. Diverges **3.** Converges **5.** Converges **7.** Converges **9.** Converges
11. Converges **13.** Diverges **15.** Diverges **17.** Diverges **19.** Converges
21. Diverges **23.** Diverges **25.** Diverges **27.** Converges **29.** Converges
31. Converges **33.** Converges **35.** Converges **37.** Converges **39.** Converges
41. Diverges **43.** Converges **45.** Converges **47.** Diverges **49.** Diverges
51. Converges **53.** Converges **55.** Converges **57.** Converges

59. If $p = 0$, then the series is the divergent harmonic series. The series also diverges if $p = 1$ (Exercise 21). The function $f(x) = \frac{1}{x(\ln x)^p}$ is continuous and positive for $x \geq 2$. Moreover, since $f'(x) = -\frac{(\ln x)^{p-1}(p+\ln x)}{x^2(\ln x)^{2p}}$ is negative for x sufficiently large, say for $x > N, f$ *is* decreasing for $x > N$. Applying the Integral Test we have:

$$\int_N^\infty \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty}\int_N^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty}\int_N^{\ln t} u^{-p}du = \lim_{t\to\infty}\frac{1}{(1-p)(\ln x)^{p-1}}\Bigg|_N^{\ln t}$$

$$= \lim_{t\to\infty}\left[\frac{1}{(1-p)[\ln(\ln t)]^{p-1}} - \frac{1}{(1-p)(\ln N)^{p-1}}\right]$$

If $p > 1$, the above integral converges to $-\frac{1}{(1-p)(\ln N)^{p-1}}$, and so the series converges.

If $p < 1$, the above integral diverges to infinity, and so the series diverges.

61. Does not violate the Integral Test, since *f* is not a decreasing function.

63. True **65.** True

9.4 ABSOLUTE AND CONDITIONAL CONVERGENCE (PAGE 362)

1. Conditionally Convergent **3.** Conditionally Convergent **5.** Absolutely Convergent
7. Conditionally Convergent **9.** Absolutely Convergent **11.** Absolutely Convergent
13. Absolutely Convergent **15.** Divergent **17.** Absolutely Convergent **19.** Divergent
21. Conditionally Convergent **23.** Conditionally Convergent **25.** Absolutely Convergent
27. Conditionally Convergent **29.** Conditionally Convergent **31.** Diverges **33.** Divergent

35. If $\sum|a_n|$ converges, then so must $\sum a_n$ (Theorem 9.20). Consequently, if $\sum a_n$ diverges, then so must $\sum|a_n|$.

37. (a) Since $\sum|a_n|$ and $\sum|b_n|$ so does $\sum|a_n| + |b_n|$; as must $\sum|a_n + b_n|$, since $|a_n + b_n| \leq |a_n| + |b_n|$. (b) No (c) No (d) Yes (e) No (f) No

39. (a) Let M be such that $|y_n| \leq M$. Since $|a_n y_n| \leq M|a_n|$, the convergence or $\sum|a_n|$ implies the convergence of $\sum|a_n y_n|$. (b) No unique answer.

9.5 POWER SERIES (PAGE 371)

1. $R = 1, [-1, 1)$ **3.** $R = 2, (-2, 2)$ **5.** $R = 5, (-5, 5)$ **7.** $R = 2, (-6, -2)$

9. $R = \frac{1}{2}, \left(\frac{1}{2}, \frac{3}{2}\right)$ **11.** $R = 1, [-5, -3]$ **13.** $R = 1, (-1, 1]$ **15.** $R = 1, [-1, 1]$

17. $\sum_{n=0}^{\infty} (-1)^n x^n, R = 1, (-1, 1)$ **19.** $2\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}, R = 3, (-3, 3)$

21. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}}, R = 3, (-3, 3)$ **23.** $\ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}, R = 5, (-5, 5)$

25. $\sum_{n=0}^{\infty} (-1)^{n+1} x^n - \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}, R = 1, (-1, 1)$ **27.** $-\sum_{1}^{\infty} \frac{2^n}{n} x^n, R = \frac{1}{2}, \left(-\frac{1}{2}, \frac{1}{2}\right)$

29. $-\sum_{1}^{\infty} \frac{x^{2n}}{n}, R = 1, (-1, 1)$ **31.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}, R = 1, (-1, 1)$

33. $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \Rightarrow f''(x) = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{(n-1)!}$ **35.** $R = \infty$

$$= \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Since $f(x) = f''(x), (f''(x) - f(x) = 0)$

37. $\lim_{n\to\infty} |c_n x^n|^{1/n} = \lim_{n\to\infty} |c_n|^{1/n}|x| = |x|(\lim_{n\to\infty} |c_n|^{1/n}) = |x|L$. By the Root Test, the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x|L < 1 \Rightarrow |x| < \frac{1}{L}$.

9.6 TAYLOR SERIES (PAGE 384)

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad R = 1$ **3.** $\sum_{n=0}^{\infty} (-1)^n x^n, \quad R = 1$ **5.** $\sum_{n=0}^{\infty} 3^n x^n, \quad R = \frac{1}{3}$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}, \quad R = \infty$ **9.** $\sum_{n=0}^{\infty} \frac{(-1)^n 100\pi^{2n}}{(2n)!} x^{2n}, \quad R = \infty$ **11.** $\sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}, \quad R = \infty$

13. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\left(x - \frac{\pi}{6}\right)^{2n+1}}{2(2n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}\left(x - \frac{\pi}{6}\right)^{2n}}{2(2n)!}, R = \infty$ **15.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}, R = 1$

17. $\sum\limits_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}\left(x-\frac{1}{2}\right)^{2n}$, $R = \infty$ **19.** $\frac{1}{3}+\sum\limits_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n3^{2n+1}n!}(x-9)^n$, $R = 9$

21. $\sum\limits_{n=0}^{\infty}\frac{e(x-1)^n}{n!}$, $R = \infty$ **23.** $\sum\limits_{n=0}^{\infty}(-1)^nx^{2n}$, $R = 1$ **25.** $\sum\limits_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{2n+1}$, $R = 1$

27. $\frac{1}{a+b}\sum\limits_{n=0}^{\infty}\frac{(-1)^n(x-a)^n}{(a+b)^n}$, $R = |a+b|$ **29.** $\sum\limits_{n=0}^{\infty}(-1)^nx^n$, $R = 1$ **31.** $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{n+1}$, $R = \infty$

33. $\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}$, $R = \infty$ **35.** $\sum\limits_{n=0}^{\infty}\frac{(-1)^nx^{2n+3}}{(2n+1)!}$, $R = \infty$ **37.** $\sum\limits_{k=0}^{\infty}\binom{-3}{k}\frac{x^k}{2^{k+3}}$, $R = 2$

39. 7 terms **41.** 7 terms

47. (a) $\sum\limits_{n=0}^{\infty}\frac{(-1)^nx^{4n+2}}{(2n+1)!}$ (b) $\sum\limits_{n=0}^{\infty}\frac{(-1)^nx^{4n+3}}{(2n+1)!(4n+3)}+C$ (c) and (d) $\frac{1}{3}-\frac{1}{42}\approx 0.3095$

49. As suggested, we first show that $\frac{d^k}{dx^k}(x-a)^n$ is (1) 0 if $n<k$, (2) $k!$ if $n = k$, and (3) $n(n-1)\dots[n-k+1](x-a)^{n-k}$ if $n>k$:

(1) Every time you differentiate $(x-a)^n$ the power of $x-a$ is reduced by 1. So, the n^{th} derivative is a constant, and higher order derivatives are 0.

(2) (Using Mathematical Induction, page 83)

I. Valid at $n = k = 1$: $(x-a)' = 1 = 1!$

II. Assume validity at $n = k = m$: $\frac{d^m}{dx^m}(x-a)^m = m!$

III. We establish validity at $n = k = m+1$: $\frac{d^{m+1}}{dx^{m+1}}(x-a)^{m+1} = (m+1)!$:

$$\frac{d^{m+1}}{dx^{m+1}}(x-a)^{m+1} = \frac{d^m}{dx^m}\left[\frac{d}{dx}(x-a)^{m+1}\right] = (m+1)\frac{d^m}{dx^m}(x-a)^m$$

$$\text{By II:} = (m+1)m! = (m+1)!$$

(3) (Using Mathematical Induction)

I. Valid at $k = 1$: $[(x-a)^n]' = n(x-a)^{n-1}$

II. Assume validity at $k = m$: $\frac{d^m}{dx^m}(x-a)^n = n(n-1)\dots(n-m+1)(x-a)^{n-m}$

III. We verify that $\frac{d^{m+1}}{dx^{m+1}}(x-a)^n = n(n-1)\dots(n-m)(x-a)^{n-(m+1)}$:

$$\frac{d^{m+1}}{dx^{m+1}}(x-a)^n = \frac{d}{dx}\left[\frac{d^m}{dx^m}(x-a)^n\right] \overset{\text{II}}{=} \frac{d}{dx}[n(n-1)\dots(n-m+1)(x-a)^{n-m}]$$
$$= n(n-1)\dots(n-m+1)\frac{d}{dx}(x-a)^{n-m}$$
$$= n(n-1)\dots(n-m+1)(n-m)(x-a)^{n-m-1}$$

And so we have: $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{k-1} c_n(x-a)^n + c_k(x-a)^k + \sum_{n=k+1}^{\infty} c_n(x-a)^n$

$$\Rightarrow f^{(k)}(x) = \frac{d^k}{x^k}\left(\sum_{n=0}^{k-1} c_n(x-a)^n + c_k(x-a)^k + \sum_{n=k+1}^{\infty} c_n(x-a)^n\right)$$
$$= \sum_{n=0}^{k-1} c_n\frac{d^k}{x^k}(x-a)^n + c_k\frac{d^k}{x^k}(x-a)^k + \sum_{n=k+1}^{\infty} c_n\frac{d^k}{x^k}(x-a)^n$$
$$= \quad 0 \quad + c_k k! \quad + \sum_{n=k+1}^{\infty} c_n n(n-1)\dots(n-k+1)(x-a)^{n-k}$$

Evaluating at $x = a$ we have $f^{(k)}(a) = c_k k!$, or: $c_k = \frac{f^{(k)}(a)}{k!}$

51. Applying the Ratio Test, we first show that the Binomial series converges for $|x| < 1$:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{r(r-1)\cdots(r-k)}{(k+1)!}x^{k+1}}{\frac{r(r-1)\cdots(r-k+1)}{k!}x^k}\right| = \left|\frac{(r-k)x}{k+1}\right| = \left|\frac{\frac{r}{k}-1}{1+\frac{1}{k}}\right||x| \to |x| \text{ as } k \to \infty$$

We now establish the fact that $g(x) = \sum_{n=0}^{\infty}\binom{r}{k}x^k = (1+x)^r$ by showing that the function

$h(x) = (1+x)^{-r}g(x) = 1$.

Step 1: $\boldsymbol{(1+x)g'(x)} = (1+x)\left[\sum_{k=0}^{\infty}\frac{r(r-1)\dots(r-k+1)}{k!}x^k\right]' = (1+x)\sum_{k=1}^{\infty}\frac{r(r-1)\dots(r-k+1)}{k!}kx^{k-1}$

$$= \sum_{k=1}^{\infty}\frac{r(r-1)\dots(r-k+1)}{k!}kx^{k-1} + x\cdot\sum_{k=1}^{\infty}\frac{r(r-1)\dots(r-k+1)}{k!}kx^{k-1}$$
$$= \sum_{k=1}^{\infty}\frac{r(r-1)\dots(r-k)}{k!}kx^{k-1} + \sum_{k=1}^{\infty}\frac{r(r-1)\dots(r-k+1)}{k!}kx^k$$

lower the index by 1 in the first series
and add 0 to the sum in the second series: $= \sum_{k=0}^{\infty}\frac{[r(r-1)\dots(r-k)]\cancel{(k+1)}}{k!\cancel{(k+1)}}x^k + \sum_{k=0}^{\infty}\frac{[r(r-1)\dots(r-k+1)]k}{k!}x^k$

$$= \sum_{k=0}^{\infty}\frac{[r(r-1)\dots(r-k+1)](r-k+k)}{k!}x^k = \boldsymbol{rg(x)}$$

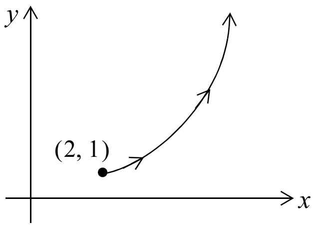
Step 2: $h'(x) = (1+x)^{-r}g'(x) - (1+x)^{-r-1}g(x) = 0$ (by Step 1).

Step 3: $h(x) = c$ (by Step 2).

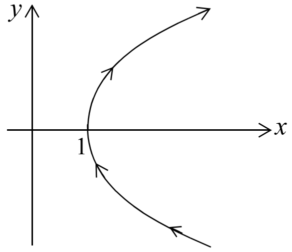
Step 4: $h(x) = h(0) = g(0) = 1$.

10.1 PARAMETRIZATION OF CURVES (PAGE 403)

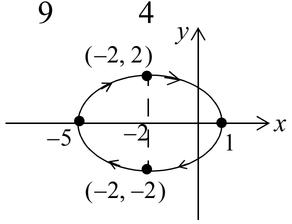
1. $y = (x-1)^3, x \ge 2, y \ge 1$



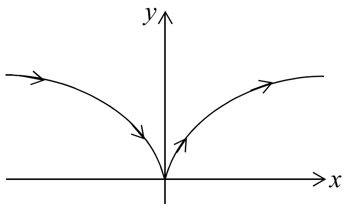
3. $x^2 - y^2 = 1, x \ge 1$



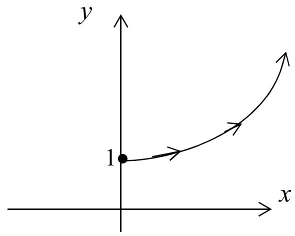
5. $\dfrac{(x+2)^2}{9} + \dfrac{y^2}{4} = 1$



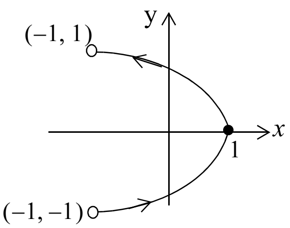
7. $y = x^{2/3}$



9. $y = \sqrt{x^2+1}$



11. $x = 1 - 2y^2$



13. $y = -x$

15. $y = -\dfrac{4}{3}x + 4\sqrt{2}$

17. $y = \dfrac{e^2}{2}x + \dfrac{e^2}{2}$

19. Horizontal: $\left(4, \pm\dfrac{16\sqrt{3}}{9}\right)$, Vertical: $(0, 0)$

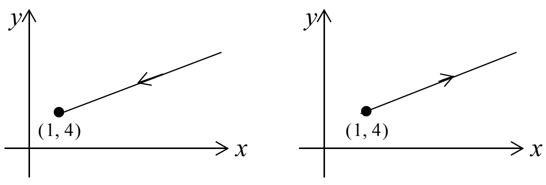
21. Horizontal: $(\pm\sqrt{2}, -1)$, $(\pm\sqrt{2}, 1)$, Vertical: $(\pm 2, 0)$

23. Horizontal: none , Vertical: none

25. Increasing: $t > -\dfrac{2}{3}$, Concave up: $t > 0$

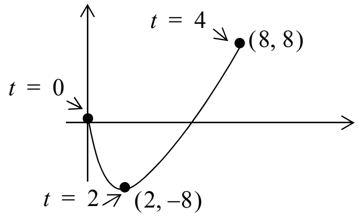
27. Increasing: nowhere , Concave up: $t < 0$

29.



$-\infty < t \le 0$ $\qquad$ $0 \le t < \infty$

31.



33. $4\sqrt{2} - 2$

35. 12

37. $\displaystyle\int_0^2 \sqrt{1 + \left(\frac{65}{4}\right)t + 4t^2}\,dt$

39. $\displaystyle\int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\,dt$

10.2 POLAR COORDINATES (PAGE 414)

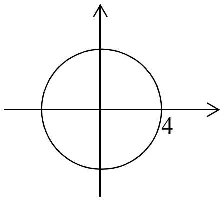
1. $(2, 0)$ **3.** $(3, 0)$ **5.** $(-1, \sqrt{3})$ **7.** $(\sqrt{2}, -\sqrt{2})$ **9.** $\left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$

11. $\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$ **13.** $\left(2, \frac{2\pi}{3} + 2k\pi\right), \left(-2, \frac{5\pi}{3} + 2k\pi\right)$ **15.** $\left(8, \frac{\pi}{6} + 2k\pi\right), \left(-8, \frac{7\pi}{6} + 2k\pi\right)$

17. $r = 3$ **19.** $r = -\cot\theta\csc\theta$ **21.** $r = -4\cos\theta$ **23.** $x^2 + y^2 = 16$

25. $x^2 + y^2 - 3y = 0$ **27.** $y = 1$

29.



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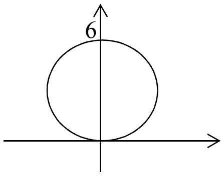

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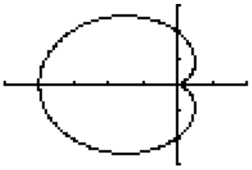

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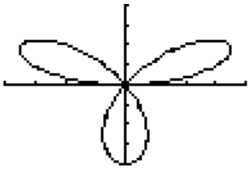

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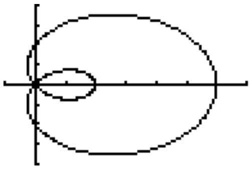

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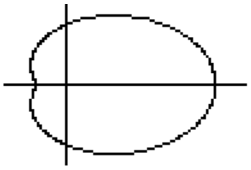
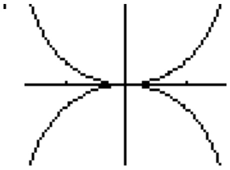
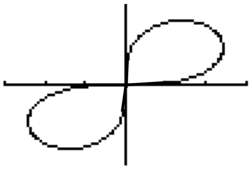


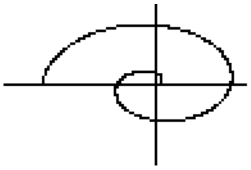
41.



43.



45.



47. $\sqrt{3}$ **49.** 1 **51.** $-\pi$ **53.** Maximum point: $(3, 3\sqrt{3})$, Minimum point: $(3, -3\sqrt{3})$

55. Maximum points: $\left(\pm\frac{2\sqrt{6}}{9}, \frac{2\sqrt{3}}{9}\right)$, Minimum points: $\left(\pm\frac{2\sqrt{6}}{9}, -\frac{2\sqrt{3}}{9}\right)$.

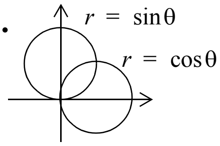
57. Maximum points: $\left(\frac{3 + \sqrt{33}}{4}, 53.6°\right), \left(\frac{3 - \sqrt{33}}{4}, 212.5°\right)$,

Min. points: $\left(\frac{3 + \sqrt{33}}{4}, 306.4°\right), \left(\frac{3 - \sqrt{33}}{4}, 147.5°\right)$ **59.** $(0, 0), \left(-\frac{1}{2}, \frac{1}{2}\right)$ **61.** $(0, 0), \left(\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)$

63. $r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow x^2 - bx + y^2 - ay = 0$

$$\Rightarrow x^2 - bx + \frac{\boldsymbol{b^2}}{\boldsymbol{4}} + y^2 - ay + \frac{\boldsymbol{a^2}}{\boldsymbol{4}} = \frac{\boldsymbol{b^2}}{\boldsymbol{4}} + \frac{\boldsymbol{a^2}}{\boldsymbol{4}} \Rightarrow \underset{\uparrow}{\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}}$$

circle centered at $\left(\frac{b}{2}, \frac{a}{2}\right)$ of radius $\frac{1}{2}\sqrt{a^2 + b^2}$

65. 

From the figure, we see that the circles $r = \cos\theta$ and $r = \sin\theta$ intersect at the origin and at the point with polar coordinates $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$:

$$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow r = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

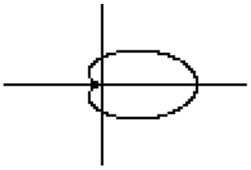
For $r = \cos\theta = f(\theta)$: $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-\sin^2\theta + \cos^2\theta}{-2\sin\theta\cos\theta} = -\frac{\cos 2\theta}{\sin 2\theta}$ (*)

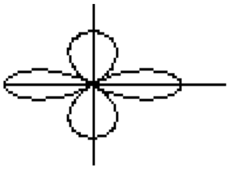
And, for $r = \sin\theta = g(\theta)$: $\frac{dy}{dx} = \frac{g'(\theta)\sin\theta + g(\theta)\cos\theta}{g'(\theta)\cos\theta - g(\theta)\sin\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{\sin 2\theta}{\cos 2\theta}$ (**)

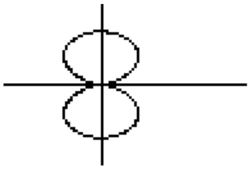
At the point $(0, 0)$, the curve $r = \cos\theta$ has a vertical tangent line [denominator in (*) is 0], whereas the curve $r = \sin\theta$ has a horizontal tangent line [numerator in (**) is 0].

At the point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$, the curve $r = \cos\theta$ has a horizontal tangent line [numerator in (*) is 0], while that of $r = \sin\theta$ is vertical [denominator in (**) is 0].

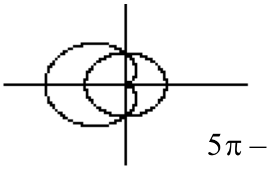
10.3 AREA AND LENGTH (PAGE 422)

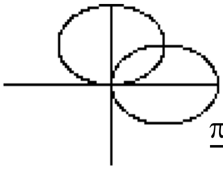
1. 
$\frac{3\pi}{2}$

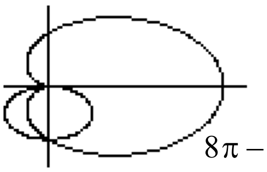
3. 
2π

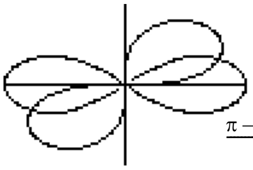
5. 
2π

7. $\frac{(\pi/4)^5}{10}$ **9.** $\frac{4-\pi}{16}$ **11.** $\frac{\pi+2}{4}$

13. 
$5\pi - 8$

15. 
$\frac{\pi-2}{8}$

17. 
$8\pi - 16$

19. 
$\frac{\pi - 2\sqrt{2}}{8}$

21. $18\sqrt{3} - 4\pi$ **23.** $\frac{2\pi - 3\sqrt{3}}{2}$ **25.** 12π **27.** $\frac{40\pi + 42\sqrt{3}}{3}$ **29.** $\frac{6\sqrt{3} - 2\pi}{3}$

31. 8 **33.** $\frac{\pi+3}{8}$ **35.** $3[\sqrt{2} + \ln(1 + \sqrt{2})]$ **37.** 38.75