If $h(x) = (x-3)^2$ then $h(8) = 25$. You get that answer by first subtracting 3 from 8: $(x-3) = (8-3) = \mathbf{5}$, and then squaring the result: $(\mathbf{5})^2 = 25$. In other words, you **first** apply the function $f(x) = x-3$, **and then** apply the function $g(x) = x^2$ to that result. This operation of first performing one function, and then another on that result, is called composition, and is denoted by $(g \circ f)(x)$:

DEFINITION: The **composition** $(g \circ f)(x)$ is given by:

$$(g \circ f)(x) = g(f(x))$$

↑ **first** apply f

↑ and **then** apply g

Here is a solved composition problem for your consideration:

EXAMPLE 1 Determine $(g \circ f)(2)$ and $(f \circ g)(2)$ for:

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = 2x - 3$$

SOLUTION: $(f \circ g)(2) = f(g(2)) = f(2 \cdot 2 - 3) = f(1) = 1^2 + 1 = 2$

$$(g \circ f)(2) = g(f(2)) = g(2^2 + 1) = g(5) = 2 \cdot 5 - 3 = 7$$

YOUR TURN.

1. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for: $f(x) = 3x - 1$ and $g(x) = -2x^2 - 3x + 1$

2. Express the function $h(x) = \sqrt{x^2 - 1} + 5$ as a composition $h = g \circ f$.

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DEFINITION: A function f is **one-to-one** if for all a and b in D_f:

$$\text{If } f(a) = f(b) \text{ then } a = b$$

EXAMPLE 2 Show that the function $f(x) = \dfrac{x}{5x+2}$ is one- to-one

SOLUTION: Appealing to the above definition, we begin with $f(a) = f(b)$, and show that this can only hold if $a = b$:

$$\begin{aligned} f(a) &= f(b) \\ \frac{a}{5a+2} &= \frac{b}{5b+2} \\ a(5b+2) &= b(5a+2) \\ 5ab + 2a &= 5ab + 2b \\ 2a &= 2b \\ a &= b \end{aligned}$$

YOUR TURN.

Show that the function $f(x) = \dfrac{x}{x+1}$ is one-to-one.

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DEFINITION: The inverse of a one-to-one function f with domain D_f and range R_f is that function f^{-1} with domain R_f and range D_f such that:

$$(f^{-1} \circ f)(x) = x \text{ for every } x \text{ in } D_f \quad \text{and} \quad (f \circ f^{-1})(x) = x \text{ for every } x \text{ in } R_f$$

EXAMPLE 3 Find the inverse of the one-to-one function:

$$f(x) = \frac{x}{5x+2}$$

SOLUTION:

Start with: $f(f^{-1}(x)) = x$

For notational convenience, substitute t for $f^{-1}(x)$: $f(t) = x$

Since $f(x) = \frac{x}{5x+2}$: $\frac{t}{5t+2} = x$

Solve for t:

$$t = (5t+2)x$$
$$t = 5tx + 2x$$
$$t - 5tx = 2x$$
$$t(1-5x) = 2x$$
$$t = \frac{2x}{1-5x}$$

Substituting $f^{-1}(x)$ back for t: $f^{-1}(x) = \frac{2x}{1-5x}$

YOUR TURN.

Determine the inverse of the one-to-one function $f(x) = \frac{x}{x+1} + 3$

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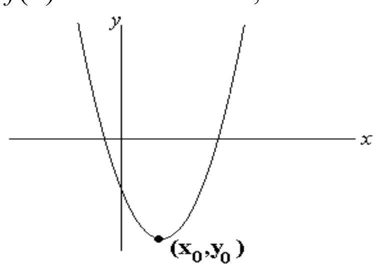
DEFINITION: A **quadratic function** is a function which can be expressed in the form:

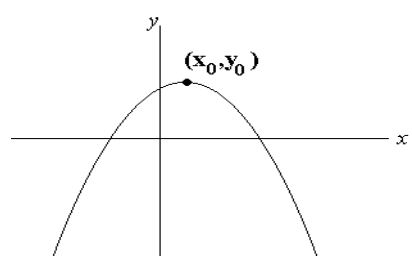
$$f(x) = ax^2 + bx + c, \text{ with } a \neq 0$$

Fact: The graph of quadratic functions are parabolas that opens upward if $a > 0$, and downward if $a < 0$:

$f(x) = ax^2 + bx + c$, with $a > 0$



$f(x) = ax^2 + bx + c$, with $a < 0$



The x-coordinate of the vertex (x_0, y_0) of the parabola $f(x) = ax^2 + bx + c$ occurs at $x_0 = -\frac{b}{2a}$.

EXAMPLE 4 Determine the maximum or minimum value of the given function, and indicate whether it is a maximum or a minimum:

(a) $f(x) = 4(x-5)^2 - 7$ (b) $f(x) = -2x^2 - 4x + 1$

SOLUTION:

(a) The graph of $f(x) = 4(x-5)^2 - 7$ opens upward. From the given standard form, we see that the minimum value of -7 occurs at $x = 5$.

(b) The graph of $f(x) = -2x^2 - 4x + 1$ opens downward, with maximum value occurring at $x = -\frac{b}{2a} = -\frac{-4}{2(-2)} = -1$.

YOUR TURN.

Determine the maximum or minimum value of:

(a) $f(x) = -3(x+2)^2 + 11$ (b) $f(x) = 3x^2 - 2x + 4$

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The main step in the solution process of an optimization word problem is to express the quantity to be optimized as a function of one variable. To achieve that end, we suggest the following 4-step procedure:

Step 1. See the problem.

Step 2. Express the quantity to be maximized or minimized in terms of any convenient number of variables.

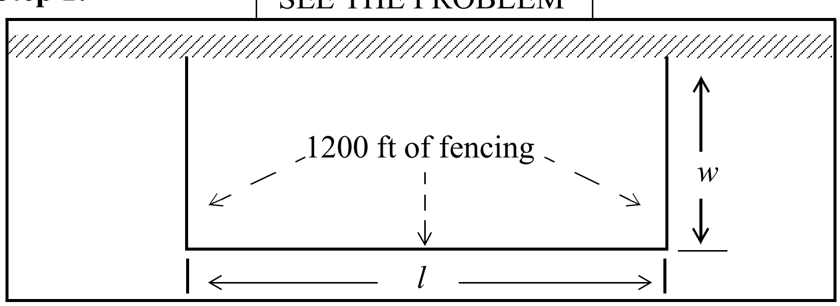
Step 3. If the expression in Step 2 involves more than one variable, use the given information to eliminate all but one variable in that expression.

Step 4. Either "by hand," or using a graphing calculator (for an approximate solution), find where the maximum or minimum of the single variable function obtained in Step 3 occurs, and then use it to determine the specified quantities.

EXAMPLE 5 A farmer has 1200 feet of fencing with which to enclose a rectangular field. By taking advantage of the straight bank of a river, only three sides of the region will have to be fenced. Find the dimensions of the region of maximum area.

SOLUTION:

Step 1: SEE THE PROBLEM



Step 2: Area is to be maximized, and we can easily express it in terms of the two indicated variables:

$$A = lw \qquad (*)$$

Step 3: To eliminate one of the variables (either l or w), we look for a relation involving those variables, and find it in the given condition that 1200 feet of fencing is to be used:

$$l + 2w = 1200 \text{ or: } l = 1200 - 2w \qquad (**)$$

By substituting $1200 - 2w$ for l in (*), we are able to express A as a function of one variable:

$$A = lw = (1200 - 2w)w = -2w^2 + 1200w$$

Step 4: The above is a quadratic function (in the variable w) with parabolic graph opening downward. Applying Theorem 6.1, we find that the maximum value of $A(w)$ occurs at:

$$w = -\frac{b}{2a} = \frac{-1200}{-4} = 300$$

Returning to (**), we obtain the length l:

$$l = 1200 - 2(300) = 600$$

Conclusion: To enclose the greatest area, the farmer should lay out a 300 ft by 600 ft region, with the 600 ft length parallel to the river.

YOUR TURN.

A rectangular field is to be enclosed on all four sides with a fence. One side of the field borders a road, and the fencing material to be used for that side costs \$8 per foot. The fencing material for the remaining sides costs \$6 per foot. Find the maximum area that can be enclosed for \$2800.

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Every polynomial function p gives rise to a polynomial equation $p(x) = 0$. As the following examples show, such an equation may easily be solved when the polynomial p is expressed as a product of linear and quadratic factors.

EXAMPLE 6 Solve:

$$x^3 + x^2 - 6x = 0$$

SOLUTION: The solution hinges on the important fact that a product is zero **if**, and **only if**, one of its factors is zero:

$$x^3 + x^2 - 6x = 0$$

Pull out the common factor, x: $\quad x(x^2 + x - 6) = 0$

factor further: $\quad x(x+3)(x-2) = 0$

A product is zero if, and **only if**, one of its factors is zero: $\quad x = 0 \text{ or } x = -3 \text{ or } x = 2$

YOUR TURN.

Solve: $(x^2 - 7)(x^4 - 81)(x^3 + 8)(x^3 - 7) = 0$

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Solving Polynomial Inequalities

When a factor $(x-c)$ occurs an odd number of times in the factorization of a polynomial $p(x)$ we will say that c is a **odd-zero** of $p(x)$. When $(x-c)$ occurs an even number of times in the factorization of $p(x)$ we will say that c is an **even-zero** of $p(x)$.

For example, 7 is an odd-zero and -5 is an even- zero of:

odd ⟶ $(x-7)^3$, even ⟶ $(x+5)^2$

$$(x-7)^3(x+5)^2$$

Note that the sign of the above polynomial **will change** about the **odd-**zero 7 [since the sign of $(x-7)^3$ will be positive for $x>7$ and negative for $x<7$], and that the sign of the polynomial will **not change** about the **even**-zero -5 [since $(x+5)^2$ is positive on either side of -5]. In general:

> The sign of a polynomial will change as one traverses an odd-zero of the polynomial and will not change as one traverses an even-zero of the polynomial.

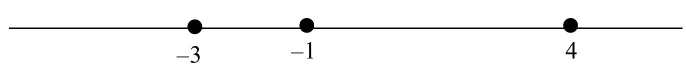
EXAMPLE 7 Solve:

$$(x+3)^3(x+1)^2(4-x)<0$$

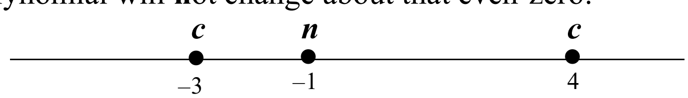
SOLUTION:

Step 1. Chart the sign of the polynomial as follows:

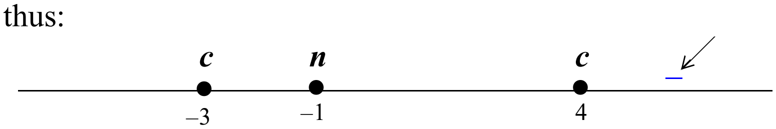
Locate the zeros of the polynomial on the number line:



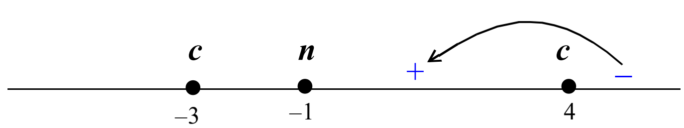
Place the letter ***c*** above the zeros of factors which are raised to an odd power [the factors $(x+3)^{\mathbf{3}}$ and $(4-x)^{\mathbf{1}}$]; this is to remind us that the sign of the polynomial will **c**hange about the odd-zeros -3 and 4. Place the letter ***n*** above the even-zero -1 to remind us that the sign of the polynomial will **n**ot change about that even-zero:



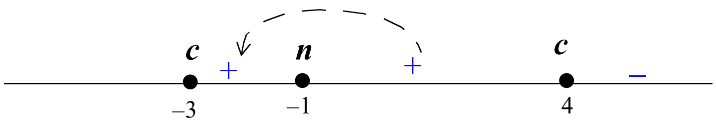
Determine the sign of the polynomial $(x+3)^3(x+1)^2\mathbf{(4-x)}$ to the right of its last zero. In this example, for $x>4$ the product is easily seen to be **negative** [$\mathbf{(4-x)}$ is the only negative factor], thus:



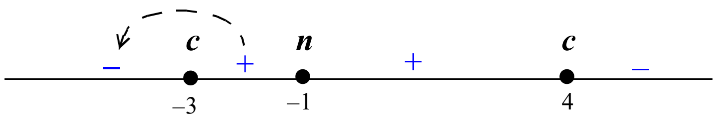
The ***c*** above 4 indicates that the sign will change about 4 (from negative to positive):



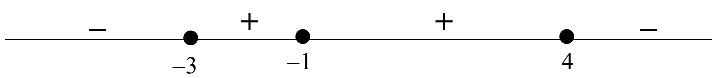
The ***n*** above -1 indicates that the sign will not change as you traverse -1 (will remain positive):



Finally, the ***c*** above -3 indicates a sign changes:



Here is the end result:



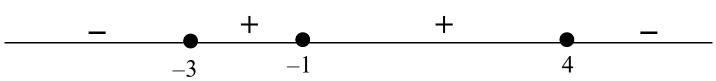
SIGN $(x+3)^3(x+1)^2(4-x)$

Step 2. Since we are solving $(x+3)^3(x+1)^2(4-x)<0$ we read off the intervals where the polynomial is negative (the "**–**" intervals): $(-\infty,-3)\cup(4,\infty)$.

NOTE: The above information also enables us to solve the inequalities

$$(x+3)^3(x+1)^2(4-x)>0: \quad (-3,-1)\cup(-1,4)$$
$$(x+3)^3(x+1)^2(4-x)\ge 0: \quad [-3,4]$$
$$(x+3)^3(x+1)^2(4-x)\le 0: \quad (-\infty,-3]\cup[4,\infty)\cup\{-1\}$$

Here is the end result:



SIGN $(x+3)^3(x+1)^2(4-x)$

YOUR TURN.

Solve: $(x+1)^2(x+2)^3(x-4)^2\ge 0$

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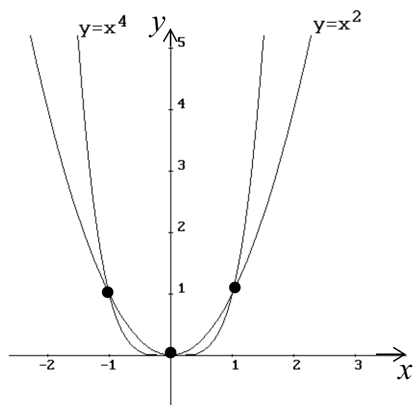
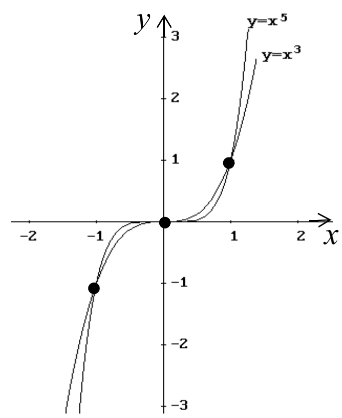
Graphing Polynomial Functions

There is no question that graphing calculators can graph most functions better and faster than any of us, but this does not diminish the importance of this section. As you will see, being able to graph a function, requires an understanding of important concepts

As is depicted below, the graphs of polynomial functions of the form $f(x) = x^n$ ($n>1$, an integer) fall into two categories:

***n* even**	***n* odd**
The graph of every $y = x^{\text{even}}$ is similar to those of the functions $y = x^2$, and $y = x^4$ (below). Each such graph passes through the origin, and the points $(-1,1)$ and $(1,1)$. The larger the exponent, the flatter is the graph over $(-1, 1)$ and the steeper outside of $(-1, 1)$.	The graph of every $y = x^{\text{odd}}$ is similar to those of the functions $y = x^3$, and $y = x^5$ (below). Each such graph passes through the origin, and the points $(-1,-1)$ and $(1,1)$. The larger the exponent, the flatter is the graph over $(-1, 1)$ and the steeper outside of $(-1, 1)$
	

A graphing calculator enables you to view only a portion of the graph of a function. To determine how a graph behaves far from the origin (as x tends to $\pm\infty$, written: $x \to \pm\infty$), use the following fact:

> Far away from the origin the graph of the polynomial function:
> $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
> resembles, in shape, that of its leading term $g(x) = a_n x^n$

For example, as $x \to \pm\infty$, the graph of $p(x) = 6x^4 - 3x^3 - 15x^2 - 10$ resembles that of $g(x) = 6x^4$. This makes sense, since as x gets larger and larger in magnitude, the term $6x^4$ becomes more and more dominant.

The following procedure will be used to graph polynomial functions that can readily be expressed as a product of linear and quadratic factors.

Step 1. **Factor the polynomial**.

Step 2. Near the origin:

Determine and plot x- and y-intercepts.

Chart the sign of the function to see where the graph lies above the x-axis and where it lies below the x-axis.

Step 3. Far from the origin:

Determine the shape of the graph as $x \to \pm\infty$.

Step 4. Sketch the anticipated graph.

EXAMPLE 8 Sketch the graph of:

$$f(x) = x^4 - x^3 - 6x^2$$

SOLUTION:

Step 1. Factor:

$$f(x) = x^4 - x^3 - 6x^2 = x^2(x^2 - x - 6) = x^2(x+2)(x-3)$$

Step 2. y-intercept: $f(0) = 0$ [see Figure (a) below]

x-intercepts: $f(x) = x^2(x+2)(x-3) = 0$ at 0, -2, and 3 [].

SIGN $f(x)$: From the sign information [top of Figure 6.10(a)], we see that moving from left to right, the graph crosses from above the x-axis to below the x-axis at -2; touches the origin but is negative on both sides of 0, and then crosses from below the x-axis to above the x-axis at 3 [see Figure (a) below].

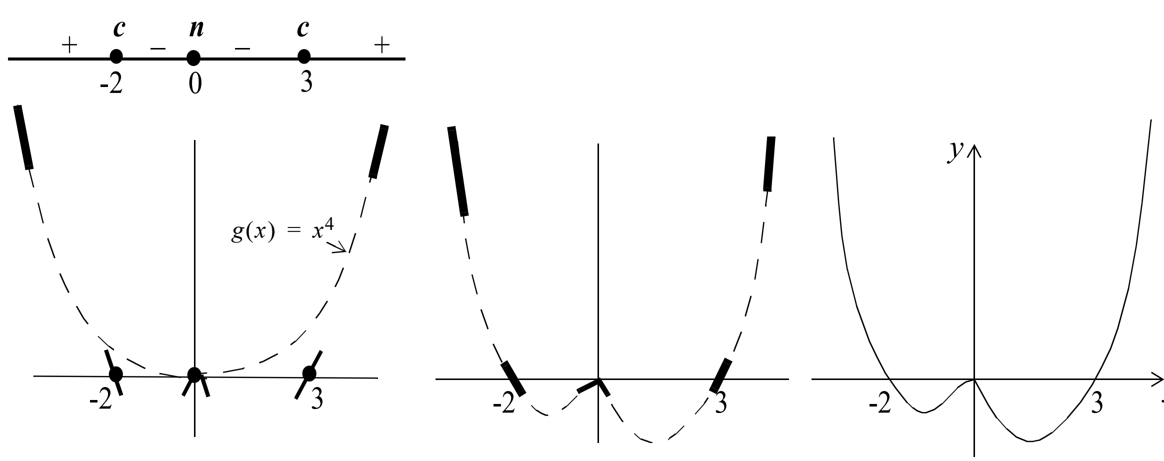
Step 3. As $x \to \pm\infty$: The farther away from the origin, the more the graph of f resembles that of its leading term $g(x) = x^4$ [see Figure (b) below]

.

Step 4. Sketch the anticipated graph: See Figure (b) and the (anticipated) final graph in Figure (c) below.

Note: Without the calculus, we can not determine where the two local minima occur, nor do we know that the local minimum between 0 and 3 is smaller than the local minimum between -2 and 0.

SIGN $x^2(x+2)(x-3)$:



(a) (b) Graph of $f(x) = x^4 - x^3 - 6x^2$ (c)

YOUR TURN.

Sketch the graph of: $f(x) = x^3 - 3x^2 - 13x + 15$

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Definition: A **rational function** is a function of the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials, and $q(x)$ is not the zero polynomial

A rational equation of the form $\frac{p(x)}{q(x)} = \frac{p_1(x)}{q_1(x)}$ can be solved by multiplying both sides of the equation by the least common denominator (LCD) of the rational expressions in that equation, and then solving the resulting polynomial equation. It is important to remember, however, that while you can't "lose" a root of an equation by multiplying both sides by any quantity:

> MULTIPLYING BOTH SIDES OF AN EQUATION BY A QUANTITY WHICH CAN BE ZERO MAY INTRODUCE EXTRANEOUS SOLUTIONS. **CHECK YOUR ANSWERS.**

EXAMPLE 9 Solve:

$$\frac{2x^2}{x^2 - x - 6} = \frac{2}{x^2 + 2x} - \frac{1}{x}$$

SOLUTION: Factor all expressions:

$$\frac{2x^2}{(x+2)(x-3)} = \frac{2}{x(x+2)} - \frac{1}{x}$$

Clear denominators by multiplying both sides of the equation by $\boldsymbol{x(x+2)(x-3)}$, the LCD of the three rational expressions:

$$\frac{2x^2}{(x+2)(x-3)} \cdot \boldsymbol{x(x+2)(x-3)} = \frac{2}{x(x+2)} \cdot \boldsymbol{x(x+2)(x-3)} - \frac{1}{x} \cdot \boldsymbol{x(x+2)(x-3)}$$

$$2x^2(\boldsymbol{x}) = 2(\boldsymbol{x-3}) - 1(\boldsymbol{x+2})(\boldsymbol{x-3})$$

$$2x^3 = 2x - 6 - (x^2 - x - 6)$$

$$2x^3 + x^2 - 3x = 0$$

$$x(2x+3)(x-1) = 0$$

$$x = 0,\ x = -\frac{3}{2},\ x = 1$$

At this point, we see that the only **possible** solutions are 0, $-\frac{3}{2}$, and 1. Any candidate which causes a denominator in the original equation to be zero must be discarded. Discarding 0 (as it renders the denominator of $\frac{1}{x}$ to be zero), we find that $-\frac{3}{2}$ and 1 are the only solutions of the given equation.

YOUR TURN.

Solve: $$\frac{x-2}{x^2-4}-\frac{5}{4}=\frac{1}{x-3}$$

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One can use the SIGN method introduced on page 8 to solve rational inequalities where both the numerator and denominator are polynomials expressed in factored form containing only linear and quadratic factors. Consider the following examples.

EXAMPLE 10 Solve:

$$\frac{2}{x-2}+\frac{x}{x+1}+1\geq 0$$

SOLUTION: Combine terms, and factor:

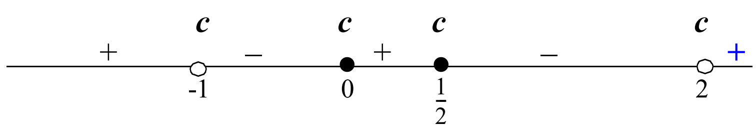
$$\frac{2(x+1)+x(x-2)+(x-2)(x+1)}{(x-2)(x+1)}\geq 0$$

$$\frac{2x+2+x^2-2x+x^2-x-2}{(x-2)(x+1)}\geq 0$$

$$\frac{2x^2-x}{(x-2)(x+1)}\geq 0$$

$$\frac{x(2x-1)}{(x-2)(x+1)}\geq 0$$

Locate the zeros of either the numerator or denominator on the number line, positioning a ***c*** above each, as all are odd zeros. Noting that the rational expression is positive to the right of 2, we placed a "+" over the right-most interval, and then moved to the left, changing the sign each time we crossed over those odd zeros:



SIGN $\frac{x(2x-1)}{(x-2)(x+1)}$

Reading off the intervals with "+" signs, and adding the numbers where the numerator is zero (the black dots), we see that:

$$\frac{2}{x-2}+\frac{x}{x+1}+1\geq 0:\quad (-\infty,-1)\cup[0,\tfrac{1}{2}]\cup(2,\infty)$$

YOUR TURN.

Solve: $\frac{3}{x} - 2 \le 5x$

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Graphing Rational Functions

In many ways, the procedure for sketching the graph of a rational function (which readily factors) is quite similar to that for polynomial functions. In particular, since the leading term of a polynomial function dominates its behavior far away from the origin, to anticipate the behavior of the graph of a rational function as $x \to \pm\infty$ you will want to consider the leading terms of both the numerator and denominator of that function:

Far away from the origin the graph of the rational function:

$$f(x) = \frac{\boldsymbol{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{\boldsymbol{b_m x^m} + b_{m-1}x^{m-1} + \cdots + b_0}$$

resembles, in shape, that of: $g(x) = \frac{\boldsymbol{a_n x^n}}{\boldsymbol{b_m x^m}}$

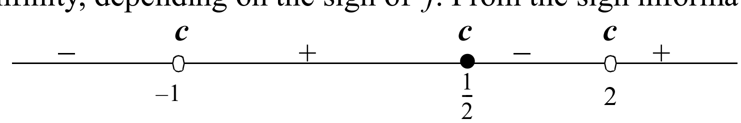
The main difference between graphing polynomial functions and rational functions is that vertical asymptotes might come into play when graphing a rational function; where:

A **vertical asymptote** for the graph of a function *f* is a vertical line about which the graph tends to either plus or minus infinity.

Consider, for example, the function:

$$f(x) = \frac{2x-1}{(x-2)(x+1)}$$

This function is not defined at $x = 2$ and at $x = -1$ (why not?). As x gets closer and closer to 2, the numerator, $2x - 1$ approaches the value $2 \cdot 2 - 1 = 3$, while the denominator $(x-2)(x+1)$ approaches zero. The magnitude of the quotient $\frac{2x-1}{(x-2)(x+1)}$ must therefore get larger and larger, tending to plus or minus infinity, depending on the sign of *f*. From the sign information:



SIGN $f(x) = \frac{2x-1}{(x-2)(x+1)}$

we conclude that:

As x approaches 2 from the left, the values of $f(x)$, being negative, tend to $-\infty$.

As x approaches 2 from the right, the values of $f(x)$, being positive, tend to ∞.

Modifying Step 2 of the 4-step procedure for graphing polynomial functions on page 8 to accommodate vertical asymptotes, enables us to arrive at an anticipated graph of some rational functions. Consider the following example:

EXAMPLE 11 Sketch the graph of the function:

$$f(x) = \frac{4x-7}{2x+5}$$

SOLUTION:

Step 1. Factor: Already in factored form.

Step 2. y-intercept: $y = f(0) = -\frac{7}{5}$ [see Figure (a) below].

x-intercepts: $f(x) = 0$ at $x = \frac{7}{4}$ [see Figure (a) below].

Vertical Asymptotes: The line $x = -\frac{5}{2}$ [see Figure (a) below].

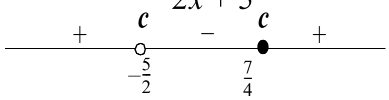
SIGN $f(x)$: From the sign information at the top of Figure (a), we conclude that the graph goes from below the x-axis to above the x-axis as you move from left to right across the x-intercept at $x = \frac{7}{4}$. Since the function is positive to the left of the vertical asymptote at $x = -\frac{5}{2}$, the graph must tend to $+\infty$ as x approaches $-\frac{5}{2}$ from the left. Since the function is negative just to the right of $x = -\frac{5}{2}$, the graph tends to $-\infty$ as x approaches $-\frac{5}{2}$ from the right.

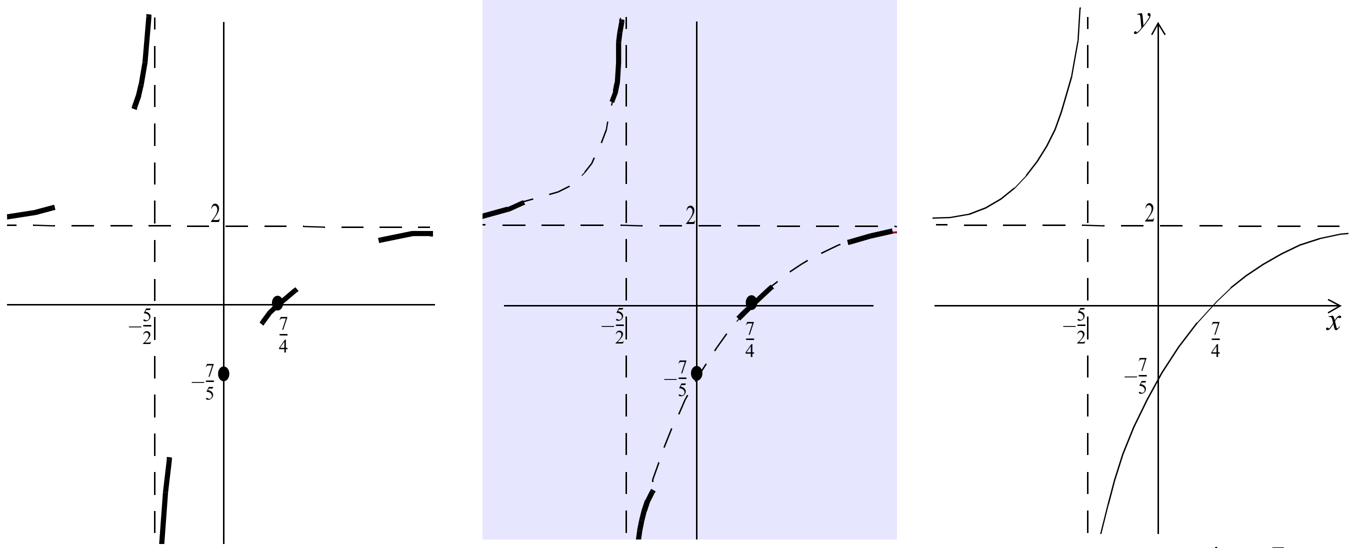
Step 3. As $x \to \pm\infty$: The graph of $f(x) = \frac{\mathbf{4x}-7}{\mathbf{2x}+5}$ approaches the (horizontal) asymptote $y = \frac{\mathbf{4x}}{\mathbf{2x}} = 2$ [see Figure (a) below].

Step 4. Sketch the anticipated graph: See Figure (b) and the (anticipated) final graph in Figure (c) below.

Figure 7.1(b) and the final graph in (c).

SIGN $\frac{4x-7}{2x+5}$:





(a) (b) Graph of $f(x) = \frac{4x-7}{2x+5}$ (c)

YOUR TURN.

Sketch the graph of the function: $f(x) = \frac{x}{x^2+4x-5}$

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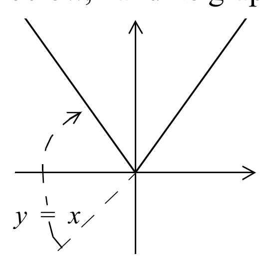
ABSOLUTE VALUE FUNCTIONS, EQUATIONS, AND INEQUALITIES

The absolute value function is defined if Figure (a) below, and its graph appears in Figure (b)

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(a)



(b)

Note that $|a|$ denotes the distance between a and the 0 on the number line, and that $|a-b|$ denotes the distance between a and b. Note also that:

For any $c \geq 0$, $|a| = c$ if and only if $a = c$ or $a = -c$ T

EXAMPLE 12 Solve:

$$|x^2 - 3x - 3| = 1$$

SOLUTION:

$$|x^2 - 3x - 3| = 1$$

$$x^2 - 3x - 3 = 1 \qquad \textbf{OR} \qquad x^2 - 3x - 3 = -1$$

$$x^2 - 3x - 4 = 0 \qquad\qquad x^2 - 3x - 2 = 0$$

$$(x-4)(x+1) = 0 \qquad\qquad x = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

$$x = 4 \text{ or } x = -1$$

We see that the absolute value equation $|x^2 - 3x - 3| = 1$ has four solutions.

YOUR TURN.

Solve: $|x^2 - x - 4| = 2$

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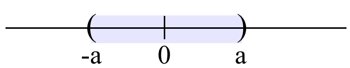
Absolute Value Inequalities

Let $a > 0$. Since $|x|$ represents the distance between x and 0 on the number line, to say that $|x| < a$ is to say that x is contained in the interval $(-a, a)$ [see Figure (a) below]. By the same token, to say that $|x| > a$ is to say that x is outside the interval $[-a, a]$ [see Figure (b) below].



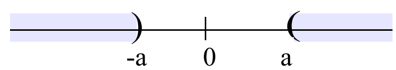
$|x| < a$

x is within a units of 0

$-a < x < a$, which is to say:

$x > -a$ AND $x < a$

(a)



$|x| > a$

x is more than a units from 0

which is to say:

$x < -a$ OR $x > a$

(b)

EXAMPLE 13 Solve:

(a) $|2x - 7| < 5$ (b) $|-4x + 3| \geq 7$

SOLUTION: (a) Turning to figure (a), we have:

$$2x - 7 > -5 \qquad \text{AND} \qquad 2x - 7 > 5$$

$$2x > 2 \qquad\qquad 2x > 12$$

$$x > 1 \qquad\qquad x > 6$$

We conclude that the interval $(1, 6)$ is the solution set of $|2x - 7| < 5$.

(b) Appealing to figure (b), we have: $-4x+3 \le -7$ OR $-4x+3 \ge 7$

$$-4x \le -10 \qquad -4x \ge 4$$

$$x \ge \frac{5}{2} \qquad x \le -1$$

We conclude that $(-\infty, -1\,] \cup [\tfrac{5}{2}, \infty)$ is the solution set of $|-4x+3| \ge 7$.

YOUR TURN.

Solve: (a) $|2x-3| \le -5x+1$ (b) $|2x-3| > 5$

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EQUATIONS INVOLVING ROOTS

We begin by noting that raising both sides of an equation to an odd power yields an equivalent equation. However:

> Raising both sides of an equation to an **even** power can introduce extraneous solutions.
>
> **CHECK YOUR ANSWERS**

EXAMPLE 14 Solve: (a) $\sqrt{2x+13}+1 = x$ (b) $\sqrt{2x+1} - 2\sqrt{x} = -1$

SOLUTIONS:

(a) $\sqrt{2x+13}+1 = x$

Isolate the radical: $\sqrt{2x+13} = x-1$

Square both sides: $(\sqrt{2x+13})^2 = (x-1)^2$

And solve for x: $2x+13 = x^2-2x+1$

$$x^2-4x-12 = 0$$

$$(x-6)(x+2) = 0$$

$$x = 6 \text{ or } x = -2$$

The above argument shows that 6 and -2 are the only **possible** solutions.

Below, we find that while 6 is a solution, -2 is not:

For $x = \mathbf{6}$:	For $x = \mathbf{-2}$:
$\sqrt{2\cdot\mathbf{6}+13}+1 \overset{?}{=} \mathbf{6}$	$\sqrt{2\cdot(\mathbf{-2})+13}+1 \overset{?}{=} \mathbf{-2}$
$\sqrt{25}+1 \overset{?}{=} 6$ — yes	$\sqrt{9}+1 \overset{?}{=} -2$ — no

(b)

$$\sqrt{2x+1} - 2\sqrt{x} = -1$$

$$\sqrt{2x+1} = 2\sqrt{x} - 1$$

Square both sides:

$$2x + 1 = 4x - 4\sqrt{x} + 1$$

$$-2x = -4\sqrt{x}$$

$$x = 2\sqrt{x}$$

Square again:

$$x^2 = 4x$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0 \text{ or } x = 4$$

For $x = \mathbf{0}$:	For $x = \mathbf{4}$:
$\sqrt{2 \cdot \mathbf{0} + 1} - 2\sqrt{\mathbf{0}} \stackrel{?}{=} -1$	$\sqrt{2 \cdot \mathbf{4} + 1} - 2\sqrt{\mathbf{4}} \stackrel{?}{=} -1$
$\sqrt{1} \stackrel{?}{=} -1$ — no	$\sqrt{9} - 4 \stackrel{?}{=} -1$ — yes

We see that the given equation has only one solution: $x = \mathbf{4}$.

YOUR TURN.

Solve: (a) $\sqrt{x+1} = -2$ (b) $\sqrt{2x+1} - 2\sqrt{x} = -1$

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EXPONENTIAL FUNCTIONS

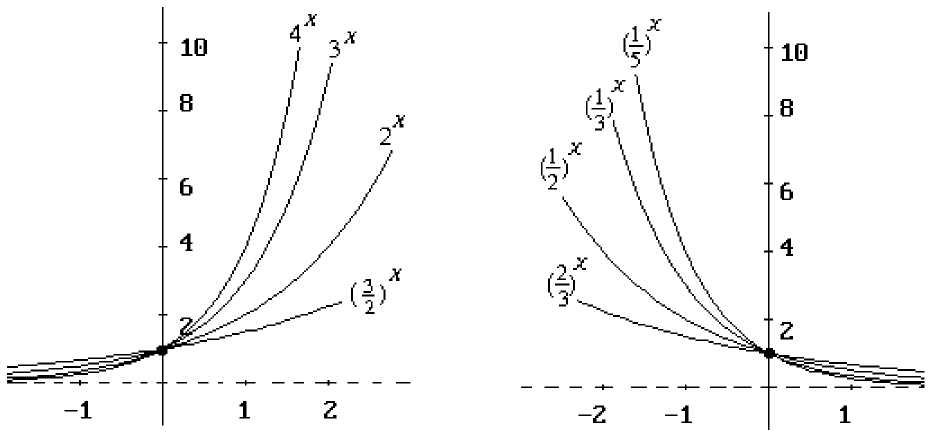
DEFINITION: Let b be a positive number other than 1. The function f given by:

$$f(x) = b^x$$

is said to be the exponential fuction of base b.

In the event that $b > 1$, the exponential function increases dramatically (exponentially),

IIf $b > 1$, then the exponential function increases dramatically (exponentially). If $b < 1$, then the function decrease (see figure below).



Note that all exponential functions are one-to-one. In particular, for any $b \neq 1$:

$$b^s = b^t \text{ if and only if } s = t$$

EXAMPLE 15 Solve:

$$3^{5x+1} = 9^{3x-5}$$

SOLUTION:

$$3^{5x+1} = 9^{3x-5}$$
$$3^{5x+1} = (3^2)^{3x-5}$$
$$3^{5x+1} = 3^{6x-10} \quad \leftarrow \text{common base of 3}$$
$$5x + 1 = 6x - 10$$
$$-x = -11$$
$$x = 11$$

YOUR TURN.

Solve: $\left(\frac{1}{8}\right)^{x^2+1} = \left(\frac{1}{2}\right)^{4x+2}$

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LOGARITHMIC FUNCTIONS

For any positive number b other than one, the exponential function b^x is one-to-one and therefore has an inverse. That inverse function is called the **logarithmic function of base *b***, and is denoted by the symbol $\log_b x$ (read: *log base b of x*).

DEFINITION: For any positive number b, other than 1, and any $x > 0$:

$$\log_b x = y \text{ if and only if } b^y = x$$

IN WORDS: To say that "log base *b* of *x* equals *y*" is the same as saying that "b to the y equals x."

For example:

Logarithmic Form		Exponential Form
$\log_3 9 = 2$	since	$3^2 = 9$
$\log_4(\frac{1}{16}) = -2$	since	$4^{-2} = \frac{1}{16}$
$\log_{\frac{1}{5}} 125 = -3$	since	$(\frac{1}{5})^{-3} = 125$

Two logarithmic functions deserve special mention:

the **common logarithm function:** $\log_{10} x$, or simply $\log x$, and

the **natural logarithm function**: $\log_e x$, or simply $\ln x$, where $e \approx 2.71828$.

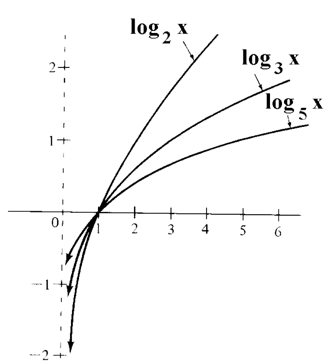
Both the common logarithmic function and the natural logarithmic function are accommodated by calculators. For example, using your calculator you will find that:

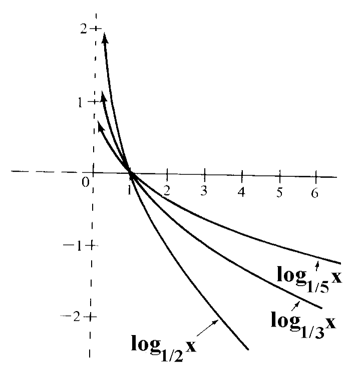
$$\log 22 \approx 1.3424 \quad (\text{i.e. } 10^{1.3424} \approx 22)$$

and

$$\ln 22 \approx 3.0910 \quad (\text{i.e. } e^{3.0910} \approx 22)$$

In the event that $b > 1$, the logarithmic function increases. If $b > 1$, then the logarithmic function decrease (see figure below).



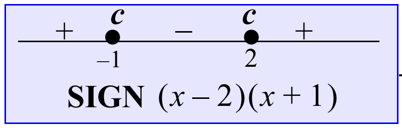


EXAMPLE 16 Find the domain of the function:

$$f(x) = \ln(x^2 - x - 2)$$

SOLUTION: Since logarithmic functions are only defined on positive numbers (see above figure), the problem reduces to solving the inequality:

$$x^2 - x - 2 > 0$$
$$(x-2)(x+1) > 0:$$



$\longrightarrow D_f = (-\infty, -1) \cup (2, \infty)$

YOUR TURN.

Find the domain of the function:

$$f(x) = \frac{\ln(x-3)}{x-5}$$

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Since logarithmic functions are one-to-one, we have:

For $s, t > 0$: $\log_b s = \log_b t$ if and only if $s = t$

EXAMPLE 17 Solve:

$$\ln(x^2 - x) = \ln(-6x + 6)$$

SOLUTION:

$$\ln(x^2 - x) = \ln(-6x + 6)$$

Theorem 9.4: $x^2 - x = -6x + 6$

$$x^2 + 5x - 6 = 0$$
$$(x+6)(x-1) = 0$$
$$x = -6 \text{ or } x = 1$$

We cannot conclude that since -6 is negative and 1 is positive, that -6 is not a solution and that 1 is a solution of the given equation. Indeed, it's the other way around, for $\log_b(\textbf{expression})$ is defined when **expression** > 0, and:

For $x = -6$ both $x^2 - x$ and $-6x + 6$ are **positive**

while for $x = 1$, $x^2 - x = 0$ [Also: $-6(1) + 6 = 0$]

Conclusion: -6 is the only solution of $\ln(x^2 - x) = \ln(-6x + 6)$.

YOUR TURN.

Solve: $\log_5(-7x - 2) = \log_5(8x + 3)$

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LOGARITHMIC PROPERTIES

(a) For any x: $\log_b b^x = x$

(b) For any $x > 0$: $b^{\log_b x} = x$

For any base b, and any r, s, and t with $s > 0, t > 0$:

(c) $\log_b (st) = \log_b s + \log_b t$

(d) $\log_b \left(\frac{s}{t}\right) = \log_b s - \log_b t$

(e) $\log_b s^r = r\log_b s$

Properties (a) and (b) are direct consequences of the fact that logarithmic and exponential functions of the same base are inverses of each other: Applying one, and then the other, brings you back to where you started.

To establish (c) we let:

$$m = \log_b s \quad \text{and} \quad n = \log_b t.$$

Changing to exponential form: $s = b^m$ and $t = b^n$. Then:

$$\log_b st = \log_b b^m b^n = \log_b b^{m+n} = m + n = \log_b s + \log_b t$$

YOUR TURN.

Establish properties (d) and (e)
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One can solve certain exponential equations in which both sides cannot readily be expressed as powers of the same base by applying a logarithmic function [typically, $\ln(x)$] to both side.

EXAMPLE 18 Solve:

$$2^{2x-1} = 3^{3x-7}$$

SOLUTION:

To bring the variable exponents "down," take the natural log of both sides:

$\ln s^r = r\ln s$:

$$\begin{aligned} 2^{2x-1} &= 3^{3x-7} \\ \ln(2^{2x-1}) &= \ln(3^{3x-7}) \\ (2x-1)\ln 2 &= (3x-7)\ln 3 \\ 2x\ln 2 - \ln 2 &= 3x\ln 3 - 7\ln 3 \\ 2x\ln 2 - 3x\ln 3 &= \ln 2 - 7\ln 3 \\ (2\ln 2 - 3\ln 3)x &= \ln 2 - 7\ln 3 \\ x &= \frac{\ln 2 - 7\ln 3}{2\ln 2 - 3\ln 3} \end{aligned}$$

YOUR TURN.

Solve: $3 \cdot 5^{x+1} = 2^{4x-1}$
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Unlike exponential functions which are defined everywhere, logarithmic functions are only defined for positive numbers. Consequently:

When solving logarithmic equations,
CHECK YOUR ANSWER
by verifying that all logarithms in the original equation are defined.

EXAMPLE 19 Solve:

(a) $\log_3(x-1)+\log_3(2x+3) = 1$ (b) $2\ln 2x - \ln(x+1) = \ln x$

SOLUTIONS:

(a)

$$\log_3(x-1)+\log_3(2x+3) = 1$$

$\log_b s + \log_b t = \log_b st$:
$$\log_3[(x-1)(2x+3)] = 1$$

Apply the inverse function of $\log_3 x$, the exponential function of base 3, to both sides:
$$3^{\log_3[(x-1)(2x+3)]} = 3^1$$

$b^{\log_b x} = x$:
$$(x-1)(2x+3) = 3$$
$$2x^2+x-3 = 3$$
$$2x^2+x-6 = 0$$
$$(2x-3)(x+2) = 0$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -2$$

We have two solution candidates, and now challenge each of them, to make sure that each logarithm in the original equation is defined:

Since $x-1$ is negative when $x = -2$, -2 is **not** a solution of the equation (you can only take logs of positive numbers). Since $x-1$ and $2x+3$ are positive for $x = \frac{3}{2}$, $\frac{3}{2}$ **is** a solution of the equation.

(b)

$$2\ln 2x - \ln(x+1) = \ln x$$

$r\ln s = \ln s^r$:
$$\ln(2x)^2 - \ln(x+1) = \ln x$$

$\ln s - \ln t = \ln\left(\frac{s}{t}\right)$:
$$\ln\frac{4x^2}{x+1} = \ln x$$

one-to-one property of logarithmic functions:
$$\frac{4x^2}{x+1} = x$$
$$4x^2 = x^2+x$$
$$3x^2-x = 0$$
$$x(3x-1) = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{1}{3}$$

Since one can only take logs of positive numbers, and since $2x$ is zero when $x = 0$, 0 is **not** a solution of the equation. Since $2x$, $x+1$, and x are all positive for $x = \frac{1}{3}$, $\frac{1}{3}$ **is** a solution of the equation.

INDEX