Math for the Modern World

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PREFACE

This book is designed to introduce the reader to several interesting and useful topics in mathematics.

Chapter 1 offers an introduction to logic, along with some basic algebraic concepts. Systems of equations are addressed in Chapter 2, concluding with a section on linear programming. Exponential and logarithmic functions surface in Chapter 3, which includes a section on exponential growth and decay. The differential calculus is introduced in Chapter 4. Investment topics, such as annuities, are discussed in Chapter 5. Chapter 6 is devoted to probability, and Chapter 7 to the methods of proof, which includes the concept of mathematical induction.

Included in the text are numerous ***Check Your Understanding*** boxes with problems that challenge the student's understanding of newly introduced concepts. Detailed solutions of those problems appear in Appendix A. Still, when attempting a Check Your Understanding problem, we suggest that you only turn to its solution in the appendix after making a valiant effort to solve it by yourself or with others. In the words of Descartes:

> ***We never understand a thing so well, and make it our own, when we learn it from another, as when we have discovered it for ourselves.***

Graphing calculator glimpses appear in the text. In the final analysis, however, one can not escape the fact that

MATHEMATICS DOES NOT RUN ON BATTERIES

CONTENTS

Chapter 4 DIFFERENTIAL CALCULUS

Chapter 5 PERSONAL FINANCE

Chapter 6 PROBABILITY

CHAPTER 1
A Logical Beginning

Mathematics strives, as best it can, to transpose our physical universe into a non-physical form—a thought-universe, as it were, where definitions are the physical objects, and where the reasoning process rules. At some point, one has to ask whether we humans are creating mathematics or whether, through some kind of wonderful mental telescope, we are capable of discovering a meager portion of the mathematical universe which, in all of its glory, may very well be totally independent of we the spectators. Whichever; but one thing appears to be clear: the nature of mathematics, as we observe it or create it, appears to be inspired by what we perceive to be intuitive logic.

§1. PROPOSITIONS

We begin by admitting that "True" and "False" are undefined concepts, and that the word "statement" is also undefined. Nonetheless:

DEFINITION 1.1
PROPOSITION

A **proposition** is a statement that is either True or False.

Fine, we have a definition, but its meaning rests on undefined words! We feel your frustration. Still, mathematics may very well be the closest thing to perfection we have. So, let's swallow our pride and proceed, using our intuitive logic

We can all agree that $5+3 = 8$ is a True proposition, and that $5+3 = 9$ is a False proposition. But how about this statement:

This statement will be proven in Chapter 7

THE SUM OF ANY TWO ODD INTEGERS IS EVEN.

First off, in order to understand the statement one has to know all of its words, including: sum, integer, odd, and even. Only then can one hope to determine if the statement is a proposition; and, if it is, to establish whether it is true or false. All in good time. For now, let us consider two other statements:

> ***HE IS AN AMERICAN CITIZEN.*** This statement cannot be evaluated to be True or False without first knowing exactly who "***He***" is. It is a **variable proposition** (more formally called a **predicate**), involving the variable "***He***." Once ***He*** is specified, then we have a proposition which is either True or False.
>
> ***THE TWO OF US ARE RELATED***. Even if we know precisely who the "*two of us*" are, we still may not have a proposition. Why not? Because the word "*related*" is just too vague.

Compound Propositions

The two numbers 3 and 5 can be put together in several ways to arrive at other numbers: $3+5$, $3-5$, $3 \cdot 5$, and so on. Similarly, two propositions, p and q, can be put together to arrive at other propositions, called compound propositions. One way is to "**and**" them:

In truth table form:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

DEFINITION 1.2
p* and *q
$\boldsymbol{p \wedge q}$

Let p, q be propositions. The **conjunction of *p* and *q*** (or simply: ***p* and *q***), written $p \wedge q$, is that proposition which is True if both p and q are True, and is False otherwise.

For example, the proposition:

$7>5$ **and** $3+5 = 8$ is True.

(since **both** $7>5$ and $3+5 = 8$ are True)

The proposition:

$7>5$ **and** $3+5 = 9$ is False.

(since $7>5$ and $3+5 = 9$ are not **both** True)

Another way to put two propositions together is to "**or**" them:

In truth table form:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that ***p* or *q*** is True when both ***p*** and ***q*** are True. As such, it is said to be the **inclusive or**. The **exclusive or** is typically used in conversation, as in: *Do you want coffee or tea* (one or the other, but not both)

DEFINITION 1.3
p* or *q
$\boldsymbol{p \vee q}$

Let p, q be propositions. The **disjunction of *p* and *q*** (or simply: ***p* or *q***), written $p \vee q$, is that proposition which is False when both p and q are False, and is True otherwise.

For example, the proposition: $7>5$ **or** $3+5 = 8$ is True, as is the proposition $7>5$ **or** $3+5 = 9$. The proposition $7<5$ **or** $3+5 = 9$ is False, since neither $7<5$ nor $3+5 = 9$ is True.

CHECK YOUR UNDERSTANDING 1.1

Let p be the proposition $7 = 5$, and q be the proposition $3+5 = 8$.

(a) Is $p \vee q$ True or False? (b) Is $p \wedge q$ True or False?

(a) True (b) False

The above *and/or* operators, which act on **two** given propositions, are said to be **binary** operators. The following operator is a **unary** operator in that it deals with only **one** given proposition:

In truth table form:

p	$\sim p$
T	F
F	T

DEFINITION 1.4
NOT *P*
$\sim p$

Let p be a proposition. The **negation of *p***, read "**not *p*,**" and written $\sim p$, is that proposition which is False if p is True, and True if p is False.

For example, the proposition:

$\sim(7<5)$ is True, since the proposition $7<5$ is False.

The proposition:

$\sim(3+5=8)$ is False, since the proposition $3+5 = 8$ is True.

CHECK YOUR UNDERSTANDING 1.2

Indicate if the proposition $\sim p$ is True or False, given that

(a) p is the proposition $5 > 3$.

(b) p is the proposition $\sim q$, where q is the proposition $3 = 5$.

(a) False (b) False

The negation operator "~" takes precedence (is performed prior) to both the "$\wedge$" and the "$\vee$" operators.

For example, the proposition:

$$\sim(3 > 5) \vee (3 + 5 = 9) \text{ is True.}$$
$$[\text{since } \sim(3 > 5) \text{ is True}]$$

As is the case with algebraic expressions, the order of operations in a logical expression can be overridden by means of parentheses.

Brackets"[ ]"play the same role as parentheses, and are used to enclose an expression which itself contains parentheses.

For example, the proposition:

$$\sim[(5 > 3) \wedge (3 + 5 = 8)] \text{ is False}$$
$$[\text{since } (5 > 3) \wedge (3 + 5 = 8) \text{ is True}]$$

CHECK YOUR UNDERSTANDING 1.3

Assume that ***p*** and ***q*** are True propositions, and that ***s*** is a False proposition. Determine if the given compound proposition is True or False.

(a) $\sim(p \wedge q)$ (b) $\sim(p \vee q)$ (c) $\sim(s \vee q)$ (d) $\sim(s \wedge q)$

(e) $\sim p \wedge q$ (f) $\sim p \vee q$ (g) $\sim s \vee q$ (h) $\sim s \wedge q$

(i) $(p \vee s) \wedge (q \wedge s)$ (j) $(p \vee s) \vee (q \wedge s)$ (k) $\sim(p \vee s) \wedge (q \vee s)$

True: d, f, g, h, j
False: a, b, c, e, i, k

CONDITIONAL STATEMENT

Consider the statement:

IF IT IS MONDAY, THEN JOHN WILL CALL HIS MOTHER.

For it to become a proposition, we must attribute logical values to it (True or False); and so we shall:

DEFINITION 1.5
if p* then *q
$p \to q$

Let p, q be propositions. The **conditional of *q* by *p***, written $p \to q$, and read ***if p then q***, is False if p is True and q is False, and is True otherwise.

p is called the **hypothesis** and q is called the **conclusion** of $p \to q$.

In truth table form:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

You probably feel comfortable with the first two rows in the adjacent truth table but maybe not so much with the last two. Why is it that for p False, the proposition $p \to q$ is specified to be True for every proposition q, whether q is True or False?

In an attempt to put your mind a bit at ease, let's reconsider the conditional proposition:

IF IT IS MONDAY, THEN JOHN WILL CALL HIS MOTHER.

Suppose it is not Monday. Then John may or may not call his mother. So, if p is False (it is not Monday), then $p \to q$ should not be assigned a value of False.

> Yes, but can't the same "argument" be given to support the assertion that if p is False, then $p \to q$ should not be assigned a value of True? It could, but that option would lead us up an intuitively illogical path (see Exercise 63)

ORDER OF OPERATIONS

First "$\sim$", then "$\wedge$, $\vee$", and finally "$\to$".
In all cases, parentheses take priority.

DEFINITION 1.6 A **tautology** is a proposition which is always True.

EXAMPLE 1.1 Verify that the given proposition is a tautology.

(a) $p \vee \sim p$

(b) $[(p \to q) \wedge p] \to q$

(c) $[(p \to q) \wedge (q \to s)] \to (p \to s)$

SOLUTION:

(a) $p \vee \sim p$:

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Definition 1.3

(b) $[(p \to q) \wedge p] \to q$:

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Definition 1.5

(c) $[(p \to q) \wedge (q \to s)] \to (p \to s)$:

p	q	s	$p \to q$	$q \to s$	$(p \to q) \wedge (q \to s)$	$p \to s$	$[(p \to q) \wedge (q \to s)] \to (p \to s)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Definition 1.5

CHECK YOUR UNDERSTANDING 1.4

Show that $[(p \to q) \vee (q \to p)]$ is a tautology.

See page A-1

LOGICAL IMPLICATION AND EQUIVALENCE

Subjected to the logical implication " $\to$ " the proposition:
If pigs can fly then Donald Duck is the president of the United States
is True, as pigs cannot fly.

Mathematicians start with a True proposition in the hope of being able to establishing the validity of another; they primarily deal with $p \Rightarrow q$

We use the symbol $p \Rightarrow q$, read ***p* implies *q***, to mean that if the proposition p is True, then q must also be True. **Equivalently:**

That $p \to q$ is a tautology.

It is important to note that $p \Rightarrow q$ is **NOT** a proposition for it does not assume values of True or False. It is an assertion (sometimes called a *meta-proposition*).

EXAMPLE 1.2 Verify that $\sim q \wedge (p \to q) \Rightarrow \sim p$.

SOLUTION: We do so by verifying that $\sim q \wedge (p \to q) \to \sim p$ is a tautology:

p	q	$\sim q$	$(p \to q)$	$\sim q \wedge (p \to q)$	$\sim p$	$[\sim q \wedge (p \to q)] \to \sim p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

CHECK YOUR UNDERSTANDING 1.5

Show that

$$[(p \to q) \vee q] \Rightarrow (p \vee \sim p)$$

See page A-1

Suppose you want to show that a proposition p is True, and you know that $p \Leftrightarrow q$. Suppose you like the looks of q better. Fine —deal with q, for p will be True if and only if q is True.

DEFINITION 1.7 LOGICALLY EQUIVALENT Two propositions p and q are **logically equivalent**, written $p \Leftrightarrow q$, if $p \Rightarrow q$ and $q \Rightarrow p$.

In other words:
p and q have identical Truth values.

Augustus DeMorgan (1806-1871).

THEOREM 1.1 DEMORGAN'S LAWS

(a) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

(b) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

PROOF:

(a)

The driving force in the table was to arrive at the truth values of the compound propositions: $\sim(p \wedge q)$ and $\sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

same

Observing that the values of $\sim(p \wedge q)$ and $\sim p \vee \sim q$ coincide, we conclude that $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$.

(b)

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

same

There are several distributive properties involving propositions. Here is one of them:

THEOREM 1.2 $p \vee (q \wedge s) \Leftrightarrow (p \vee q) \wedge (p \vee s)$

PROOF: The eight possible value-combinations of p, q, and s, are listed in the first three columns of the table below. The remaining columns speak for themselves.

p	q	s	$q \wedge s$	$\boldsymbol{p \vee (q \wedge s)}$	$p \vee q$	$p \vee s$	$\boldsymbol{(p \vee q) \wedge (p \vee s)}$
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**F**	T	F	**F**
F	F	T	F	**F**	F	T	**F**
F	F	F	F	**F**	F	F	**F**

same

In general, a truth table involving n propositions will contain 2^n rows. That being the case, they quickly become "unmanageably tall":
A four-proposition-table calls for $2^4 = 16$ rows, while a five-proposition-table has 32 rows.

EXAMPLE 1.3 Are $(p \wedge q) \to \sim p$ and $p \wedge (q \to \sim p)$ logically equivalent?

SOLUTION: No:

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \to \sim p$	$q \to \sim p$	$p \wedge (q \to \sim p)$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	**T**	T	**F**
F	F	F	T	**T**	T	**F**

not the same truth values, so not logically equivalent

CHECK YOUR UNDERSTANDING 1.6

Show that:

(a) $p \to q \Leftrightarrow \sim p \vee q$ (b) $\sim(p \to q) \Leftrightarrow \sim(\sim p \vee q)$

Answer: See page A-1.

THEOREM 1.3 $p \to q \Leftrightarrow \sim q \to \sim p$

PROOF:

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

same

This is a very useful result. It tells you that:

as $p \to q$ goes
so does $\sim q \to \sim p$

Getting a bit ahead of the game, we point out that by virtue of Theorem 1.3, one can establish the validity of the proposition:

If n^2 is even then n is even
by verifying that:
If n is odd, then n^2 is not even

EXERCISES

Exercises 1-10. State whether the given proposition is True or False.

1. 10 is an even number.
2. 15 is an even number.
3. 10 or 15 is an even number.
4. 10 and 15 are even numbers.
5. $3<2$ or $[5>7$ or 9 is odd].
6. $3<2$ and $[5>7$ or 9 is odd].
7. $3<2$ or $[5>7$ and 9 is odd].
8. $[3<2$ and $5>7]$ or 9 is odd.
9. $[3<2$ or $5>7]$ and 9 is odd.
10. $[3<2$ or $5>7]$ or 9 is not odd

Exercises 11-31. Assume that p and q are True propositions, and that r and s are False propositions. Determine if the given compound proposition is True or False.

11. $p \wedge s$
12. $\sim(p \wedge s)$
13. $\sim(p \wedge \sim s)$
14. $r \wedge s$
15. $\sim(r \wedge s)$
16. $\sim(r \wedge \sim s)$
17. $r \vee s$
18. $\sim(r \vee s)$
19. $\sim(r \vee \sim s)$
20. $(p \vee s) \wedge (r \wedge s)$
21. $(p \vee s) \vee (r \wedge s)$
22. $(p \wedge q) \wedge (\sim r \wedge \sim s)$
23. $(p \wedge q) \vee (r \wedge s)$
24. $(p \wedge q) \wedge (r \vee s)$
25. $(p \wedge \sim r) \wedge (q \vee s)$
26. $\sim(p \wedge q) \vee \sim(r \wedge s)$
27. $\sim[(p \wedge q) \vee (r \wedge s)]$
28. $\sim(p \wedge q) \wedge (p \wedge s)$
29. $(p \wedge q) \wedge \sim(r \wedge s)$
30. $\sim[(p \vee r) \wedge \sim(p \wedge s)]$
31. $\sim[(p \wedge q) \wedge \sim(r \wedge \sim s)]$

Exercises 32-39. Determine if the given statement is a tautology.

32. $p \vee \sim p$
33. $p \vee (\sim p \vee q)$
34. $p \wedge (\sim p \vee q)$
35. $p \vee (\sim p \wedge q)$
36. $(\sim p \vee q) \vee (p \wedge \sim q)$
37. $(p \wedge q) \vee [\sim p \vee (p \wedge \sim q)]$
38. $[(p \wedge q) \to s] \to [p \to (q \to s)]$
39. $[(p \vee q) \to s] \to [\sim p \to (q \to s)]$

Exercises 40-43. Use the fact that $p \Rightarrow q$ if and only if $p \to q$ is a tautology to demonstrate that each is indeed a tatutology.

40. $p \wedge q \Rightarrow p$
41. $p \wedge q \Rightarrow p \vee \sim q$
42. $(p \vee q) \wedge (\sim p \wedge q) \Rightarrow q$
43. $p \wedge (\sim p \vee q) \Rightarrow p \vee q$

Exercises 44-59. Determine if the given pair of statements are logically equivalent.

44. $p \vee (p \wedge q)$, p
45. $p \vee (p \vee q)$, $p \vee q$
46. $(p \wedge \sim q) \vee p$, p
47. $(p \wedge \sim q) \vee p$, q

48. $\sim(p \to q)$, $\sim q \to \sim p$

49. $\sim[(p \vee \sim q) \vee (\sim p \wedge \sim q)]$, $\sim q$

50. $\sim[p \vee (\sim p \wedge q)]$, $\sim p \wedge \sim q$

51. $p \vee q$, $\sim(\sim p \wedge \sim q)$

52. $\sim(p \to q)$, $p \wedge (\sim q)$

53. $\sim(p \wedge \sim q)$, $q \vee \sim p$

54. $(p \wedge q) \vee (p \wedge s)$, $p \vee (q \wedge s)$

55. $(p \vee q) \vee (p \wedge s)$, $(p \vee q) \wedge s$

56. $(p \to r) \wedge (q \to s)$, $(p \vee q) \to s$

57. $(p \to s) \wedge (q \to s)$, $p \vee (q \to s)$

58. $(p \to q) \wedge (q \to s)$, $p \to s$

59. $(p \to q) \leftrightarrow (q \to s)$, $p \leftrightarrow s$

Exercises 60-62. Establish the given logical equivalences.

60. **Commutative Laws:**
(a) $p \wedge q \Leftrightarrow q \wedge p$
(b) $p \vee q \Leftrightarrow q \vee p$

61. **Associative Laws:**
(a) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
(b) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

62. **Distributive Laws:**
(a) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
(b) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Already proven (Theorem 1.2)

63. **In defense of Definition 1.5.**

(a) Prove that if we chose to define:

p	q	$p \to q$
T	T	T
T	F	F
F	T	**F**
F	F	T

Then $p \to q \Leftrightarrow q \to p$
(which runs contrary to our logical instincts)

(b) Prove that if we chose to define:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	**F**

Then $p \to q \not\Leftrightarrow \sim q \to \sim p$
(which runs contrary to our logical instincts)

(c) What if we chose to define:

p	q	$p \to q$
T	T	T
T	F	F
F	T	**F**
F	F	**F**

Anything "bad" happens?

§2. EXPONENTS AND FACTORING

Mathematics is, in part, a visual art, and much of its power stems from the fact that a great deal of information can be represented compactly. An example of this is the familiar exponent form:

DEFINITION 1.8 Integer Exponents

For any positive integer n and any number a:

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\text{-times}}$$

a raised to the n^{th} power

CANCELLATION LAW

For $c \neq 0$

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}$$

Note: The cancellation law is often abused. You can only cancel a common **factor** in the numerator and denominator of a fraction. **DON'T** do this:

$$\frac{2\cancel{a}+b}{\cancel{a}} = 2+b$$

WRONG!

From its very definition, we see that:

$$9^2 \cdot 9^3 = (9 \cdot 9)(9 \cdot 9 \cdot 9) = 9^5 = 9^{2+3}$$

$$\frac{9^5}{9^2} = \frac{\cancel{9^2} \cdot 9^3}{\cancel{9^2}} = 9^3 = 9^{5-2} \quad \text{(see margin)}$$

$$(9^2)^5 = 9^2 \cdot 9^2 \cdot 9^2 \cdot 9^2 \cdot 9^2 = 9^{10} = 9^{2 \cdot 5}$$

$$(2 \cdot 5)^3 = (2 \cdot 5)(2 \cdot 5)(2 \cdot 5) = (2 \cdot 2 \cdot 2)(5 \cdot 5 \cdot 5) = 2^3 \cdot 5^3$$

$$\left(\frac{2}{5}\right)^3 = \frac{2 \cdot 2 \cdot 2}{5 \cdot 5 \cdot 5} = \frac{2^3}{5^3}$$

In general:

THEOREM 1.4

EXPONENT RULES

Let m and n be positive integers. When all expressions are defined, we have:

(i) $a^m a^n = a^{m+n}$ — When multiplying add the exponents

(ii) $\dfrac{a^m}{a^n} = a^{m-n}$ — When dividing subtract the exponents

(iii) $(a^m)^n = (a^n)^m = a^{mn}$ — A power of a power: multiply the exponents

(iv) $(ab)^n = a^n b^n$ — A power of a product equals the product of the powers

(v) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ — A power of a quotient equals the quotient of the powers

It is important to note that these exponent rules deal with products, and not with sums. In particular, though it is true that:

$$(3 \cdot 4)^2 = 3^2 \cdot 4^2$$

it is **NOT TRUE** that

$$(3+4)^2 \neq 3^2 + 4^2$$

WRONG:

$(3+4)^2 = 49$ while

$3^2 + 4^2 = 25$.

In (ii) we stated *When dividing, subtract the exponents*. But if we try this with the quotient $\frac{9^2}{9^5}$, we end up with the expression $\frac{9^2}{9^5} = 9^{2-5} = 9^{-3}$, which at this point is meaningless. Observing that

$\frac{9^2}{9^5} = \frac{\cancel{9^2}}{\cancel{9^2} \cdot 9^3} = \frac{1}{9^3}$, we breathe meaning into the expression 9^{-3} by defining it to be $\frac{1}{9^3}$. By the same token, since $\frac{9^2}{9^2} = 1$, if we want the "*subtract exponent*" rule $\frac{9^2}{9^2} = 9^{2-2} = 9^0$ to hold, we are forced to define 9^0 to be 1.

In general:

One can show Theorem 1.4 holds for all real numbers. For example:

$$2^{-5} \cdot 2^{\sqrt{3}} = 2^{-5+\sqrt{3}}$$

While we're at it we also note that for any $a \ge 0$ we define $a^{1/2}$ to be $\sqrt{a}$. This makes sense, since:

$$\sqrt{a} \cdot \sqrt{a} = a$$

and

$$a^{1/2} \cdot a^{1/2} = a^{\frac{1}{2}+\frac{1}{2}} = a$$

DEFINITION 1.9 For any positive integer n and any $a \neq 0$:

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}$$

EXAMPLE 1.4 Simplify:

(a) $\frac{(2a)^2 b^3}{(a^{-3}b^2)^5}$ (b) $\frac{(ab^2c^{-3})^4\left(\frac{a}{b}\right)^3}{(a^{-2}b^2)^3}$

SOLUTION: There are many ways of simplifying the given expressions. Our approach is designed to highlight the exponent rules of Theorem 1.4:

(a) $\frac{(2a)^2 b^3}{(a^{-3}b^2)^5} = \frac{2^2a^2b^3}{a^{-15}b^{10}} = 4a^{2+15}b^{3-10} = 4a^{17}b^{-7} = \frac{4a^{17}}{b^7}$

(b)
$$\frac{(ab^2c^{-3})^4\left(\frac{a}{b}\right)^3}{(a^{-2}b^2)^3} = \frac{a^4b^8c^{-12}(ab^{-1})^3}{a^{-6}b^6}$$
$$= \frac{a^4b^8c^{-12}a^3b^{-3}}{a^{-6}b^6}$$
$$= a^{4+3+6}b^{8-3-6}c^{-12} = a^{13}b^{-1}c^{-12} = \frac{a^{13}}{bc^{12}}$$

CHECK YOUR UNDERSTANDING 1.7

Simplify:

(a) $(-2ax^2)^4$ (b) $\left(\frac{-x^2}{2}\right)^3$ (c) $(-x)^2\left(\frac{x}{2y}\right)^3$

Answers: (a) $16a^4x^8$ (b) $-\frac{x^6}{8}$ (c) $\frac{x^5}{8y^3}$

FACTORING POLYNOMIALS

$(a+b)(a-b)$
$= a(a-b)+b(a-b)$
$= a^2-ab+ba-b^2$
$= a^2-b^2$

Factoring is the reverse process of multiplication. By reading the following equation from right to left:

$$(a+b)(a-b) = a^2-b^2 \quad \text{(see margin)}$$

we obtain the formula for factoring a difference of two squares:

THEOREM 1.5 $\quad a^2-b^2 = (a+b)(a-b)$

For example: $\quad 4x^2-9 = (2x+3)(2x-3)$

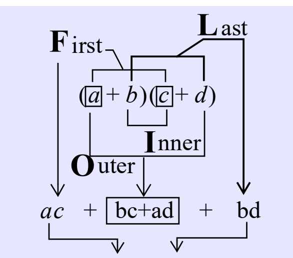
And: $\quad 4x^2-5 = (2x+\sqrt{5})(2x-\sqrt{5})$

CHECK YOUR UNDERSTANDING 1.8

Factor:

(a) $25x^2-1$ (c) x^4-16

Answers:
(a) $(5x+1)(5x-1)$
(c) $(x^2+4)(x+2)(x-2)$



“FOIL”

Some trinomials (three terms) can be factored by reversing the so-called “**FOIL**” multiplication method (see margin). To gain a factorizations of $5x^2+27x-18$, for example, we start with the template

$$(5x \quad\quad)(x \quad\quad) \qquad (*)$$

which gives us the **F**irst term $5x^2$. We then envision pairs of integers in the template whose product equals the magnitude of the **L**ast term 18, and position them in (*) in the hope that the sum of the **O**uter-**I**nner terms has a magnitude of 27:

$(5x \quad 9)(x \quad 2)$	No luck: “in 9 and 10 does not sit a 27” (there “sits a 1 and a 19”
$(5x \quad 2)(x \quad 9)$	No luck: “in 2 and 45 does not sit a 27”
$(5x \quad 6)(x \quad 3)$	No luck: “in 6 and 15 does not sit a 27”
$(5x \quad 3)(x \quad 6)$	Some luck: “27 does sit in 3 and 30,” and the signs can also be accommodated: To end up with -18 one of the numbers 3 and 6 has to be negative and the other positive. In addition, the “OI” combination must be a positive 27. Success: $(5x^2+27x-18) = (5x-3)(x+6)$

CHECK YOUR UNDERSTANDING 1.9

Factor:

(a) $2x^2+7x-4$ (b) $18x^4-6x^3-60x^2$

Answers: (a) $(2x-1)(x+4)$
(b) $6x^2(3x+5)(x-2)$

POLYNOMIAL EQUATIONS

In the following examples we illustrate how factorization can be used to solve certain polynomial equations.

EXAMPLE 1.5 Solve:

$$x^3 + x^2 - 6x = 0$$

SOLUTION: The solution again hinges on the fact that a product is zero **if** and **only if** one of its factors is zero:

$$x^3 + x^2 - 6x = 0$$

Pull out the common factor, x: $$x(x^2 + x - 6) = 0$$

factor further: $$x(x+3)(x-2) = 0$$

solutions: $$x = 0 \text{ or } x = -3 \text{ or } x = 2$$

Answers: (a) $x = \pm\frac{1}{5}$

(b) $x = \frac{1}{2}, x = -4$

CHECK YOUR UNDERSTANDING 1.10

Solve:

(a) $25x^2 - 1 = 0$ (b) $2x^2 + 7x - 4 = 0$

Exercises

Exercises 1-9. Evaluate each of the following.

1. $-2^2(2+3)^2$
2. $(-2)^2(2\cdot 3)^2$
3. $\dfrac{(2+3)^2}{3^2}$
4. $\dfrac{\left(\frac{1}{2}\right)^2+\frac{5}{4}}{2}$
5. $\dfrac{\left(\frac{1}{2}+\frac{1}{4}\right)^2}{8^{-1}}$
6. $2^3\left(\frac{1}{2}+\frac{1}{4}\right)^2$
7. $\dfrac{2^{-1}+2}{9\cdot 3^{-1}}$
8. $\dfrac{3(3-\frac{1}{2})^2}{4^0\cdot 3^2}$
9. $\dfrac{3^{-2}-2^{-3}}{-3^2}$

Exercises 10-12. Simplify.

10. $\dfrac{\left(\frac{a}{b}\right)^2}{(2ab)^2}$
11. $\dfrac{(ab)^2}{a^3(-b)^3}$
12. $\dfrac{(-ab)^2a^{-3}}{(2ab)^3}$

Exercises 13-30. Factor the polynomial.

13. $9x^2-4$
14. $-x^2+4$
15. $4x^2-25$
16. $100x^2-1$
17. $-4x^2+1$
18. $4x^2-5$
19. $2x^2-5$
20. x^2+3x+2
21. $x^2+7x+12$
22. x^2-x-12
23. $x^2-7x+12$
24. $2x^2+5x+2$
25. $6x^2+7x+2$
26. $6x^2+13x+6$
27. $6x^2-7x-5$
28. $9x^3-x$
29. $-6x^3+x^2+12x$
30. $6x^3+31x^2+40x$

Exercises 31-44. (Equations) Solve the polynomial equation.

31. $2x^2+9x-35=0$
32. $x^2+2x-35=0$
33. $25x^2-16=0$
34. $x^2-25=0$
35. $5x^2-6=0$
36. $x^2-5=0$
37. $x^2+10x+25=0$
38. $4x^2-4x+1=0$
39. $3x^3-14x^2-5x=0$
40. $3x^3+16x^2=-5x$
41. $2x^3-5x^2-3x=0$
42. $3x^3+2x^2=5x$
43. $(2x^2+5x-25)(5x^2+9x-2)=0$
44. $(4x^2-25)(x^2-2)=0$

§3. UNITS CONVERSION

More than a number is required to represent a measurement. One cannot, for example, say that a board is two long. Units must also be specified: Is it two feet long? Two inches? Two yards? Two meters? The length of the board is certainly constant, but the numbers describing that length differ, depending on the unit of measurement used. For example, a board that is 2 feet long is also 24 inches long, and we write:

$$2\text{ feet} = 24\text{ inches} \quad \text{or} \quad 2\text{ ft} = 24\text{ in.}$$

The units in the above equation are critical; removing them would result in the ridiculous assertion that $2 = 24$.

A unit of measure, such as a "foot," is a quantity that can be cancelled when it appears as a common factor of the numerator and denominator of a fraction. For example,

$$\frac{12\cancel{\text{ ft}}}{3\cancel{\text{ ft}}} = \frac{12}{3} = 4$$

In short, units can be treated like nonzero constants; for indeed, they are:

$7a + 2a = 9a$		$7ft + 2ft = 9ft$
$(360\cancel{a})\left(3 \cdot \frac{b}{\cancel{a}}\right) = 1080b$		$(360\cancel{yd})\left(3\frac{ft}{\cancel{yd}}\right) = 1080ft$
$\left(9 \cdot \frac{a}{\cancel{b}}\right)\left(40 \cdot \frac{\cancel{b}}{\cancel{c}}\right)\left(52 \cdot \frac{\cancel{c}}{d}\right) = 18{,}720\frac{a}{d}$		$\left(9\frac{\$}{\cancel{hr}}\right)\left(40\frac{\cancel{hr}}{\cancel{wk}}\right)\left(52\frac{\cancel{wk}}{yr}\right) = 18{,}720\frac{\$}{yr}$
$\dfrac{3a}{16 \cdot \frac{a}{b}} = \dfrac{3\cancel{a}}{16} \cdot \dfrac{b}{\cancel{a}} = \dfrac{3}{16} \cdot b$ invert and multiply		$\dfrac{3\,oz}{16\frac{oz}{lb}} = \dfrac{3\cancel{oz}}{16} \cdot \dfrac{lb}{\cancel{oz}} = \dfrac{3}{16}lb$ invert and multiply

Suppose you want to convert 3 pounds to ounces. Knowing that $1\text{ lb} = 16\text{ oz}$ allows you to write:

$$\frac{1\text{ lb}}{16\text{ oz}} = 1 \text{ and } \frac{16\text{ oz}}{1\text{ lb}} = 1$$

Since you can multiply an expression by 1 without changing its value:

$$3\text{ lb} = (3\text{ lb})(1) = (3\cancel{\text{ lb}})\left(\frac{16\text{ oz}}{1\cancel{\text{ lb}}}\right) = 48\text{oz}$$

Let us agree to call a relation of the type $1\text{ lb} = 16\text{oz}$ a conversion bridge. Here are some of the more familiar conversion bridges:

$1\text{ lb} = 16\text{ oz}$	$1\text{ ft} = 12\text{ in.}$	$1yd = 3\text{ ft}$
$1\text{ mi} = 5280\text{ ft}$	$1\text{ gal.} = 4\text{ qt}$	$1\text{ qt} = 2\text{ pt}$
$1\text{ hr} = 60\text{ min}$	$1\text{ min} = 60\text{ sec}$	$1\$ = 100\,¢$

EXAMPLE 1.6 Convert 2.3 gallons to pints.

SOLUTION: To convert 2.3 gallons to pints, we take the path

$$\text{gal.} \to \cancel{\text{gal.}}\left(\frac{\cancel{\text{qt}}}{\cancel{\text{gal.}}}\right)\left(\frac{\text{pt}}{\cancel{\text{qt}}}\right) \to \text{pt}$$

along with the bridges $1\text{ gal.} = 4\text{ qt}$ and $1\text{ qt} = 2\text{ pt}$:

$$2.3\text{ gal.} = (2.3\text{ gal.})\left(\frac{4\text{ qt}}{1\text{ gal.}}\right)\left(\frac{2\text{ pt}}{1\text{ qt}}\right) = (2.3)(4)(2)\text{pt} = 18.4\text{ pt}$$

EXAMPLE 1.7 Convert 50 feet per minute to miles per hour.

SOLUTION: To convert 50 feet per minute to miles per hour we take the path

$$\frac{\text{ft}}{\text{min}} \to \frac{\cancel{\text{ft}}}{\cancel{\text{min}}}\cdot\frac{\text{mi}}{\cancel{\text{ft}}}\cdot\frac{\cancel{\text{min}}}{\text{hr}} \to \frac{\text{mi}}{\text{hr}}$$

Along with the bridges $1\text{ min} = 60\text{ sec}$ and $1\text{ mi} = 5280\text{ ft}$:

$$50\frac{\text{ft}}{\text{min}} = 50\frac{\text{ft}}{\text{min}}\cdot\frac{1\text{mi}}{5280\text{ft}}\cdot\frac{60\text{min}}{1\text{hr}} = \frac{50\cdot 60}{5280}\frac{\text{mi}}{\text{hr}} \approx 0.57\frac{\text{mi}}{\text{hr}}$$

approximately equal

CHECK YOUR UNDERSTANDING 1.11

Convert 12 ounces per minute to pounds per hour.

Answer: $45\frac{\text{lb}}{\text{hr}}$

As is illustrated in the following example, one has to be careful when converting units that are raised to a power other than one.

EXAMPLE 1.8 Convert 7 pounds per cubic foot to ounces per cubic inch.

SOLUTION: The following path will take you from pounds per cubic foot to ounces per cubic inch:

$$\frac{\text{lb}}{\text{ft}^3} \to \frac{\cancel{\text{lb}}}{\cancel{\text{ft}^3}} \cdot \frac{\cancel{\text{ft}^3}}{\text{in.}^3} \cdot \frac{\text{oz}}{\cancel{\text{lb}}} \to \frac{\text{oz}}{\text{in.}^3}$$

Noting that, $(1\text{ ft})^3 = (12\text{ in.})^3 = 12^3\text{in.}^3$, we have:

$$7\frac{\text{lb}}{\text{ft}^3} = 7\frac{\text{lb}}{\text{ft}^3} \cdot \frac{1^3\text{ft}^3}{12^3\text{ in.}^3} \cdot \frac{16\text{ oz}}{1\text{ lb}} = \frac{7 \cdot 16}{12^3}\frac{\text{oz}}{\text{in.}^3} \approx 0.065\frac{\text{oz}}{\text{in.}^3}$$

Answer: $\approx 0.13\frac{\text{qt}}{\text{ft}^2}$

CHECK YOUR UNDERSTANDING 1.12

Convert 0.3 gallons per square yard to quarts per square foot.

One meter was originally intended to be, and is very nearly equal to, one ten-millionth of the distance between the equator and the north pole, measured along a meridian. Presently, it is defined as 1,650,733.73 wavelengths of the orange-red radiation of krypton86 under specified conditions.

It should be noted that the conversion bridge $1\text{ g} = 0.0353\text{ oz}$ only holds near the surface of the earth, as gram is a mass-unit while ounce is a force-unit.

THE METRIC SYSTEM

The three basic units of measure in the metric system are the meter (m), gram (g), and liter (l). By combining these basic units with the following prefixes, multiple and fractional units are obtained:

kilo = one thousand
deci = one tenth
centi = one hundreth
milli = one thousandth

For Example:

one kilogram = 1000 grams, and we write: $1\text{ kg} = 1000\text{ g}$ and $\frac{1}{1000}\text{kg} = 1\text{ g}$

one decimeter $= \frac{1}{10}$ meter, and we write: $1\text{ dm} = \frac{1}{10}\text{ m}$ and $10\text{ dm} = 1\text{ m}$

one centimeter $= \frac{1}{100}$ meter, and we write: $1\text{ cm} = \frac{1}{100}\text{ m}$ and $100\text{ cm} = 1\text{ m}$

one milliliter $= \frac{1}{1000}$ liter, and we write: $1\text{ mL} = \frac{1}{1000}\text{ L}$ and $1000\text{ mL} = 1\text{ L}$

Here are some (approximate) conversion bridges between the English system and the metric system:

$$1\text{ m} = 3.28\text{ ft} \qquad 1\text{ g} = 0.0353\text{ oz} \qquad 1\text{ L} = 0.264\text{ gal.}$$

EXAMPLE 1.9 Convert 13 pounds to kilograms.

SOLUTION: To convert 13 pounds to kilograms without a direct bridge from pounds to grams, first convert from pounds to ounces, and then from ounces to grams. Here is the path:

$$\text{lb} \to \cancel{\text{lb}} \cdot \frac{\cancel{\text{oz}}}{\cancel{\text{lb}}} \cdot \frac{\cancel{\text{g}}}{\cancel{\text{oz}}} \cdot \frac{\text{kg}}{\cancel{\text{g}}} \to \text{kg}$$

And here is the completed journey:

$$\begin{aligned} 13\text{ lb} &= (13\text{ lb})\left(\frac{16\text{ oz}}{1\text{ lb}}\right)\left(\frac{1\text{ g}}{0.0353\text{ oz}}\right)\left(\frac{1\text{ kg}}{1000\text{ g}}\right) \\ &= \frac{(13)(16)}{(0.0353)(1000)}\text{kg} \approx 5.89\text{ kg} \end{aligned}$$

EXAMPLE 1.10 The currency in Venezuela is the Bolivar (Bs), and in 1974 the exchange rate was 4.23 Bs per dollar. At that time, a liter of gasoline sold for 0.15 Bs. What was the cost in dollars of a gallon of gasoline?

SOLUTION: Taking the path:

$$\frac{\text{Bs}}{\text{L}} \to \frac{\cancel{\text{Bs}}}{\cancel{\text{L}}}\,\frac{\$}{\cancel{\text{Bs}}}\cdot\frac{\cancel{\text{L}}}{\text{gal.}} \to \frac{\$}{\text{gal.}}$$

we have:

$$\begin{aligned} 0.15\frac{\text{Bs}}{\text{L}} &= \left(0.15\frac{\text{Bs}}{\text{L}}\right)\left(\frac{1\,\$}{4.23\text{ Bs}}\right)\left(\frac{1\text{L}}{0.264\text{ gal.}}\right) \\ &= \frac{0.15}{(4.23)(0.26)}\frac{\$}{\text{gal.}} \approx 0.13\frac{\$}{\text{gal.}} \end{aligned}$$

Conclusion: A gallon of gasoline cost \$0.13.

CHECK YOUR UNDERSTANDING 1.13

Convert 4 grams per cubic centimeter to ounces per cubic inch.

Answer: $\approx 2.32\frac{\text{oz}}{\text{in.}^3}$

MEASUREMENTS

The carton in Figure 1.1 has a length of 2 feet, a width of 1.2 feet, and a height of 1.3 feet.

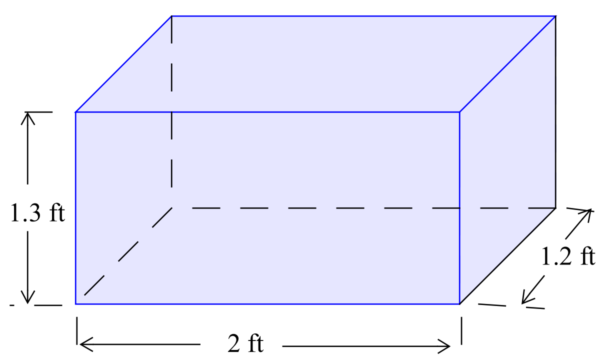


Figure 1.1

The side facing you in the figure has an area of 2(1.3) square feet (2.6 ft.^{2}). The volume of the carton is 2(1.2)(1.3) cubic feet (3.12 ft.^{3}).

EXAMPLE 1.11 Convert the above two measurements into meters.

SOLUTION: Taking the conversion bridge: $1\ m = 3.28\text{ ft}$ we have:

$$\text{Area of } 2.6\text{ ft.}^2 = (2.6\ \cancel{\text{ft.}}^2)\left(\frac{1^2\text{m}^2}{3.28^2\cancel{\text{ft}}^2}\right) = \frac{2.6}{3.28^2}\text{m}^2 \approx 2.42\text{m}^2$$

$$\text{Volume of: } 3.12\text{ ft.}^3 = (3.12\ \cancel{\text{ft.}}^3)\left(\frac{1^3\text{m}^3}{3.28^3\cancel{\text{ft}}^3}\right) = \frac{3.12}{3.28^3}\text{m}^3 \approx 0.08\text{m}^3$$

The volume of the cylindrical drum in Figure 1.2 is the area of its circular base: $A = \pi r^2 = \pi 2^2\text{ in}^2 = 4\pi\text{ in}^2$, times its height: $h = 8\text{ in}$:

$$V = Ah = (4\pi\text{ in}^2)(8\text{ in}) = 32\pi\text{ in}^3$$

As is depicted in Figure 1.2, the material, M, for the side of the drum is the product of the circumference of its base, time its height:

$$M = 2\pi rh = (2\pi)(2\text{ in})(8\text{ in}) = 32\pi\text{ in}^2$$

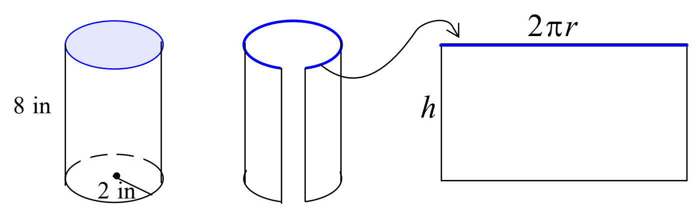


Figure 1.2

CHECK YOUR UNDERSTANDING 1.14

Referring to the cylindrical drum in Figure 1.2, assume that the metal for the top and bottom of the drum costs \$3 per square foot, and that the metal for the side of the drum costs \$2 per square foot. What is the total material cost in Euros if the current exchange rate is 1.10 EUR for \$1.

Answer: 295.81 EUR

EXERCISES

Exercises 1-14. Convert:

1. 3.2 feet to miles
2. 20 grams to kilograms
3. 5 feet per hour to yards per minute
4. 55 miles per hour to kilometers per hour
5. 5 centimeters to inches
6. 8 feet to yards
7. 8 square feet to square meters
8. 8 square meters to square feet
9. 8 square meters to square yards
10. 8 square yards to square meters

11. The speed of light in a vacuum is 2.9979×10^8 m/sec . Convert this to miles per hour.

12. How many seconds are there in a week? a month? a year?

13. A nautical mile is 1.852 kilometers. How many miles are there in 150 nautical miles?

14. A fathom is 6 feet. How many meters are there in 10 fathoms? How many fathoms in a mile?

15. Oil flow through a pipe at the rate of 50 gal./min. Find the rate in liters/sec.

16. A furlough is $\frac{1}{8}$ of a mile, and a league is 3 miles. How many furloughs to a league?

17. How many square feet of a certain material can you buy for \$6, if a square yard costs \$18.25?

18. What is the cost of a cubic foot of gravel that sells for \$3.25 per cubic yard?

19. When the planer Mars is closest to the Earth it is about 35 million miles away. How many minutes does it take light from Mars to reach the Earth when it is closest to the Earth, given that light travels at 3×10^8 meters per second?

20. If you are driving at 50mph, how many meters do you travel in 2 minutes?

CHAPTER SUMMARY

TRUTH TABLES

p and q

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p or q

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

not p

p	$\sim p$
T	F
F	T

if *p then q*

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p iff q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TAUTOLOGY

A **tautology** is a proposition which is necessarily always True.

p implies *q*

$p \Rightarrow q$

$p \Rightarrow q$ is **not** a proposition for it does not assume values of True or False. It is an assertion. We do have:

$p \Rightarrow q$ is valid if and only if the proposition $p \rightarrow q$ is a tautology

p and *q* are logically equivalent

$p \Leftrightarrow q$

$p \Leftrightarrow q$ if and only if $p \leftrightarrow q$ is a tautology.

That is, ***p*** is True if and only if ***q*** is True.

DEMORGAN'S LAWS

$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ and $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

EXPONENTS

For any positive integer n and any number a:

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\text{-times}}$$

In addition, if $a \neq 0$: $a^0 = 1$ and $a^{-n} = \dfrac{1}{a^n}$

EXPONENT THEOREMS

(i) $a^m a^n = a^{m+n}$ — When multiplying add the exponents

(ii) $\dfrac{a^m}{a^n} = a^{m-n}$ — When dividing subtract the exponents

(iii) $(a^m)^n = (a^n)^m = a^{mn}$ — A power of a power: multiply the exponents

(iv) $(ab)^n = a^n b^n$ — A power of a product equals the product of the powers

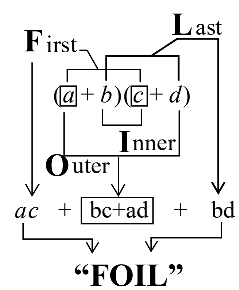
(v) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ — A power of a quotient equals the quotient of the powers

FACTORING POLYNOMIALS

Difference of two squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Some trinomials (three terms) can be factored by reversing the so-called "**FOIL**" multiplication method:



TREAT UNITS LIKE CONSTANTS

$(360\,\cancel{yd})\left(3\dfrac{ft}{\cancel{yd}}\right) = 1080\,ft$ and $\dfrac{3\,oz}{16\dfrac{oz}{lb}} = \dfrac{3\,\cancel{oz}}{16}\cdot\dfrac{lb}{\cancel{oz}} = \dfrac{3}{16}lb$

invert and multiply

CHAPTER 2
EQUATIONS AND INEQUALITIES

§1. LINEAR EQUATIONS AND INEQUALITIES

The **solution set** of an equation is simply the set of numbers that satisfy the equation (left side of equation equals right side of equation). When solving equations, the following result plays a dominant role:

THEOREM 2.1 Adding (or subtracting) the same quantity to (or from) both sides of an equation, or multiplying (or dividing) both sides of an equation by the same nonzero quantity will not alter the solution set of the equation.

The fact that you can add the same quantity to both sides of an equation without altering its solution set allows you to move terms from one side of the equation to the other as long as you change the sign of those terms. For example:

$$\begin{aligned} 7x+\mathbf{3} &= 5x-6 \\ (7x+3)-\boxed{3} &= (5x-6)-\boxed{3} \\ 7x &= (5x-6)-\mathbf{3} \end{aligned}$$

As you can see the **plus 3** on the left side of the equation ended up being a **minus 3** on the right side of the equation.

EXAMPLE 2.1 Solve:

(a) $7x+3 = 5x-6$

(b) $3x-x+2-9 = 4x+6+x$

(c) $\dfrac{3x}{5}-\dfrac{2-x}{3}+1 = \dfrac{x-1}{15}$

SOLUTION:

(a)
$$\begin{aligned} 7x+3 &= 5x-6 \\ 7x-5x &= -6-3 \\ 2x &= -9 \\ x &= -\frac{9}{2} \end{aligned}$$

(b)
$$\begin{aligned} 3x-x+2-9 &= 4x+6+x \\ 2x-7 &= 5x+6 \\ 2x-5x &= 6+7 \\ -3x &= 13 \\ x &= -\frac{13}{3} \end{aligned}$$

(c)
$$\begin{aligned} \frac{3x}{5}-\frac{2-x}{3}+1 &= \frac{x-1}{15} \\ 15\left(\frac{3x}{5}-\frac{2-x}{3}+1\right) &= 15\frac{(x-1)}{15} \\ {}^{3}\cancel{15}\left(\frac{3x}{\cancel{5}}\right)-{}^{5}\cancel{15}\left(\frac{2-x}{\cancel{3}}\right)+15 &= x-1 \\ 3(3x)-5(2-x)+15 &= x-1 \\ 9x-10+5x+15 &= x-1 \\ 9x+5x-x &= -1+10-15 \\ 13x &= -6 \\ x &= -\frac{6}{13} \end{aligned}$$

Cancellation Property

For $c \neq 0$:

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}$$

In particular:

$$15\left(\frac{3x}{5}\right) = \frac{3\cdot\cancel{5}(3x)}{\cancel{5}} = 3(3x)$$

CHECK YOUR UNDERSTANDING 2.1

Solve:

(a) $5 - 8x + 2 = -12x - 3 + 6x$ (b) $\frac{3x}{5} - \frac{2-x}{3} + 1 = x - 1$

Answers: (a) 5 (b) 20

The following example illustrates two methods that can be used to solve a system of two linear equations in two unknowns.

To solve a system of several equations in several variables is to determine values of the variables which simultaneously satisfy each equation in the system. For example, $x = 1, y = 2$ is a solution of the following system of two equations in two unknowns:

$$\left.\begin{aligned} 2x + y &= 4 \\ x - 3y &= -5 \end{aligned}\right\}$$

since both equations are satisfied when the indicated values are substituted for the variables:

$2 \cdot 1 + 2 = 4$ Check!
$1 - 3 \cdot 2 = -5$ Check!

The ***ordered pair*** $\left(\frac{1}{8}, \frac{19}{8}\right)$ is interpreted to mean:

$$x = \frac{1}{8}, y = \frac{19}{8}$$

EXAMPLE 2.2 Solve the following system of two equations in two unknowns.

$$\left.\begin{aligned} (1)\!: \quad -3x + y &= 2 \\ (2)\!: \quad 2x + 2y &= 5 \end{aligned}\right\}$$

SOLUTION:

<u>ELIMINATION METHOD:</u> Add (or subtract) a multiple of one equation to possibly a multiple of the other, so as to eliminate one of the variables:

$$\begin{aligned} \text{multiply (1) by 2:} \quad -6x + 2y &= 4 \\ (2)\!: \quad 2x + 2y &= 5 \\ -8x \quad &= -1 \\ \text{divide:} \quad x &= \frac{1}{8} \end{aligned}$$

Substituting $x = \frac{1}{8}$ in (1) [we could have chosen (2)], we have:

$$-3\left(\frac{1}{8}\right) + y = 2 \Rightarrow y = 2 + \frac{3}{8} = \frac{19}{8}$$

$$\text{Answer: } \left(\frac{1}{8}, \frac{19}{8}\right).$$

<u>SUBSTITUTION METHOD:</u>

$$\left.\begin{aligned} (1)\!: \quad -3x + y &= 2 \\ (2)\!: \quad 2x + 2y &= 5 \end{aligned}\right\}$$

Solving for y in (1), we have:

$$y = 3x + 2 \qquad (*)$$

Substituting this value in (2) yields:

$$\begin{aligned} 2x + 2(3x + 2) &= 5 \\ 2x + 6x + 4 &= 5 \\ 8x &= 1 \\ x &= \frac{1}{8} \end{aligned}$$

Returning to (*), we find the corresponding y-value:

$$y = 3 \cdot \frac{1}{8} + 2 = \frac{19}{8}$$

$$\text{Answer: } \left(\frac{1}{8}, \frac{19}{8}\right).$$

CHECK YOUR UNDERSTANDING 2.2

Solve:

$$\left.\begin{aligned} 3x + 4y &= -1 \\ x + 2y &= 0 \end{aligned}\right\}$$

Answer: $\left(-1, \frac{1}{2}\right)$

EXAMPLE 2.3

A larger number is 2 less than 3 times a smaller number, and their sum equals 18.

Find the numbers.

SOLUTION: Let the variables x and y denote the larger and smaller numbers, respectively. We are given that:

$$\left.\begin{aligned} (1)\quad x &= 3y - 2 \\ (2)\quad x + y &= 18 \end{aligned}\right\}$$

Substituting $3y - 2$ for x in (2):
$$\begin{aligned} 3y - 2 + y &= 18 \\ 4y &= 20 \\ y &= 5 \end{aligned}$$

Substituting 5 for y in (1): $x = 3(5) - 2 = 13$

Answer: The two numbers are 13 and 5.

EXAMPLE 2.4

A farmer bought feed for his 20 chickens and 10 cows. Each bag of feed can feed one of each respective animal. Each bag of cow feed costs three times as much as a bag of chicken feed. The farmer spent \$210. How much did each bag of chicken and cow feed cost?

The real challenge, of course, is to translate the given information into equations. Compressing the information will help:

x: cost per cow
y: cost per chicken

Total cost: $10x + 20y$

Given: $x = 3y$

Given: Total cost = \$210

SOLUTION: Let the variables x denote the cost for a bag of cow feed and y denote the cost for a bag of chicken feed. We then have:

$$\begin{aligned} (1)\quad x &= 3y \\ (2)\quad 10x + 20y &= 210 \end{aligned}$$

Substituting $3y$ for x in (2):
$$\begin{aligned} 10(3y) + 20y &= 210 \\ 50y &= 210 \\ y &= \frac{210}{50} = 4.20 \end{aligned}$$

Substituting 4.20 for y in (1): $x = 3(4.20) = 12.60$

Answer: Bag of chicken feed: \$4.20. Bag of cow feed: \$12.60

CHECK YOUR UNDERSTANDING 2.3

Currently, a rectangle is 4 times as long as it is wide. If the length is increased by 4 inches and the width is decreased by 1 inch, the area will be 32 square inches. What are the dimensions of the current rectangle?

Answer:
Width: 3 inches
Length: 12 inches

LINEAR INEQUALITIES

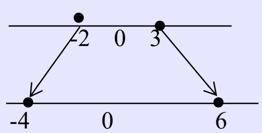
One solves linear inequalities in exactly the same fashion as one solves linear equations, with one notable exception:

WHEN MULTIPLYING OR DIVIDING BOTH SIDES OF AN INEQUALITY BY A **NEGATIVE** QUANTITY, **REVERSE** THE DIRECTION OF THE INEQUALITY SIGN.

If you multiply both sides of the inequality $-2<3$ by the positive number 2, then the inequality sign remains as before:

$$-2<3$$

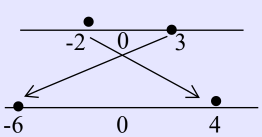
multiply by 2: $-4<6$



But if you multiply both sides by a negative quantity, then the sense of the inequality is reversed:

$$-2<3$$

multiply by -2 : $4>-6$



To illustrate:

Equation	Inequality
$3x-5 = 5x-7$	$3x-5<5x-7$
$3x-5x = -7+5$	$3x-5x<-7+5$
$-2x = -2$	$-2x<-2$
$x = 1$	reverse — dividing by a negative number
	$x>1$

EXAMPLE 2.5

Solve:

$$-3 \le 4x+2<9$$

SOLUTION: The above is shorthand for writing:

$$-3 \le 4x+2 \quad \text{and} \quad 4x+2<9$$

You can solve the two inequalities separately (below left), or you can solve the two inequalities simultaneously (below right).

$$-3 \le 4x+2 \quad \text{and} \quad 4x+2<9$$
$$-4x \le 2+3 \quad \text{and} \quad 4x<9-2$$
$$-4x \le 5 \quad \text{and} \quad 4x<7$$
$$x \ge -\frac{5}{4} \quad \text{and} \quad x<\frac{7}{4}$$
$$-\frac{5}{4} \le x<\frac{7}{4}$$

$$-3 \le 4x+2<9$$
Subtract 2: $-3-2 \le (4x+2)-2<9-2$
$$-5 \le 4x<7$$
Divide by 4: $-\frac{5}{4} \le x<\frac{7}{4}$

CHECK YOUR UNDERSTANDING 2.4

Solve:

$$\frac{3x}{5} - \frac{2-x}{3} + 1 < \frac{x-1}{15}$$

Suggestion: Begin by multiplying both sides of the inequality by 15 so as to eliminate all denominators.

Answer: $x < -\frac{6}{13}$

EXAMPLE 2.6 Find all pairs of consecutive odd positive integers, both of which are smaller than 18, such that their sum is at least 20.

SOLUTION: Let x denote the smaller odd number, and y the larger.

Since both are smaller than 18 the larger must be smaller than 18:

$$y < 18 \quad (*)$$

Since their sum is at least 20:

$$x + y \geq 20 \quad (**)$$

Since they are consecutive odd numbers:

$$y = x + 2 \quad (***)$$

Substituting this value of y in (**):

$$x + x + 2 \geq 20$$

$$2x \geq 20 - 2 = 18$$

$$x \geq \frac{18}{2} = 9$$

Possible values of y (see (*) and (***)): 11, 13, 15, 17

Possible pairs of consecutive odd integers:

(9,11) (11, 13) (13, 15) (15, 17)

CHECK YOUR UNDERSTANDING 2.5

Mary scored 82%, 87%, and 85% on her three tests, which constitute 60% of the course grade. The grade on the final accounts for 40% of the course grade. Is it possible for Mary to receive an A in the course, given that a course grade of at least 90% is required for an A?

Answer: No

EXERCISES

Exercises 1-6. Solve the given linear equation.

1. $2x + 8 = 7x - 5$
2. $5x + 2 = -7x - 12$
3. $-3x + 7 = 4x - 5 + \frac{x}{2}$
4. $-(-2y - 4) + \frac{y}{3} + 3y = 5$
5. $\frac{x-2}{3} = 2(-x + 1)$
6. $7x + \frac{2x - 4 - 3x}{2} = \frac{7x - 5 + 2x}{4}$

Exercises 7-12. Solve the given system of equations.

7. $\left.\begin{aligned} x + 2y &= 3 \\ x - 2y &= 0 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x + y &= -1 \\ x + 2y &= 0 \end{aligned}\right\}$

9. $\left.\begin{aligned} x + y &= 4 \\ 2x - y &= 1 \end{aligned}\right\}$

10. $\left.\begin{aligned} x + y &= 1 \\ 4x + y &= 2 \end{aligned}\right\}$

11. $\left.\begin{aligned} \frac{x}{3} + y &= 1 \\ 2x - \frac{y}{2} &= 2 \end{aligned}\right\}$

12. $\left.\begin{aligned} \frac{x+y}{2} &= \frac{1}{4} \\ \frac{4x+y}{3} &= 0 \end{aligned}\right\}$

Exercises 13-18. Solve the given linear inequality.

13. $3x - 5 > 2x + 4$
14. $-4x + 3 > x + 2$
15. $\frac{x}{2} - 3x \geq 4x + 5$
16. $-(-2x + 4) \leq 5x - 5$
17. $\frac{x+2}{4} + x - 1 < x + 2$
18. $\frac{x+4}{-4} + x - 1 < x + 2$

19. The sum of two numbers is 25. One of the numbers exceeds the other by 9. Find the numbers.

20. Find three consecutive odd positive integers so that the sum of twice the first integer, the second integer, and three times the third integer is 152.

21. In a child's piggy bank are quarters and dimes for a value of \$1.80. How many coins of each does the child have if he has a total of 18 coins?

22. A man has a collection of stamps made up of 5-cent stamps and 8-cent stamps. There are three times as many 8-cent stamps as 5-cent stamps. The total value of all the stamps is \$3.48. How many of each stamp does the man have?

23. The length of a rectangle is twice its width. If the perimeter is 72 inches, find the length and width of the rectangle.

24. Joe is presently 5 years younger than Mary. Four years from now, Mary will be twice as old as Joe. What is their present ages?

25. Sally's father is 4 times as old as Sally. After 5 years, father will be three times as old as Sally. Find their present ages.

26. The sum of two consecutive multiples of 5 is 55. What tare the numbers?

27. The cost of two tables and three chairs is \$705. If a table costs \$40 more than a chair, what is the cost of a chair?

28. If 75 gallons of water were added to a pool that is half full, the pool would then be $\frac{2}{3}$ full. How many gallons of water does the pool hold when it is full?

29. Three friends split their dinner bill. Mike paid \$7 less than half the total bill, Tom paid \$8 more than $\frac{1}{4}$ the total bill, and Linda paid the remaining \$20. How much was the total bill?

30. Sam is deciding whether or not he should become a member gym to use a squash court. The membership cost is \$135. Members pay \$5 to rent out a court. Non-members can rent the court also, but they have to pay \$12 each time. how many times would Sam need to rent the court in order for it be cheaper to be a member than a non member?

31. A pipe is 21 feet long, and you want to cut it into three pieces. The second piece is to be twice as long as the first piece, and the third piece is to be 1 foot longer than the second piece. What is the length of the first piece?

32. A rope is 100 feet long. You need to cut the rope into three pieces. The second piece is to be three times as long as the first piece, and the third piece must be 18 feet long. What is the length of the second piece?

33. Your grades on tests 1, 2, and 4 are 82, 76, and 90. Unfortunately, you missed the third test and received a 0. If you have one test left to take, and if the passing grade for the course is 70, can you still pass the course?

§2. Systems of Linear Equations

To solve a system of equations, say the system:

$$\left.\begin{aligned} 2x+4y-4z &= 6 \\ 2x+6y+4z &= 0 \\ x+y+2z &= -2 \end{aligned}\right\}$$

is to determine values of the **variables** x, y, and z for which each of the three equations is satisfied

Here are three operations that can be performed on a system of equation, **without altering the solution set**:

Elementary Operations on Systems of Linear Equations
Interchange the order of any two equations in the system.
Multiply both sides of an equation by a nonzero number.
Add a multiple of one equation to another equation.

MATRICES

Matrices are arrays of numbers arranged in rows and columns:

$$\begin{bmatrix} 2 & 13 & 4 \\ -9 & 7 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 7 \\ 10 & 6 \\ -8 & \sqrt{3} \end{bmatrix} \qquad \begin{bmatrix} 1 & 7 & 0 \\ 3 & 6 & 5 \\ 11 & 2 & -12 \end{bmatrix}$$

(i) (ii) (iii)

Matrix (i) contains 2 rows and 3 columns and is said to be a 2×3 (two-by-three) matrix. Similarly, (ii) is a 3×2 matrix, and (iii) is a 3×3 matrix (a **square matrix**).

It is often convenient to represent a system of equations in matrix form. The rows of the matrix in Figure 2.1(b), for example, concisely represents the equations in Figure 2.1(a). Note that the variables x, y, and z are suppressed in the matrix form, and that the vertical line represents the equal sign in the equations. Such a matrix is said to be the **augmented matrix** associated with the given system of equations.

$$\left.\begin{aligned} 2x+4y-4z &= 6 \\ 2x+6y+4z &= 0 \\ x+y+2z &= -2 \end{aligned}\right\} \longleftrightarrow \left[\begin{array}{ccc|c} 2 & 4 & -4 & 6 \\ 2 & 6 & 4 & 0 \\ 1 & 1 & 2 & -2 \end{array}\right]$$

System of Equations (a) **Augmented Matrix** (b)

Figure 2.1

HERE IS WHERE WE ARE GOING:

Suppose you want to solve the system of equations **[1]** in Figure 2.2.

Assume, for the time being, that you can go from its augmented matrix (**[2]**) to the augmented matrix **[3]**, via elementary row operations.

System **[4]**, which is associated with augmented matrix **[3]**, is easily seen to have the solution: $(x = -1, y = 1, z = -1)$. But this must also be the solution of system **[1]**, since the two systems of equations have the same solution set!

[1] $\left.\begin{aligned} 2x + 4y - 4z &= 6 \\ 2x + 6y + 4z &= 0 \\ x + y + 2z &= -2 \end{aligned}\right\}$ $\xrightarrow{\text{augmented matrix}}$ $\left[\begin{array}{ccc|c} 2 & 4 & -4 & 6 \\ 2 & 6 & 4 & 0 \\ 1 & 1 & 2 & -2 \end{array}\right]$ **[2]**

Same Solution Set (between [1] and [4])

Elementary Row Operations (from [2] to [3])

[4] $\left.\begin{aligned} x + 0y + 0z &= -1 \\ 0x + y + 0z &= 1 \\ 0x + 0y + z &= -1 \end{aligned}\right\}$ $\xleftarrow{\text{system of equations}}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$ **[3]**

Figure 2.2

We note that two systems of equations are **equivalent** if one can go from one to the other via elementary equation operations. In particular, system [1] and [4] in Figure 2.2 are equivalent.

Similarly, two matrices are said to be **equivalent** if one can go from one to the other via elementary row operations. In particular, matrix [2] and [3] in Figure 4.2 are equivalent.

The remainder of this section is designed to illustrate a method which can be used to go from matrix **[2]** of Figure 2.2 to matrix **[3]**, via elementary row operations.

ELEMENTARY ROW OPERATION

Capital letters are typically used to represent matrices, and double subscripted lower case letters for their entries; as in:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note that the first subscript of the element a_{ij} denotes its row i, and the second subscript, its column j.

The following notation will be used to represent elementary row operations:

ELEMENTARY ROW OPERATION	NOTATION
Switch row i with row j:	$R_i \leftrightarrow R_j$
Multiply each entry in row i by a nonzero number c:	$cR_i \rightarrow R_i$
Multiply each entry in row i by a number c, and then add the resulting row i to row j:	$cR_i + R_j \rightarrow R_j$

Below, we go from $\begin{bmatrix} 2 & 4 & 2 \\ 6 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$ to $\begin{bmatrix} \mathbf{1} & 2 & 1 \\ \mathbf{0} & \mathbf{1} & \frac{2}{3} \\ \mathbf{0} & -5 & -3 \end{bmatrix}$ via row operations:

$$\begin{bmatrix} \mathbf{2} & 4 & 2 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \rightarrow R_2} \begin{bmatrix} \mathbf{1} & 2 & 1 \\ \mathbf{0} & -3 & -2 \\ 4 & 3 & 1 \end{bmatrix} \xrightarrow{-4R_1 + R_3 \rightarrow R_3} \begin{bmatrix} \mathbf{1} & 2 & 1 \\ \mathbf{0} & -3 & -2 \\ \mathbf{0} & -5 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2 \rightarrow R_2} \begin{bmatrix} \mathbf{1} & 2 & 1 \\ \mathbf{0} & \mathbf{1} & 2/3 \\ \mathbf{0} & -5 & -3 \end{bmatrix}$$

CHECK YOUR UNDERSTANDING 2.6

Perform row operations to go from the above matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & -5 & -3 \end{bmatrix}$ to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Answer: See page A-4

ROW-REDUCED-ECHELON FORM

A matrix may have many different equivalent forms. Here is the nicest of them all:

DEFINITION 2.1
ROW-REDUCED ECHELON FORM

A matrix is in **row-reduced-echelon form** when it satisfies the following three conditions:

(i) The first non-zero entry in any row is 1 (called its **leading-one),** and all of the entries above or below that leading-one are 0.

(ii) In any two successive rows, not consisting entirely of zeros, the leading-one in the lower row appears further to the right than the leading-one in the row above it.

(iii) All of the rows that consist entirely of zeros are at the bottom of the matrix.

These three matrices are in row-reduced-echelon form:

$$A = \begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

CHECK YOUR UNDERSTANDING 2.7

Determine if the given matrix is in row-reduced-echelon form. If not, list the condition(s) of Definition 2.1 which are not satisfied.

$$(a)\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (b)\ \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (c)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix} \quad (d)\ \begin{bmatrix} 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer: Yes: (a), (c), and (d).
No: (b) [fails (ii)]

Though a bit tedious, reducing a matrix to its row-reduced-echelon form is a routine task. Just focus on getting those all-important **leading-ones** (which are to be positioned further to the right as you move down), with **zeros above and below** them. Consider the following example:

EXAMPLE 2.7 Perform elementary row operations to obtain the row-reduced-echelon form for the matrix:

$$\begin{bmatrix} 2 & 4 & -4 & 6 \\ 2 & 6 & 4 & 0 \\ 1 & 1 & 2 & -2 \end{bmatrix}$$

SOLUTION: Leading-one Step. We could divide the first row by 2 to get a leading-one in that row, but choose to switch the first row and third row instead:

$$\begin{bmatrix} 2 & 4 & -4 & 6 \\ 2 & 6 & 4 & 0 \\ 1 & 1 & 2 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & -2 \\ 2 & 6 & 4 & 0 \\ 2 & 4 & -4 & 6 \end{bmatrix}$$

Zeros-above-and-below Step:

$$\begin{bmatrix} 1 & 1 & 2 & -2 \\ 2 & 6 & 4 & 0 \\ 2 & 4 & -4 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 2 & -2 \\ \mathbf{0} & 4 & 0 & 4 \\ 2 & 4 & -4 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 1 & 2 & -2 \\ \mathbf{0} & 4 & 0 & 4 \\ \mathbf{0} & 2 & -8 & 10 \end{bmatrix}$$

Next leading-one Step:

$$\begin{bmatrix} 1 & 1 & 2 & -2 \\ \mathbf{0} & 4 & 0 & 4 \\ \mathbf{0} & 2 & -8 & 10 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2 \to R_2} \begin{bmatrix} 1 & 1 & 2 & -2 \\ \mathbf{0} & 1 & 0 & 1 \\ \mathbf{0} & 2 & -8 & 10 \end{bmatrix}$$

Zeros-above-and-below Step:

$$\begin{bmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -8 & 10 \end{bmatrix} \xrightarrow{-1R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -8 & 10 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -8 & 8 \end{bmatrix}$$

Next leading-one Step:

$$\begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -8 & 8 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_3 \to R_3} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Zeros-above-and-below Step:

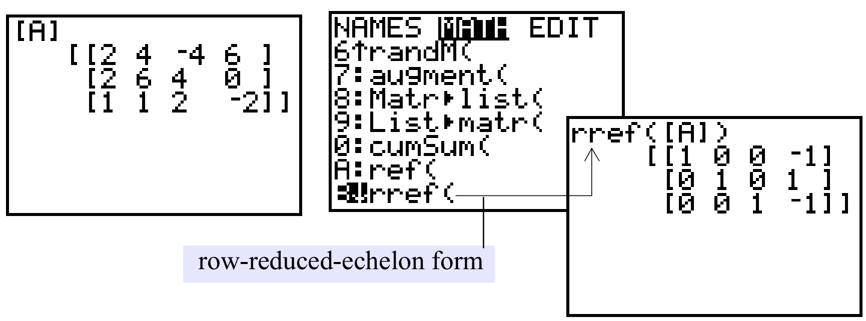
$$\begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We are now at a row-reduced-echelon form, and so we stop.

While not difficult, the above example illustrates that obtaining the row-reduced-echelon form of a matrix can be a bit tedious. It's a dirty job, but someone has to do it:

We are utilizing a TI-84+ calculator

GRAPHING CALCULATOR GLIMPSE 2.1



In harmony with graphing calculators, we will adopt the notation $\text{rref}(\boldsymbol{A})$ to denote the row-reduced-echelon form of a matrix $\boldsymbol{A}$.

EXAMPLE 2.8

Solve the system:

$$\left.\begin{aligned} 2x+4y-4z &= 6 \\ 2x+6y+4z &= 0 \\ x+y+2z &= -2 \end{aligned}\right\}$$

SOLUTION: All the work has been done:

augmented matrix

Example 2.7

$$\left.\begin{aligned} 2x+4y-4z &= 6\\ 2x+6y+4z &= 0\\ x+y+2z &= -2 \end{aligned}\right\} \leftrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 2 & 4 & -4 & 6\\ 2 & 6 & 4 & 0\\ 1 & 1 & 2 & -2 \end{array}\right] \end{array} \rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1 \end{array}\right] \end{array} \leftrightarrow \left.\begin{aligned} x &= -1\\ y &= 1\\ z &= -1 \end{aligned}\right\}$$

These two systems of equations are equivalent (same solution sets)

From the above we can easily spot the solution of the given system:

$$(x = -1, y = 1, z = -1)$$

CHECK YOUR UNDERSTANDING 2.8

Proceed as in Example 2.7 to solve the given system of equations.

$$\left.\begin{aligned} x+y+z &= 6\\ 3x+2y-z &= 4\\ 3x+y+2z &= 11 \end{aligned}\right\}$$

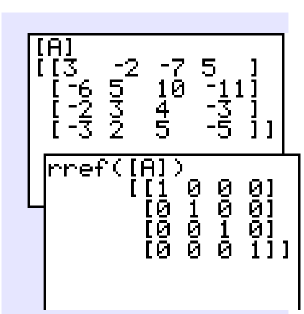
Answer:
$x = 1, y = 2, z = 3$

A system of equations may have a unique solution, infinitely many solutions, or no solution whatsoever. If it has no solution, then the system is said to be **inconsistent**, otherwise it is said to be **consistent**. As is illustrated in the following examples, the solution set of any system of equations can be spotted from the row-reduced-echelon form of its augmented matrix.

EXAMPLE 2.9 Determine if the following system of equations is consistent. If so, find its solution set.

$$\left.\begin{aligned} 3x-2y-7z &= 5\\ -6x+5y+10z &= -11\\ -2x+3y+4z &= -3\\ -3x+2y+5z &= -5 \end{aligned}\right\}$$

SOLUTION: Proceeding as in the previous section, we have:



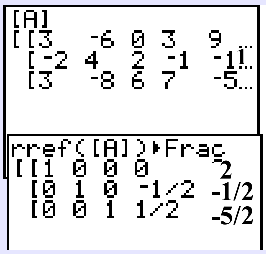
$$\left.\begin{aligned} 3x-2y-7z &= 5\\ -6x+5y+10z &= -11\\ -2x+3y+4z &= -3\\ -3x+2y+5z &= -5 \end{aligned}\right\} \longleftrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 3 & -2 & -7 & 5\\ -6 & 5 & 10 & -11\\ -2 & 3 & 4 & -3\\ -3 & 2 & 5 & -5 \end{array}\right] \end{array} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 0\\ 0 & \mathbf{1} & 0 & 0\\ 0 & 0 & \mathbf{1} & 0\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right] \end{array}$$

$$\boxed{\mathbf{0x+0y+0z = 1}}$$

Since the equation represented by the last row in the above rref-matrix cannot be satisfied, the given system of equations is inconsistent.

EXAMPLE 2.10 Determine the solution set of the system:

$$\left.\begin{aligned} 3x-6y+3w &= 9\\ -2x+4y+2z-w &= -11\\ 3x-8y+6z+7w &= -5 \end{aligned}\right\}$$



SOLUTION:

$$\left.\begin{aligned} 3x-6y+3w &= 9\\ -2x+4y+2z-w &= -11\\ 3x-8y+6z+7w &= -5 \end{aligned}\right\} \longleftrightarrow \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array}\\ \left[\begin{array}{cccc|c} 3 & -6 & 0 & 3 & 9\\ -2 & 4 & 2 & -1 & -11\\ 3 & -8 & 6 & 7 & -5 \end{array}\right] \end{array} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array}\\ \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & 0 & 2\\ 0 & \mathbf{1} & 0 & -\frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & \mathbf{1} & \frac{1}{2} & -\frac{5}{2} \end{array}\right] \end{array}$$

Figure 2.3

We know that the solution set of the above system of equations coincides with that of the one stemming from the row-reduced-echelon form of its augmented matrix;

$$\left.\begin{aligned} x+0y+0z+0\boldsymbol{w} &= 2\\ 0x+y+0z-\frac{1}{2}\boldsymbol{w} &= -\frac{1}{2}\\ 0x+0y+z+\frac{1}{2}\boldsymbol{w} &= -\frac{5}{2} \end{aligned}\right\} \text{ or: } \left.\begin{aligned} x &= 2\\ y &= -\frac{1}{2}+\frac{1}{2}\boldsymbol{w}\\ z &= -\frac{5}{2}-\frac{1}{2}\boldsymbol{w} \end{aligned}\right\}$$

Any variable that is not associated with a leading one in the row-reduced echelon form of an augmented matrix is said to be a **free variable**. In the current setting, the variable $\boldsymbol{w}$ is a free variable (see rref in Figure 2.3).

As you can see, that variable $\boldsymbol{w}$, which we moved to the right end of the equations, can be assigned any value whatsoever, after which the values of y and z (the variables associated with leading-ones in Figure 2.3) are determined. For example, setting $\boldsymbol{w} = \mathbf{0}$ leads to the particular solution:

$$\left(x=2,\ y=-\frac{1}{2},\ z=-\frac{5}{2},\ \boldsymbol{w}=\mathbf{0}\right)$$

We can generate another solution by letting $\boldsymbol{w} = \mathbf{1}$:

$$(x=2,\ y=0,\ z=-3,\ \boldsymbol{w}=\mathbf{1})$$

and so on and so forth (there are infinitely many solutions).

CHECK YOUR UNDERSTANDING 2.9

Determine if the system associated with the given row-reduced-echelon augmented matrix is consistent or inconsistent. If consistent indicate whether it has a unique solution or infinitely many solutions.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1\\ 0 & 1 & 5 & 4\\ 0 & 0 & 0 & 2 \end{array}\right]$ (b) $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array}\\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1\\ 0 & 1 & 5 & 4\\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

This equation: $0x+0y+0z = 0$, can be discarded as it is valid for all x, y, z. We are therefore faced with a system of two equations in three unknowns. The matrix will have a free variable.

Answer:
(a) Inconsistent
(b) Infinitely many.

EXERCISES

Exercises 1-2. Write down the augmented matrix associated with the given system of equations.

1. $$\left.\begin{aligned} 3x-3y+z &= 2\\ 5x+5y-9z &= -1\\ -3x-4y+z &= 0 \end{aligned}\right\}$$

2. $$\left.\begin{aligned} 2x+3y-4w &= 5\\ x-4z+w &= -1\\ x-4y &= 0\\ -x-y+z+4w &= 9 \end{aligned}\right\}$$

Exercises 3-4. Write down the system of equations associated with the given augmented matrix.

3. $$\left[\begin{array}{ccc|c} 5 & 1 & 4 & 3\\ -2 & -3 & 1 & 4\\ \frac{1}{2} & -1 & 0 & 0 \end{array}\right]$$

4. $$\left[\begin{array}{cccc|c} 2 & 4 & 1 & 0 & 9\\ 0 & 5 & 5 & 2 & 2\\ 2 & 1 & -3 & 8 & 11 \end{array}\right]$$

Exercises 5-7. Solve the system of equations corresponding to the given row-reduced-echelon matrix.

5. $$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array}\\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

6. $$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array}\\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & -3\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

7. $$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 & 2\\ 0 & 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \end{array}$$

Exercises 8-11 (Graphing Calculator) Determine the solution of the given system of equations.

8. $$\left.\begin{aligned} x-2y+z &= 1\\ -3x+5y-2z &= 4\\ 4x-8y+3z &= 6 \end{aligned}\right\}$$

9. $$\left.\begin{aligned} x-y-z &= 3\\ 4x-2y-5z &= 1\\ -x+3y+6z &= -2 \end{aligned}\right\}$$

10. $$\begin{aligned} x-2y+z &= 1\\ -x+y-2z &= 4\\ x-2y+3z &= 6 \end{aligned}$$

11. $$\left.\begin{aligned} 2x-y &= 2z+w+2\\ w-x &= y\\ 4y+3z &= 4\\ x+2y &= 3w-z \end{aligned}\right\}$$

12. Construct a system of three equations in three unknowns x, y, and z with solution $x = 1, y = 2, z = 3$ is a solution of the system, satisfying the condition that each equation is of the form $ax+by+cz+dw = d$ which satisfies the condition that each equation is of the form $ax+by+cz = d$ with a, b, and c not equal to 0..

13. Construct a system of four equations in four unknowns, x, y, z, and w with solution set $x = 1, y = 2, z = 3, w = 4$, satisfying the condition that each equation is of the form $ax + by + cz = d$ with a not equal to 0.

14. The sum of three integers is 40. Three times the smaller integer is equal to the sum of the others. Twice the larger is equal to 8 more than the sum of the other two. Find the integers.

15. The sum of three different integers is 21. The largest integer is one more than the sum of the other two, and the middle integer is nine less than the sum of the other two. Find the integers

16. A community theater sold 63 tickets to an afternoon performance for a total of \$444. An adult ticket cost \$8, a child ticket cost \$4, and a senior ticket cost \$6. If twice as many tickets were sold to adults as to children and seniors combined, how many of each ticket were sold?

17. A jar contains nickels, dimes, and quarters. The jar contains 32 coins with a total value of \$4.35. There are two less dimes then the other coins combined. How many quarters, dimes, and nickels are in th jar?

18. Bill sold 82 items at the swap meet for a total of \$504. He sold packages of socks for \$6, printed t-shirts for \$12, and hats for \$5. If he sold 5 times as many hats as he did t-shirts, how many of each item were sold?

19. The sum of four different integers is 42, and the larger integer equals the sum of the other 3. The sum of the larger two is 18 more than the sum of the smaller two, and the largest minus the smallest equals 16. Find the integers.

20. Mary is three years older than Bill, who is five years older tan Tom, who is twice the age of Judy, who is ten years younger than Bill. What are the ages of the four individuals?

§3 TWO-DIMENSIONAL LINEAR PROGRAMMING

A two-dimensional linear programming problem consists of an **objective function** $f(x, y)$ that is to be optimized, and a system of linear inequalities called **constraints**.

Let's jump right into an example:

EXAMPLE 2.11 Determine the maximum and minimum values of the objective function $z = 2x + y$, subject to the constraints:
(1): $x + y \le 4$
(2): $2x + y \le 6$
(3): $x \ge 0, y \ge 0$

The easiest way to draw a line is to locate its x and y intercepts. For $x + y = 4$: when $x = 0, y = 4$, and when $y = 0, x = 4$ so the line passes through the two points $(0, 4)$ and $(4, 0)$.

SOLUTION: Constraint (3) tells us that the constraint region lies in the first quadrant.

To address constraint (1), consider the line $x + y = 4$ in Figure 2.4(a) (see margin). To see whether the constraint region lies above or below that line, we tested its value at the origin $(0, 0)$:

If $x = 0$ and $y = 0$ then $x + y = 0 + 0 = 0 < 4$, so the constraint region $x + y \le 4$ lies **below** the line $x + y = 4$.

The line $2x + y = 6$ appears in Figure 2.3 (b). At the point $(0, 0)$, $2x + y = 0 < 6$. It follows that the constraint region $2x + y \le 6$ lies below the line $2x + y = 6$.

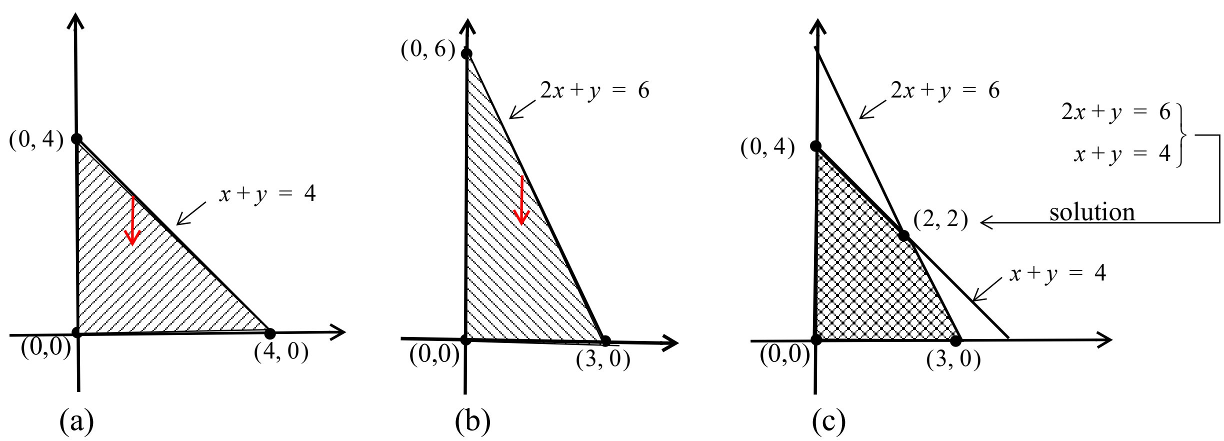


Figure 2.4

The intersection of the constraint region in (a) and (b) leads us to the region in Figure 24 (c), called the **feasible region** for the problem.

To find the point in the feasible region where the objective function $z = 2x + y$ assumes its maximum value and its minimum value, we employ the following method:

In Exercise 16 you are asked to show directly that this theorem holds for Example 2.11.

THEOREM 2.2
Simplex Method

If the feasible region of a linear programming problem is finite and closed (contains all of its boundary points), then the maximum and minimum values of the objective function occur at vertices.

It can be shown that a continuous function defined on a closed bounded set S assumes its maximum and minimum values at points in S.

Evaluating the objective function $f(x, y) = 2x + y$ of Example 2.11 at each of the four vertices of the region in Figure 2.4(c), we have:

Vertex, (x, y), of Constraint Region	$f(x, y)$
$(0, 0)$	0
$(0, 4)$	4
$(2, 2)$	6
$(3, 0)$	6

Conclusion: Maximum value: 6   Minimum value: 0

CHECK YOUR UNDERSTANDING 2.10

Determine both the maximum and minimum value of the objective function $z = x + 2y$, subject to the constraints:

$$x + y \le 3$$
$$x + 2y \le 3$$
$$2x + y \le 3$$
$$x \ge 0, y \ge 0$$

Answer: Maximum: 3
Minimum: 0

EXAMPLE 2.12 Find the maximum and minimum value of the objective function $z = 3x + 4y$, subject to the constraints:

$$x - y \le 1$$
$$x + 2y \ge 2$$
$$y \le 1$$
$$x \ge 0, y \ge 0$$

SOLUTION: Perhaps the safest way to arrive at the feasible region in Figure 2.5 is to test each constraint at $(0, 0)$, as we did in Example 2.11. Doing so, we find that:

The constraint region must lie **above** the line $x - y = 1$ for it to contain the origin—as it must, since $0 - (0) = 0 < 1$.

The constraint region must lies **above** the line $x + 2y = 2$ for it to exclude the origin—as it must, since $0 + 2 \cdot 0 = 0 < 2$.

The constraint region must lie **below** the line $y = 1$ for it to contain the origin—as it must, since $0 < 1$.

Here, then, is the feasible region for the current problem:

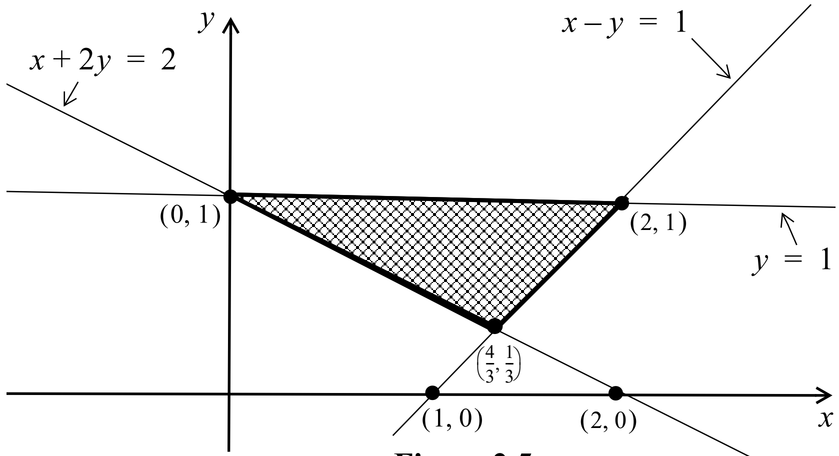


Figure 2.5

Theorem 2.2 tells us that the maximum value of the objective function is 10 and that its minimum value is 4:

Vertex, (x, y), of Constraint Region	$z = 3x + 4y$
$(0, 1)$	4
$\left(\frac{4}{3}, \frac{1}{3}\right)$	$\frac{16}{3}$
$(2, 1)$	10

.

EXAMPLE 2.13 Find the maximum and minimum value of the objective function $z = 3x + 2y$, subject to the constraints:

$$2x + 3y \geq 6$$
$$2x - y \leq 10$$
$$-x + y \leq 4$$
$$\frac{x}{2} + y \leq 6$$
$$x \geq 0, y \geq 0$$

Just to make sure, you can choose a point in the region, say $(1, 3)$, and see that it satisfies each of the given constraint inequalities.

SOLUTION: The constraint region appears in Figure 2.6. Justification:

The constraint region must lie **above** the line $2x + 3y = 6$ for it to exclude the origin—as it must, since $2 \cdot 0 + 3 \cdot 0 = 0 < 6$.

The constraint region must lie **above** the line $2x - y = 10$ for it to include the origin—as it must, since $2 \cdot 0 - 0 = 0 < 10$.

The constraint region must lie **below** the line $-x + y = 4$ for it to contain the origin— as it must, since $-0 + 0 = 0 < 4$.

The constraint region must lie **below** the line $\frac{x}{2} + y = 6$ for it to contain the origin— as it must, since $0 + 0 < 6$.

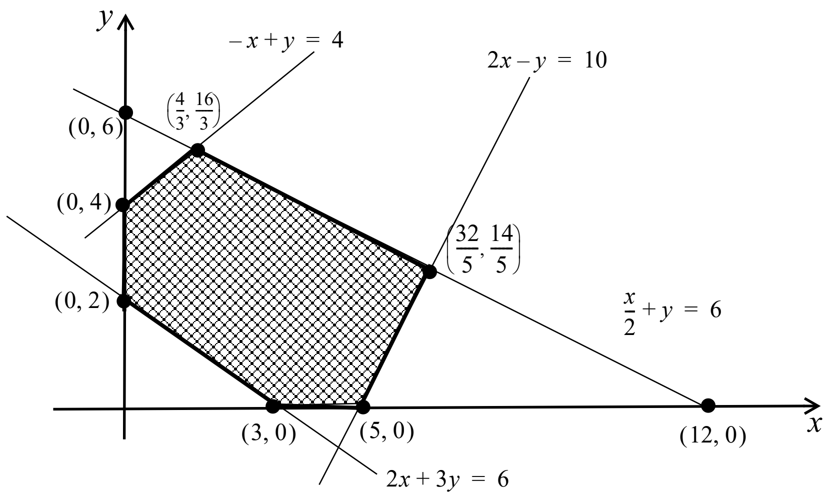


Figure 2.6

All that remains is to evaluate objective function $f(x, y) = 3x + 2y$ at the six vertices of the constraint region:

Vertex, (x, y), of Constraint Region	$z = 3x + 2y$
$(0, 2)$	4
$(0, 4)$	8
$\left(\frac{4}{3}, \frac{16}{3}\right)$	$\frac{44}{3} \approx 14.7$
$\left(\frac{32}{5}, \frac{14}{5}\right)$	$\frac{124}{5} \approx 24.8$
$(5, 0)$	15
$(3, 0)$	9

Conclusion: Maximum value of $\frac{124}{5}$ occurs at $\left(\frac{32}{5}, \frac{14}{5}\right)$.

Minimum value of 4 occurs at $(0, 2)$.

CHECK YOUR UNDERSTANDING 2.11

Determine both the maximum and minimum value of the objective function $z = 4x - y$, subject to the constraints

$$\begin{aligned} 2x + y &\ge 4 \\ x + 2y &\ge 4 \\ 2x + y &\le 6 \\ x \ge 0, y &\ge 0 \end{aligned}$$

Answer: Maximum: 10
Minimum: –6

EXAMPLE 2.14 Mary manufactures silver earrings and necklaces.

She will not spend more that 64 hours per week in the shop.

It takes her 4 hours to make a pair of earrings, and 8 hours to make a necklace.

She needs to make at least 6 pairs of earrings per week.

She realizes a profit of \$25 for every pair of earrings and a profit of \$60 for each necklace.

Find the maximum possible profit per week.

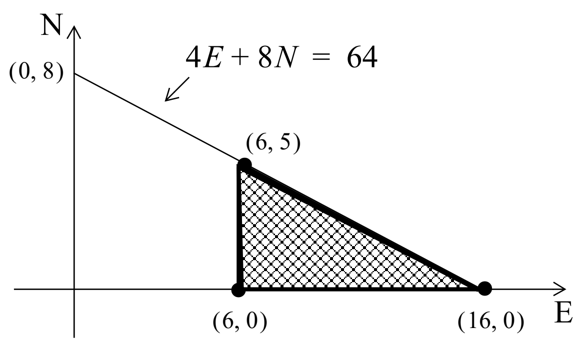
SOLUTION: Let E and N denote the weekly number of earrings and necklaces to be manufactured.

The profit function (objective function): $P = 25E + 60N$.

Constraint inequalities: $4E + 8N \le 64$

$E \ge 6, N \ge 0$

The constraint region:



Evaluating the objective function $P(E, N) = 25E + 60N$ at each of the four vertices of the constraint region we have:

Vertex (E, N)	Profit: $25E + 60N$
$(6, 0)$	\$150
$(6, 5)$	\$450
$(16, 0)$	\$400

Conclusion: A maximum profit of \$450 will be achieved with the production of 6 pairs of earrings and 5 necklaces,

EXAMPLE 2.15 A furniture company produces tables and chairs at a profit of \$75 per table and \$50 per chair.

Five hours and 35 board feet of wood are required for the construction of each table. Six hours and 10 board feet of wood are required for each chair.

A total of 700 board feet per week and of 300 work hours per week are available.

Determine production to achieve maximum profit.

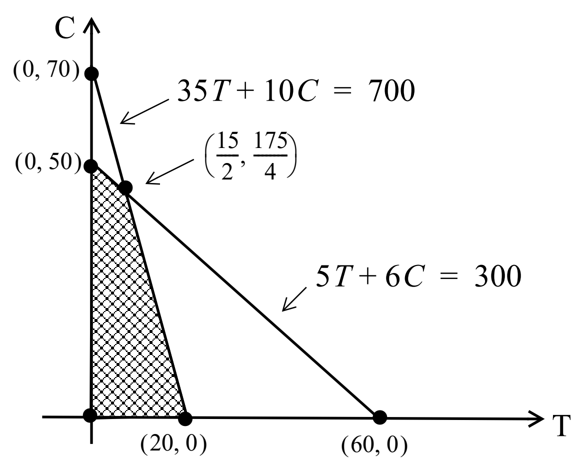
SOLUTION: Let T and C denote the weekly number of tables and chairs to be constructed.

The profit function (objective function): $P = 75T + 50C$

The constraint inequalities:

$$35T + 10C \le 700$$
$$5T + 6C \le 300$$
$$T \ge 0,\ C \ge 0$$

The constraint region:



Evaluating the objective function $P(T, C) = 75T + 50C$ at each of the four vertices of the constraint region we have:

Vertex, (x, y), of Constraint Region	$P(T, C)$
$(0, 0)$	0
$(20, 0)$	1500
$\left(\frac{15}{2}, \frac{175}{4}\right) = (7.5, 43.75)$	2750
$(0, 50)$	2500

Yes, the 2750 is the largest of the four profit numbers, but that figure needs to be adjusted since one cannot sell a fraction of a table or of a chair. Moreover, we cannot round $(7.5, 43.75)$ up to (8,44), for that point would not fall within the constraint region. We have to round down to 7 tables and 43 chairs. That \$2750 has to be changed to $\$[75(7)+50(43)] = \$2675$, which is still the largest profit number.

Conclusion: A maximum profit of \$2675 will be achieved with the production of 7 tables and 43 chairs.

Answer: A maximum profit of \$2000 will be achieved with the production of 40 tables and no chairs.

CHECK YOUR UNDERSTANDING 2.12

Solve the problem of Example 2.15 after replacing the restriction that 700 board feet are available with the restriction that 400 board feet are available.

In each of our examples, we were able to draw the constraint region for the given two-dimensional linear programming problem, and then used Theorem 2.2 to optimize the objective function. In practice, problems often involve hundreds of equations with thousands of variables, which can result in an astronomical number of “vertices”. People have been working on such problem for decades, and have developed high-quality software procedures, such as CPLEX and Gurobi.

EXERCISES

Exercises 1-15 Determine the maximum and minimum value of the given objective function, $z = f(x, y)$, subject to the indicated constraints.

1. $z = x + 2y$
$$\left.\begin{array}{r} 4x + y \le 16 \\ x + 2y \le 12 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

2. $z = 5x + 2y$
$$\left.\begin{array}{r} x + y \le 10 \} \\ x + 2y \le 12 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

3. $z = 3x - 4y$
$$\left.\begin{array}{r} 3x + 2y \le 16 \\ x + 2y \le 12 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

4. $z = 4x + 2y$
$$\left.\begin{array}{r} 5x + y \ge 5 \\ x + 3y \ge 9 \\ \frac{5}{9}x + y \le 5 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

5. $z = -5x + 30y$
$$\left.\begin{array}{r} x + y \le 6 \\ 2x + y \ge 9 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

6. $z = 2x - y$
$$\left.\begin{array}{r} x + 2y \ge 8 \\ 6x + 4y \ge 36 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

7. $z = x + 2y$
$$\left.\begin{array}{r} x + y \ge 30 \\ x + y \le 40 \\ 4x + y \ge 40 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

8. $z = -3x + 2y$
$$\left.\begin{array}{r} x + y \ge 30 \\ x + 2y \le 40 \\ 2x + 3y \ge 72 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

9. $z = 3x - 4y$
$$\left.\begin{array}{r} 2x + y \le 28 \\ 4x + y \le 48 \\ 2x + 3y \le 60 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

10. $z = 4x - 5y$
$$\left.\begin{array}{r} 2x + y \le 28 \\ 4x + y \le 48 \\ 2x + 3y \le 60 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

11. $z = 6x + y$
$$\left.\begin{array}{r} x + y \ge 30 \\ x + 2y \le 40 \\ 2x + 3y \ge 72 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

12. $z = x - 5y$
$$\left.\begin{array}{r} 2x + y \le 28 \\ 4x + y \le 48 \\ 2x + 3y \le 60 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

13. $z = 7x - 5y$

$$\left.\begin{array}{r} 2x + y \le 14 \\ x + 3y \le 15 \\ x + 3y \le 9 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

14. $z = 6x + 6y$

$$\left.\begin{array}{r} 4x + 2y \le 28 \\ 2x + 6y \le 36 \\ 4x + 6y \le 36 \\ x \ge 0, y \ge 0 \end{array}\right\}$$

15. $z = x - 9y$

$$\left.\begin{array}{c} x + 2y \le 200 \\ 4x + 3y \le 480 \\ x \ge 100 \\ y \ge 0 \end{array}\right\}$$

16. (Theory) Without using Theorem 2.2, show that the maximum value of the objective function $z = 2x + y$ of Example 2.11 must occur at the vertex $(3, 2)$.

 Suggestion: Sketch the line $y = -2x + z$ in Figure 2.3, and raise it as far as you can under the condition that part of the line remains in the constraint region.

17. A manufacturer has $750m^2$ of cotton textile and $1000m^2$ of polyester in stock. Every pair of pants requires $1m^2$ of cotton and $2m^2$ of polyester. Every jacket requires $1.5m^2$ of cotton and $1m^2$ of polyester. The manufacturer will realize a profit of $10 from the sale of each pair of pants, and a profit of $17 from the sale of each jacket. Determine production that will maximize profit.

18. An oil company owns two refineries: refinery A and refinery B. Refinery A is capable of producing 20 barrels of gasoline and 25 barrels of fuel oil per day. Refinery B is capable of producing 40 barrels of gasoline and 20 barrels of fuel oil per day. The company requires at least 1000 barrels of gasoline and at least 800 barrels of fuel oil. If it costs $300 per day to operate refinery A and $500 per day to operate refinery B, how many days should each refinery be operated by the company so as to minimize costs?

19. Food X costs $3 per pound and yields 2 units of vitamins, 10 units of starch, and 6 units of protein. Food Y costs $5 per pound and yields 6 units of vitamins, 2 units of starch, and 4 units of protein. The minimum requirements are 178 units of vitamins, 200 units of starch, and 240 units of protein. Food X is sold in bags of 89/100 pounds per bag, and food Y is sold in bags of 1 pound per bag. A maximum of 100 bags of food can be delivered. Determine the combinations of X and Y that will satisfy the minimum requirements at minimal cost.

20. Consider a chocolate manufacturing company that produces only two types of chocolate: A and B. Both the chocolates require Milk and Choco only. Each unit of A requires 1 unit of Milk and 3 units of Choco. Each unit of B requires 1 unit of Milk and 2 units of Choco. The company kitchen has a total of 5 units of Milk and 12 units of Choco. The company makes a profit of $6 per unit of A sold and $5 per unit of B sold. How many units of A and B should it produce respectively to maximize profit?

21. You need to buy some filing cabinets. You know that Cabinet X costs $50 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs $70 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given $2000 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. How many of which model should you buy, in order to maximize storage volume?

22. A shoe manufacturer produces high tops and running sneakers. Each high type pair requires 3 minutes of cutting time and 1.5 minutes of stitching time. Each Running pair requires 2.5 minutes of cutting time and 2 minute of stitching time. The cutting machine is available for a maximum of 275 minutes each day, and the stitching machine is available for a maximum of 300 minutes each day. The company nets a profit of $18 per high tops and $13 per running shoes. Determine the maximum profit that can be made with the available resources.

CHAPTER SUMMARY

INEQUALITIES	When multiplying or dividing both sides of an inequality by a negative quantity, reverse the direction of the inequality sign.

ELEMENTARY OPERATIONS ON SYSTEMS OF LINEAR EQUATIONS	Interchange the order of any two equations in the system. Multiply both sides of an equation by a nonzero number. Add a multiple of one equation to another equation. Performing any sequence of elementary operations on a system of linear equations will result in an equivalent system of equations.

ELEMENTARY MATRIX ROW OPERATIONS	Interchange the order of any two rows in the matrix. Multiply each element in a row of the matrix by a nonzero number. Add a multiple of one row of the matrix to another row of the matrix. Two matrices are equivalent if one can be derived from the other by performing elementary row operations. Systems of equations associated with equivalent matrices are themselves equivalent (same solution set).

rref-form OF A MATRIX	A matrix is in **row-reduced-echelon form** when it satisfies the following three conditions: (i) The first non-zero entry in any row is 1 (called its **leading-one),** and all of the entries above or below that leading-one are 0. (ii) In any two successive rows, not consisting entirely of zeros, the leading-one in the lower row appears further to the right than the leading-one in the row above it. (iii) All of the rows that consist entirely of zeros are at the bottom of the matrix. One can easily read off the solutions of a system of equations from its associated matrix reduced in rref-form.

CONSISTENT AND INCONSISTENT SYSTEMS OF EQUATIONS	A system of equations may have a unique solution, infinitely many solutions, or no solution whatsoever. If it has no solution, then the system is said to be **inconsistent**, otherwise it is said to be **consistent**.

LINEAR PROGRAMMING	A linear programming problem consists of an **objective function** $f(x, y)$ that is to be optimized, and a system of linear inequalities called **constraints**.

SIMPLEX METHOD	If the constraint region of a linear programming problem is finite and closed (contains all of its boundary points), then the maximum and minimum values of the objective function occur at vertices.

CHAPTER 3
Functions

§1. Basic Definitions

Roughly speaking, a **function** f from a set X to a set Y, represented by $f: X \to Y$, is a rule that assigns to each $x \in X$ exactly one element $y \in Y$: $y = f(x)$. The set X is called the **domain** of f, and the **range** of f is the set $f(X)$ composed of all the function values.

We want to emphasize the fact that the variable x in $f(x)$ is a place-holder; a "box" that can hold any meaningful expression. For example:

$$f(\boldsymbol{x}) = 2\boldsymbol{x} + 5$$

$$f\boxed{} = 2\boxed{} + 5$$

So:

$$f(\mathbf{3}) = 2 \cdot \mathbf{3} + 5 = 11$$

$$f(\boldsymbol{c}) = 2 \cdot \boldsymbol{c} + 5 = 2c + 5$$

$$f(\boldsymbol{x^2 + 3}) = 2(\boldsymbol{x^2 + 3}) + 5 = 2x^2 + 11$$

You want to remember the following two formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Proof of the top formula:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

EXAMPLE 3.1 Determine $f(x+h)$ for $f(x) = -3x^2 + 6x - 1$.

Solution:

$$\begin{aligned}f(\boldsymbol{x+h}) &= -3(\boldsymbol{x+h})^2 + 6(\boldsymbol{x+h}) - 1\\ &= -3(x^2 + 2xh + h^2) + 6x + 6h - 1\\ &= -3x^2 - 6xh - 3h^2 + 6x + 6h - 1\end{aligned}$$

CHECK YOUR UNDERSTANDING 3.1

For $f(x) = 3x^2 - 5$, determine $f(2x+1)$

Answer: $12x^2 + 12x - 2$

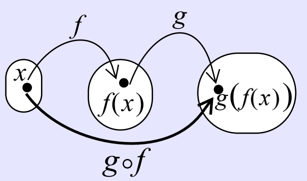
Composition of Functions

If $h(x) = (x-3)^2$, then $h(8) = 25$, right? You got that answer by first subtracting 3 from 8: $(x-3) = (8-3) = \mathbf{5}$, and then squaring the result: $(\mathbf{5})^2 = 25$. In other words, you **first** apply the function $f(x) = x - 3$, **and then** apply the function $g(x) = x^2$ to that result.

The operation of first invoking one function, and then another, is called **composition**, and is denoted by $(g \circ f)(x)$:

:

COMPOSITION



DEFINITION 3.1 The **composition** $(g \circ f)(x)$ is given by:

$$(g \circ f)(x) = g(f(x))$$

first apply f

and **then** apply g

(Assuming $f(x)$ is in the domain of g)

EXAMPLE 3.2 Determine $(g \circ f)(2)$ and $(f \circ g)(2)$ for:

$$f(x) = x^2 + 1 \text{ and } g(x) = 2x - 3$$

SOLUTION:

$$(g \circ f)(2) = g(\boldsymbol{f(2)}) = g(\mathbf{2^2 + 1}) = g(\mathbf{5}) = 2 \cdot \mathbf{5} - 3 = 7$$

$$(f \circ g)(2) = f(\boldsymbol{g(2)}) = f(\mathbf{2 \cdot 2 - 3}) = f(\mathbf{1}) = \mathbf{1}^2 + 1 = 2$$

EXAMPLE 3.3 Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for:

$$f(x) = 3x - 1 \text{ and } g(x) = -2x^2 - 3x + 1$$

SOLUTION:

Definition 3.1

$f(x) = 3x - 1$

$$\begin{aligned}(f \circ g)(x) = f(g(x)) = f(-2x^2 - 3x + 1) &= 3(-2x^2 - 3x + 1) - 1\\ &= -6x^2 - 9x + 3 - 1\\ &= -6x^2 - 9x + 2\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) = g(f(x)) = g(3x - 1) &= -2(3x - 1)^2 - 3(3x - 1) + 1\\ &= -2(9x^2 - 6x + 1) - 9x + 3 + 1\\ &= -18x^2 + 3x + 2\end{aligned}$$

CHECK YOUR UNDERSTANDING 3.2

Let $f(x) = x^2 + 2x - 2$ and $g(x) = 4x + 3$.

(a) Evaluate $(f \circ g)(-2)$ (b) Determine $(f \circ g)(x)$

Answers:
(a) 13 (b) $16x^2 + 32x + 13$

ONE-TO-ONE AND ONTO FUNCTIONS

Our next goal is to single out some functions that count more than others; beginning with one-to-one functions:

DEFINITION 3.2
ONE-TO-ONE

A function $f: X \to Y$ is said to be **one-to-one** if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Equivalently, f is one-to-one if
$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
In words:
different x's are mapped to different y's.

Figure 3.1 represents the action of two functions, f and g, from $X = \{0, 1, 2\}$ to $Y = \{a, b, c, d\}$. The function f on the left is one-to-one, while the function g is not.

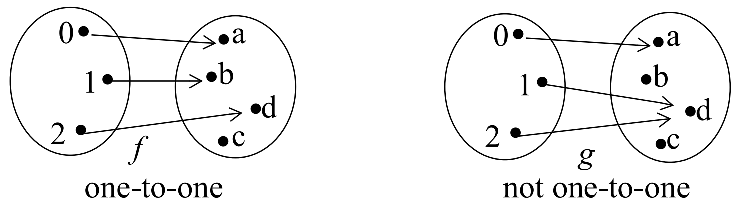


Figure 3.1

EXAMPLE 3.4 Show that the function $f(x) = \dfrac{x}{5x+2}$ is one-to-one.

SOLUTION: Appealing to Definition 3.2, we begin with $f(a) = f(b)$, and then go on to show that this can only hold if $a = b$:

$f(a) = f(b)$

$a = b$

$$\begin{aligned} f(a) &= f(b) \\ \frac{a}{5a+2} &= \frac{b}{5b+2} \\ a(5b+2) &= b(5a+2) \\ 5ab+2a &= 5ab+2b \\ 2a &= 2b \\ a &= b \end{aligned}$$

In words:
$f: X \to Y$ is onto if every element in Y is "hit" by some $f(x)$.

DEFINITION 3.3
ONTO

A function $f: X \to Y$ is **onto** if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

Figure 3.2 represents the action of two functions, f and g, from $X = \{0, 1, 2, 3\}$ to $Y = \{a, b, c\}$. The function f on the left is onto, while the function g is not..

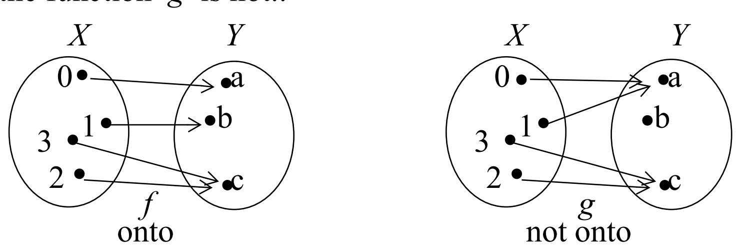


Figure 3.2

DEFINITION 3.4
BIJECTION

A function $f: X \to Y$ that is both one-to-one and onto is said to be a **bijection.**

Consider the bijection $f: \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\} \to \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ depicted in Figure 3.3(a), and the function $f^{-1}: \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\} \to \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ in Figure 3.3(b). Figuratively speaking, f^{-1}, read f *inverse*. was obtained by "reversing" the direction of the arrows in Figure 3.3(a).

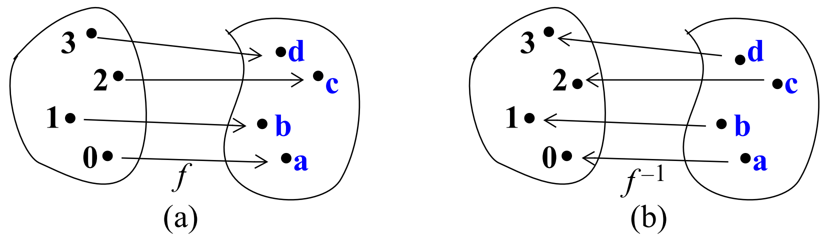


Figure 3.3

In general:

DEFINITION 3.5
INVERSE FUNCTION

The **inverse** of a bijection $f: X \to Y$, is the function $f^{-1}: Y \to X$ given by:

$$f^{-1}(y) = x, \text{ where } f(x) = y$$

Figure 3.3 certainly suggests that if f is a bijection then so is f^{-1}, and that if you apply f and then f^{-1} you are back to where you started from, and that if you apply f^{-1} and then f you are again back to where you started from. This is indeed the case:

THEOREM 3.3

If $f: X \to Y$ is a bijection, then so is $f^{-1}: Y \to X$. Moreover:

$$(f^{-1} \circ f)(x) = x \text{ and } (f \circ f^{-1})(y) = y$$

Henceforth, we will deal exclusively with functions of the form $f: X \to Y$, where both X and Y consist of real numbers, and where Y is the **range** of f. As such, if f is one-to-one, then it is a bijection.

Range of $f: X \to Y$:
$f(X) = \{f(x) | x \in X\}$

EXAMPLE 3.5

Show that the function $f(x) = 3x + 2$ is a bijection. Find its inverse and show, directly, that $(f \circ f^{-1})(x) = x$ and that $(f^{-1} \circ f)(x) = x$.

SOLUTION: Verifying that f is one-to-one (and therefore a bijection):

$$f(a) = f(b) \Rightarrow 3a + 2 = 3b + 2 \Rightarrow 3a = 3b \Rightarrow a = b$$

Finding the inverse of f:

Start with: $(f \circ f^{-1})(x) = x$ i.e: $f[f^{-1}(x)] = x$

$\boxed{\textbf{substitute } t \text{ for } f^{-1}(x):}$ $f[t] = x$

$\boxed{\text{Since } f(x) = 3x + 2:}$ $3t + 2 = x$

$\boxed{\text{Solve for } t:}$ $t = \dfrac{x-2}{3}$

$\boxed{\textbf{Substitute } f^{-1}(x) \text{ back for } t:}$ $f^{-1}(x) = \dfrac{x-2}{3}$

Verifying that $(f \circ f^{-1})(x) = x$:

$\boxed{\text{Since } f^{-1}(x) = \frac{x-2}{3}}$ $\boxed{\text{Since } f(x) = 3x+2}$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-2}{3}\right) = 3\left(\frac{x-2}{3}\right) + 2$$
$$= x - 2 + 2 = x$$

Verifying that $(f^{-1} \circ f)(x) = x$:

$\boxed{\text{Since } f(x) = 3x+2}$ $\boxed{\text{Since } f^{-1}(x) = \frac{x-2}{3}}$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x+2) = \frac{(3x+2)-2}{3}$$
$$= \frac{3x}{3} = x$$

CHECK YOUR UNDERSTANDING 3.3

(a) Show that the function $f(x) = \frac{1}{2}x - 5$ is one-to-one.

(b) Determine the inverse of f.

(c) Verify, directly, that $(f \circ f^{-1})(x) = x$.

Answer: See page A-6.

EXERCISES

Exercise 1-8. Determine $(g \circ f)(2)$ and $(f \circ g)(2)$ for the given functions.

1. $f(x) = x + 1$ and $g(x) = x - 3$
2. $f(x) = x^2 + 3$ and $g(x) = 2x - 3$
3. $f(x) = x^2 + x$ and $g(x) = 2x - 1$
4. $f(x) = x^2 + x$ and $g(x) = x^2 - 3x\overline{1}$
5. $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x$
6. $f(x) = \dfrac{x-3}{x+2}$ and $g(x) = \dfrac{1}{x}$
7. $f(x) = x + 1$ and $g(x) = \dfrac{x^2}{x-1}$
8. $f(x) = \dfrac{x-3}{x+2}$ and $g(x) = \dfrac{x+2}{x^2}$

Exercise 9-14. Determine $(g \circ f)(x)$ and $(f \circ g)(x)$ for the given functions.

9. $f(x) = x + 1$ and $g(x) = x - 3$
10. $f(x) = x + 3$ and $g(x) = 2x - 3$
11. $f(x) = x^2 + 3$ and $g(x) = 2x - 3$
12. $f(x) = x^2 + x$ and $g(x) = 3x$
13. $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x$
14. $f(x) = \dfrac{x-3}{x+2}$ and $g(x) = \dfrac{1}{x}$

Exercises 15-18. Show that the given function is one-to-one.

15. $f(x) = -5x - 1$
16. $f(x) = 6x + 5$
17. $f(x) = \dfrac{x-5}{4x}$
18. $f(x) = \dfrac{2x-5}{3x}$

Exercises 19-26: (a) Find the inverse of the given bijection f.
(b) Verify directly that $(f \circ f^{-1})(x) = x$ and that $(f^{-1} \circ f)(x) = x$.

19. $f(x) = 6x + 2$
20. $f(x) = 7x - 3$
21. $f(x) = 2x + 6$
22. $f(x) = \dfrac{7x-3}{2}$
23. $f(x) = \dfrac{1}{2}x + 1$
24. $f(x) = \dfrac{x}{3} - \dfrac{1}{2}$
25. $f(x) = \dfrac{x}{x+3}$
26. $f(x) = \dfrac{x+3}{x}$

§2. Exponential and Logarithmic Functions

Exponential and logarithmic functions play important roles throughout mathematics, and in numerous other fields.

DEFINITION 3.6
Exponential Function

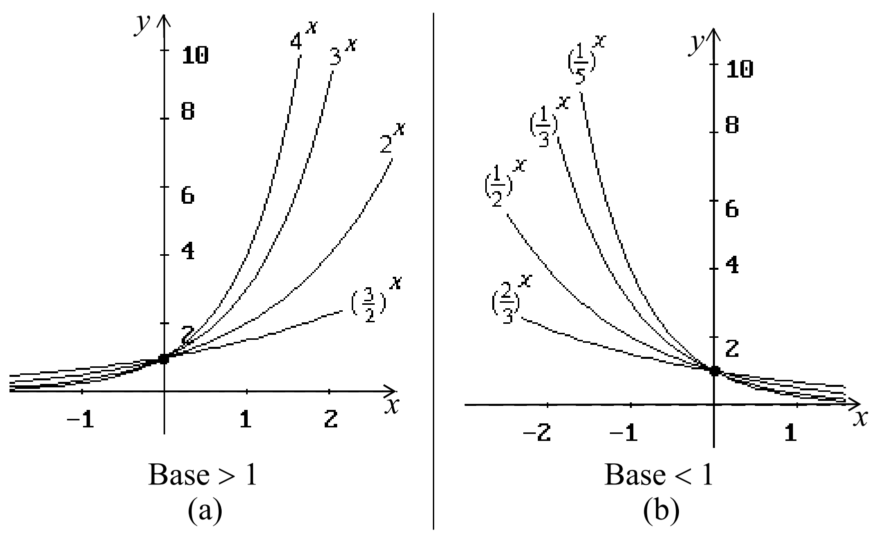
Let b be a positive number other than 1. The function f given by:

$$f(x) = b^x$$

is said to be the **exponential function with base *b***.

Note:
For any $a \neq 0$: $a^0 = 1$, and for any number r:
$$a^{-r} = \frac{1}{a^r}$$
for example:
$$(3)^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

Graphs of several exponential functions appear in Figure 3.4. Since $b^0 = 1$ for every base b, each graph contains the point $(0, 1)$. Note that the exponential functions of base greater than 1 in Figure 3.4(a) are **increasing functions** (graphs climb as you move to the right), and that those of base less than 1 in Figure 3.4(b) are **decreasing functions** (graphs fall as you move to the right).



Base > 1
(a)

Base < 1
(b)

Figure 3.4

The following result, which is evident from the graphs in Figure 3.4, can be used to solve exponential equations involving a common base:

THEOREM 3.4

Every exponential function is one-to-one:

$$b^s = b^t \text{ if and only if } s = t$$

More generally

$$b^{\text{expression}_1} = b^{\text{expression}_2}$$

if and only if

$$\textbf{expression}_1 = \textbf{expression}_2$$

EXAMPLE 3.6 Solve:

(a) $3^{5x+1} = 9^{3x}$ (b) $2^{x^2} = 4^x$

SOLUTION:

(a)

$$3^{5x+1} = 9^{3x}$$

Common base: $3^{5x+1} = (3^2)^{3x}$

$$3^{5x+1} = 3^{2(3x-5)}$$

$$3^{5x+1} = 3^{6x-10}$$

Theorem 3.4 $5x+1 = 6x-10$

$$-x = -11$$

$$\boxed{x = 11}$$

(b)

$$2^{x^2} = 4^x$$

$$2^{x^2} = (2^2)^x$$

$$2^{x^2} = 2^{2x}$$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0 \Rightarrow \boxed{x = 0, x = 2}$$

CHECK YOUR UNDERSTANDING 3.4

Solve:

$$3^{x+2} = 27^x$$

Answer: $x = 1$

Since the exponential function b^x is one-to-one, it has an inverse— it is called the **logarithmic function of base *b***, and is denoted by $\log_b x$.

DEFINITION 3.7
LOGARITHMIC FUNCTIONS

For any positive number b, other than 1, and any $x > 0$:

$$\log_b x = y \text{ if } b^y = x$$

IN WORDS:	To say that "log base b of x equals y" is the same as saying that "b to the y equals x."

In still other words:

$\log_b x$ is the exponent that b must be raised to, in order to get x.

For example:

$$\log_3 9 = 2 \text{ since } 3^2 = 9$$

The graphs of several logarithmic functions appear in Figure 3.5. Note that the graphs of logarithmic functions of base greater than one are increasing functions [see Figure 3.5(a)], while those of base less than 1 are decreasing functions [see Figure 3.5(b)].

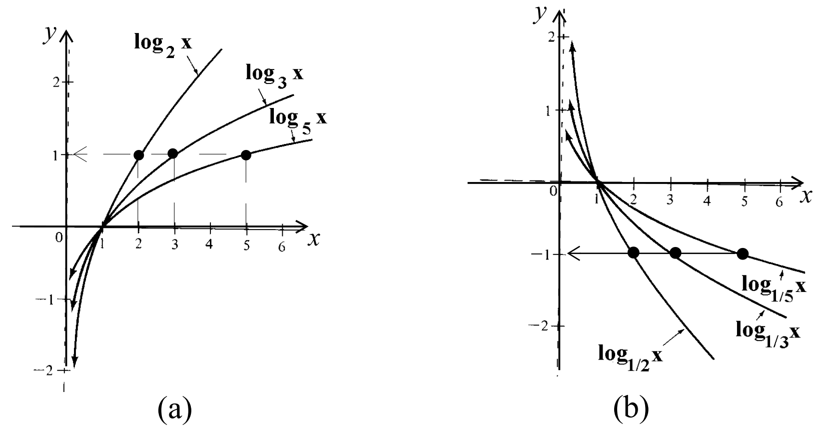


Figure 3.5

Note also that the domain of logarithmic functions is the set of **positive numbers**. In particular:

For $\log_b(\textbf{expression})$ to be defined, **expression** must be **positive.**

Moreover, since logarithmic functions are one-to-one:

THEOREM 3.5 For $s, t > 0$:

$$\log_b s = \log_b t \text{ if and only if } s = t$$

EXAMPLE 3.7 Solve:

$$\log_2(x^2 - x) = \log_2(-6x + 6)$$

SOLUTION:

$$\log_2(x^2 - x) = \log_2(-6x + 6)$$

Theorem 3.5: $x^2 - x = -6x + 6$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

$$x = -6 \text{ or } x = 1$$

For $x = -6$, $x^2 - x$ is **positive**, while for $x = 1$, $x^2 - x = 0$. It follows that -6 is the only solution of the equation.

CHECK YOUR UNDERSTANDING 3.5

Solve: $\log_2(x^2 + 2x) = \log_2 3$

Answer: $x = -3, x = 1$

Two logarithmic functions deserve special mention:

the **common logarithm:** $\log_{10} x$, or simply $\log x$

and

the **natural logarithm**: $\log_e x$, or simply $\ln x$

As it is with the number pi, $\pi \approx 3.1459$, the number $e \approx 2.7183$, arises naturally throughout mathematics and the sciences.

Graphing calculators can be used to evaluate common and natural logs. For example, using your calculator you will find that:

$$\log 22 \approx 1.3424 \quad (\text{i.e. } 10^{1.3424} \approx 22)$$

and

$$\ln 22 \approx 3.0910 \quad (\text{i.e. } e^{3.0910} \approx 22)$$

CHECK YOUR UNDERSTANDING 3.6

(a) Evaluate, without a calculator.

(i) $\log_2 16$ (ii) $\log_{\frac{1}{3}} 9$ (iii) $\log_5 1$

(b) Use a calculator to find a two decimal place approximation for:

(i) $\dfrac{\ln 37.3}{\ln 1.8 + \ln 13.2}$ (ii) $\dfrac{\log 50}{5 - \log 12}$

Answers: (a-i) 4 (a-ii) -2
(a-iii) 0
(b-i) ≈ 1.14 (b-ii) ≈ 0.43

The following properties are direct consequences of the fact that logarithmic and exponential functions of the same base are inverses of each other:

THEOREM 3.6
INVERSE PROPERTIES

(i) For any x: $\log_b b^x = x$

(ii) For any $x > 0$: $b^{\log_b x} = x$

In addition, we have:

THEOREM 3.7

For any base b, and any r, s, and t with $s > 0, t > 0$:

(i) $\log_b (st) = \log_b s + \log_b t$

(the log of a product is the sum of the logs)

(ii) $\log_b \left(\dfrac{s}{t}\right) = \log_b s - \log_b t$

(the log of a quotient is the difference of the logs)

(iii) $\log_b s^r = r \log_b s$

(the log of a power of a value is the power times the log of the value)

As is illustrated in the following examples, the above theorem can be used to solve certain exponential and logarithmic equations.

EXAMPLE 3.8 Solve:

$$2^{2x-1} = 3^{3x-7}$$

SOLUTION:

$$2^{2x-1} = 3^{3x-7}$$

We decide to apply the natural logarithmic function to both sides of the equation: $\ln(2^{2x-1}) = \ln(3^{3x-7})$

by Theorem 3.7(iii): $(2x-1)\ln 2 = (3x-7)\ln 3$

$$2x\ln 2 - \ln 2 = 3x\ln 3 - 7\ln 3$$
$$2x\ln 2 - 3x\ln 3 = \ln 2 - 7\ln 3$$
$$(2\ln 2 - 3\ln 3)x = \ln 2 - 7\ln 3$$
$$x = \frac{\ln 2 - 7\ln 3}{2\ln 2 - 3\ln 3}$$

Or, more compactly

$$x = \frac{\ln 2 - 7\ln 3}{2\ln 2 - 3\ln 3} = \frac{\ln\frac{2}{3^7}}{\ln\frac{4}{27}}$$

CHECK YOUR UNDERSTANDING 3.7

Solve:

$$2^{3x-1} = 5^{x+2}$$

Answer:

$$\frac{2\ln 5 + \ln 2}{3\ln 2 - \ln 5} = \frac{\ln 50}{\ln\frac{8}{5}}$$

.

EXAMPLE 3.9

Solve:

(a) $2\ln 2x - \ln(x+1) = \ln x$

(b) $\log_3(x-1) + \log_3(2x+3) = 1$

SOLUTION:

(a)

$$2\ln 2x - \ln(x+1) = \ln x$$

$r\ln s = \ln s^r$: $\ln(2x)^2 - \ln(x+1) = \ln x$

$\ln s - \ln t = \ln\left(\frac{s}{t}\right)$: $\ln\frac{4x^2}{x+1} = \ln x$

one-to-one property of logarithmic functions: $\frac{4x^2}{x+1} = x$

$$4x^2 = x^2 + x$$
$$3x^2 - x = 0$$
$$x(3x-1) = 0$$
$$x = 0 \text{ or } x = \frac{1}{3}$$

We challenge both candidates:
Since you can only take logs of positive numbers, and since $2x$ is zero when $x = 0$, 0 is **not** a solution of the equation.
Since $2x$, $x+1$, and x are all positive for $x = \frac{1}{3}$, $\frac{1}{3}$ **is** a solution of the equation.

(b)
$$\log_3(x-1)+\log_3(2x+3) = 1$$

$\log_b s+\log_b t = \log_b st$:
$$\log_3[(x-1)(2x+3)] = 1$$
$$3^{\log_3[(x-1)(2x+3)]} = 3^1$$

$b^{\log_b x} = x$:
$$(x-1)(2x+3) = 3$$
$$2x^2+x-3 = 3$$
$$2x^2+x-6 = 0$$
$$(2x-3)(x+2) = 0$$
$$x = \frac{3}{2} \quad\text{or}\quad x = -2$$

We have two solution candidates, and now challenge both: Since $x-1$ is negative when $x = -2$, -2 is **not** a solution of the equation (you can only take logs of positive numbers). Since $x-1$ and $2x+3$ are positive for $x = \frac{3}{2}$, $\frac{3}{2}$ **is** a solution of the equation.

CHECK YOUR UNDERSTANDING 3.8

Solve:

(a) $\log_2(x+9)-\log_2 x = 2$

(b) $\log_2(2x+5)-\log_2(2x+1) = \log_2(3)$

Answers: (a) 3 (b) 1/2

Here is a useful result:

THEOREM 3.8 CHANGE OF BASE FORMULA

For any bases a and b, and any $x>0$:
$$\log_b x = \frac{\log_a x}{\log_a b}$$

PROOF:

Begin with:
$$b^{\log_b x} = x$$

Apply $\log_a x$ to both sides:
$$\log_a b^{\log_b x} = \log_a x$$

Apply $\log_a x^r = r\log_a x$ (with $x = b$ and $r = \log_b x$):
$$(\log_b x)(\log_a b) = \log_a x$$

Divide both sides by $\log_a b$:
$$\log_b x = \frac{\log_a x}{\log_a b}$$

EXAMPLE 3.10 Approximate, to two-decimal places, the value of $\log_2 3$.

SOLUTION: $\log_2 3 = \dfrac{\log 3}{\log 2} \approx 1.58$ or $\log_2 3 = \dfrac{\ln 3}{\ln 2} \approx 1.58$

CHECK YOUR UNDERSTANDING 3.9

Use a calculator to approximate, to two decimal places, the value of:

(a) $\log_5 32$ (b) $\log_{\frac{1}{2}} 17$

Answers: (a) ≈ 2.15
(b) ≈ -4.09

EXERCISES

Exercises 1-9. Solve the given exponential equation.

1. $2^x = 8^{2x+1}$
2. $3^{2x-3} = \left(\frac{1}{3}\right)^x$
3. $3^{2x-3} = \left(\frac{1}{3}\right)^{x^2}$
4. $2^x = \left(\frac{1}{2}\right)^{2x+1}$
5. $\left(\frac{1}{2}\right)^{x-2} = 2^x$
6. $\left(\frac{1}{2}\right)^{x-2} = \left(\frac{1}{4}\right)^x$
7. $\frac{2^x}{4} = 4^{x+1}$
8. $5 \cdot 5^{x+1} = 25^x$
9. $2\left(\frac{1}{2}\right)^{x-2} = 4\left(\frac{1}{4}\right)^{2x}$

Exercise 10-14 Evaluate.

10. $\log_2\frac{1}{8}$
11. $\log_3 1$
12. $\log_{\frac{1}{2}} 32$
13. $\log_5 5^9$
14. $\log_2 2^{\log_3 9}$

Exercise 15-18 Use a calculator to approximate the value to two decimal places.

15. $\log_5 97$
16. $\log_3 5$
17. $\log_4 15$
18. $\log_5 11$

Exercise 19-24 Solve the given logarithmic equation.

19. $\log_3(x+2) = \log_3(3x+1)$
20. $\log(5x-2) = \log(x^2-2)$
21. $\ln(x^3-x) = \ln(3x)$
22. $\log_2(x-5) = \log_2(-x^2+1)$
23. $\log 3^{2x} = \log\frac{1}{27}$
24. $\log_2(x+1) = 1$

Exercise 25-29 Solve the given exponential equation. Express your answer in terms of the natural logarithmic function.

25. $3^{3x-5} = 12$
26. $3^{2x} = 4$
27. $4^{2x+1} = 3^{x-1}$
28. $2^{x+1} = 5^x$
29. $3^{2x+1} = 4^{x-1}$

Exercise 30-33 Solve the given logarithmic equation.

30. $\log_2 x - \log_2 5 = 4$
31. $\ln(2x+5) - \ln(2x-1) = -\ln 3$
32. $\log_2(x^2-16) - \log_2(x-4) = -2$
33. $\log_2(3x+8) - \log_2 x = 2$

§3. Exponential Growth and Decay

The following result, which is established in the calculus, reveals the exponential nature of the growth or decline of certain substances.

THEOREM 3.9

Exponential Growth/Decay Formula

If the rate of change of the amount of a substance is proportional to the amount present, then the amount present at a time t before or after an established initial time $(t = 0)$ is given by the formula:

$$A(t) = A_0(e^k)^t$$

where A_0 is the initial amount present, and k is a constant that depends on the particular substance.

Note that at $t = 0$, $A(t) = A_0$:

$$A(0) = A_0(e^k)^0 = A_0$$

From our knowledge of exponential functions, we know that the amount $A(t) = A_0(e^k)^t$ increases with time if $k > 0$ (since $e^k > 1$), and decreases with time if $k < 0$ (since $e^k < 1$). The former case is referred to as **exponential growth**, and the latter as **exponential decay**.

The constant k represents the rate of exponential growth or decay of the particular substance.

Population Growth

In an ideal environment, the population of living organisms (humans, rabbits, bacteria, etc.) increases exponentially. Consider for example, the bacteria cell process depicted in Figure 3.6. In a fixed period of time (called the **doubling time** of the organism), one cell divides into two; then, in the same period of time, the two become four—the four become eight—and so on:

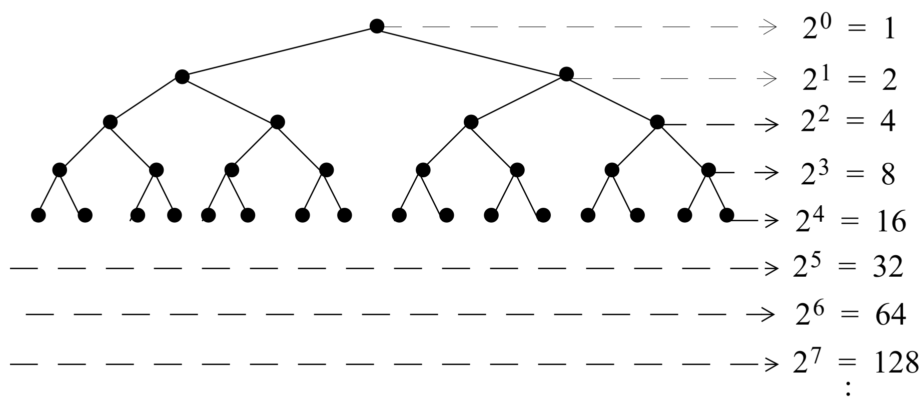


Figure 3.6

EXAMPLE 3.10 The population of Middletown was 12,500 in 2000 and it grew to 16,000 by 2020.

(a) What will the population be in 2050?

(b) What was the population in 1960?

SOLUTION: Establishing 2000 as time $t = 0$, we see, from Theorem 3.9, that the population at time t is given by:

$$A(t) = 12{,}500\,(e^k)^t \qquad (*)$$

To find the value of e^k, we use the fact that in 2020 $t = 2020 - 2000 = 20$ and $A(t) = 16{,}000$. Thus:

$$\begin{aligned} A(t) &= 12{,}500\,(e^k)^t \\ 16000 &= 12500(e^k)^{20} \\ (e^k)^{20} &= \frac{16000}{12500} = 1.28 \\ e^k &= (1.28)^{\frac{1}{20}} \end{aligned}$$

Substituting this value for e^k into (*), we have:

$$A(t) = 12500\left[(1.28)^{\frac{1}{20}}\right]^t = 12500(1.28)^{\frac{t}{20}} \qquad (**)$$

We are now able to answer (a) and (b).

(a) To approximate the population in the year 2050, we substitute $2050 - 2000 = 50$ for t in (**):

$$A(50) = 12500(1.28)^{\frac{50}{20}} \approx 23{,}170$$

(b) To approximate the population in the year 1960, we substitute $1960 - 2000 = -40$ for t in (**):

$$A(-40) = 12500(1.28)^{\frac{-40}{20}} \approx 7{,}629$$

EXAMPLE 3.11 The doubling time of E.coli bacteria is 20 minutes. If a culture of the bacteria contains one million cells, determine how long it will take before the culture increases to 9 million cells.

SOLUTION: Figure 3.7 reflects the fact that every 20 minutes the bacteria doubles. From that figure we can conclude that it will take a little longer than 60 minutes for there to be 9 million cells.

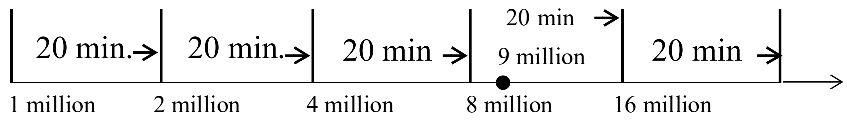


Figure 3.7

To more accurately determine the time required, we turn to the formula of Theorem 3.9, where $A(t)$ now denotes the number of cells (in millions) present at time t (in minutes):

From Theorem 3.9: $$A(t) = A_0(e^k)^t$$

Since $A(t) = 2A_0$ when $t = 20$: $$2A_0 = A_0(e^k)^{20}$$

$$2 = (e^k)^{20}$$

$$e^k = 2^{\frac{1}{20}}$$

The exponential growth formula for E.coli bacteria: $$A(t) = A_0\left(2^{\frac{1}{20}}\right)^t$$

Or, since $(x^r)^s = x^{rs}$: $$A(t) = A_0 \cdot 2^{\frac{t}{20}}$$

Since there are 1 (million) initially, $A_0 = 1$: $$A(t) = 1 \cdot 2^{\frac{t}{20}}$$

To determine how long it will take for there to be 9 million cells, set $A(t) = 9$ and solve for t: $$9 = 1 \cdot 2^{\frac{t}{20}}$$

$$2^{\frac{t}{20}} = 9$$

$$\ln 2^{\frac{t}{20}} = \ln 9$$

$\ln s^r = r \ln s$: $$\frac{t}{20}\ln 2 = \ln 9$$

$$t = \frac{20\ln 9}{\ln 2} \approx 63$$

We conclude that it will take approximately 63 minutes for the culture to increase from 1 million cells to 9 million cells. In other words, in an ideal situation, E.coli bacteria will increase by a factor of 9 approximately every 63 minutes.

CHECK YOUR UNDERSTANDING 3.10

The population of a town grows at a rate proportional to its population. The initial population of 500 increased by 15% in 9 years. How long will it take for the population to triple?

Answer: ≈ 70.75 years

RADIOACTIVE DECAY

By emitting alpha and beta particles and gamma rays, the radioactive mass of a substance decreases with time at a rate proportional to the amount present. Consequently, Theorem 3.9 applies.

A radioactive substance is represented in Figure 3.8. In a fixed period of time (called the **half-life** of the substance), one gram of the radioactive substance decreases to one-half gram; then, in the same period of time, only one-quarter gram remains radioactive, then one-eighth, and so on:

The half-life of radioactive substances varies greatly. While the half-life of uranium^{235} is 713 million years, that of polonium^{212} is less than a millionth of a second.

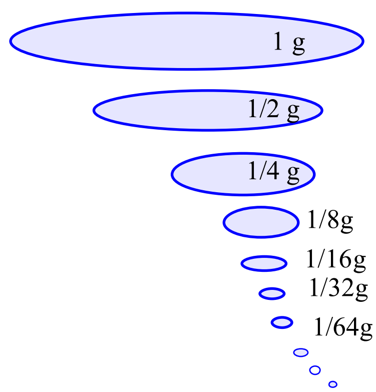


Figure 3.8

CARBON-14 DATING

Organic substances contain both carbon-14 and non-radioactive carbon in known proportions. A living organism absorbs no more carbon when it dies. The carbon-14 decays, thus changing the proportion of the two kinds of carbon in the organism.

By comparing the present proportion of carbon-14 with the assumed original proportion, one can determine how much of the original carbon-14 is present, and therefore how long the organism has been dead. The next example illustrates this method, called **carbon-14 dating**.

EXAMPLE 3.12 A skeleton is found to contain one-sixth of its original amount of carbon-14. How old is the skeleton, given that carbon-14 has a half-life of 5,730 years?

SOLUTION: Turning to Theorem 3.9, where $A(t)$ now denotes the amount of carbon-14 present at time t (in years), with $t = 0$ at the time of death, we have:

$$A(t) = A_0(e^k)^t$$

$A(t) = \frac{A_0}{2}$ when $t = 5730$:
$$\frac{1}{2}A_0 = A_0(e^k)^{5730}$$

Solve for e^k:
$$\frac{1}{2} = (e^k)^{5730}$$

$$e^k = \left(\tfrac{1}{2}\right)^{\frac{1}{5730}} = (2^{-1})^{\frac{1}{5730}} = 2^{-\frac{1}{5730}}$$

This brings us to the exponential decay formula for carbon-14:

$$A(t) = A_0 \cdot 2^{-\frac{t}{5730}} \qquad (*)$$

We are told that the skeleton contains one-sixth of its original amount of carbon-14, that is: $A(t) = \frac{A_0}{6}$. To find the skeleton's age, we substitute $\frac{A_0}{6}$ for $A(t)$ in $(*)$, and solve for t:

Divide both sides by A_0:
$$\frac{A_0}{6} = A_0 \cdot 2^{-\frac{t}{5730}}$$

$$2^{-\frac{t}{5730}} = \frac{1}{6}$$

$$\ln\left(2^{-\frac{t}{5730}}\right) = \ln\left(\frac{1}{6}\right) = \ln(6^{-1})$$

$\ln s^r = r \ln s$:
$$-\frac{t}{5730}\ln 2 = -\ln 6$$

$$t = \frac{5730 \ln 6}{\ln 2} \approx 14,812$$

We conclude that the skeleton is approximately 14,812 years old.

EXAMPLE 3.13

A certain radioactive substance loses $\frac{1}{3}$ of its original mass in four days. How long will it take for the substance to decay to $\frac{1}{10}$ of its original mass?

SOLUTION: We are told that after four days, $\frac{1}{3}$ of the substance decays, or equivalently, that the amount present 4 days later is $\frac{2}{3}A_0$. Turning to the exponential formula of Theorem 3.9, we have:

$$A(t) = A_0(e^k)^t \qquad (*)$$

Setting $t = 4$ and $A(t) = \frac{2}{3}A_0$, we solve for e^k:

$$\frac{2}{3}A_0 = A_0(e^k)^4$$

$$\frac{2}{3} = (e^k)^4$$

$$e^k = \left(\frac{2}{3}\right)^{\frac{1}{4}}$$

Substituting this value of e^k back into (*) yields the exponential decay formula for the radioactive substance under consideration:

$$A(t) = A_0\left(\frac{2}{3}\right)^{\frac{t}{4}}$$

To determine how long it will take for the substance to decay to $\frac{1}{10}$ of its original mass, we substitute $\frac{A_0}{10}$ for $A(t)$ in (*) and solve for t:

$$\frac{A_0}{10} = A_0\left(\frac{2}{3}\right)^{\frac{t}{4}}$$

$$\frac{1}{10} = \left(\frac{2}{3}\right)^{\frac{t}{4}}$$

$$\ln\frac{1}{10} = \ln\left(\frac{2}{3}\right)^{\frac{t}{4}}$$

$\ln s^r = r\ln s$: $$\ln 10^{-1} = \ln\left(\frac{2}{3}\right)^{\frac{t}{4}}$$

$$-\ln 10 = \frac{t}{4}\ln\frac{2}{3}$$

$$t = \frac{-4\ln 10}{\ln\frac{2}{3}} \approx 22.7$$

We conclude that it will take approximately 22.7 days for the substance to decay to $\frac{1}{10}$ of its original mass.

CHECK YOUR UNDERSTANDING 3.11

A certain radioactive substance decays exponentially in accordance with the formula $A(t) = A_0 e^{-\frac{t}{4}}$ where $A(t)$ is the number of grams present after t years.

(a) How many grams of the substance will there be in 2026, given that 35 grams were present in 2023?

(b) What is the half-life of the substance?

Answers: (a) ≈ 16.53 g
(b) $\ln 16 \approx 2.77$ years

EXERCISES

1. **(Bacteria Growth)** Suppose that a certain type of bacteria is subjected to the following law of exponential growth: $A(t) = A_0 e^{0.01t}$, where t is measured in minutes.
   (a) How long will it take for the bacteria to double?
   (b) If 2000 bacteria were present at 1 P.M. how many will be present at 1:25 P.M.?
   (c) If 2000 bacteria were present at 1 P.M. how many were present at 12:50 P.M.?

2. **(Ant Population)** On April 15 there were 1500 ants in a particular ant colony. On April 20th there were 2500. Assume that ants increase at a rated proportional to the number present.
   (a) How many ants will there be in April 20?
   (b) How many ants were there in April10?
   (c) How many hours will it take for a population of the ants to double?
   (d) How many hours will it take for a population of the ants to quadruple?

3. **(Town population)** The town of Pleasant had a population of 35,200 in 2015 which grew to 36,000 by 2020. Assume that the population increases at a rate proportional the current population.
   (a) What will be the population in 2030?
   (b) What was the population in 2005?
   (c) How many years will it take for a population at that town to double?

4. **(Radioactive Decay)** A radioactive substance decays exponentially in accordance with the formula $A(t) = A_0 e^{-\frac{t}{4}}$ where t is measured in years. Give that 35 grams of the substance was present in 2012, determine:
   (a) The number of grams that will be present in 2032.
   (b) The number of grams that were present in 2000.
   (c) The half-life of the substance.

5. **(Radioactive Decay)** A radioactive substance has a half-life of 74 years
   (a) How long will it take for the substance to decay to one-fourth of its initial mass?
   (b) If 300 grams are present initially, how many grams will remain after 35 years?

6. **(Iodine-131)** Iodine-131 is a dangerous residual of nuclear reactors. The half-life of iodine-131 is only 8 days. How long must waste material that contains 50 times the acceptable disposal lever of iodine-131 be stored before it can be properly disposed?

7. **(Strontium-90)** The artificial isotope Strontium-90 substance is a fission product recovered from nuclear reactors and released nuclear weapons. Because of its chemical similarity to calcium, long term exposure to the isotope results in substantial substitution of Strontium-90 for calcium in animals and humans resulting in a destruction of the blood-cell-forming bone marrow. How long will it take for the isotope to be reduced by 95% if it has a half-life of 28 years?

8. **(Dead Sea Scrolls)** Approximately 20% of the original carbon-14 remains in the Dead Sea Scrolls. How old are they, given that the half-life of carbon-14 is 5730 years.

9. **(Piltdown Skull)** The American chemist, Willard Frank Libby, was awarded the 1960 Nobel prize for his development of radioactive dating. His theory helped to expose the greatest scientific hoax ever perpetrated. Early in the twentieth century part of a human skull and an ape-like jawbone were found in a gravel pit in Piltdown, Sussex, England giving, birth to the so-called Piltdown Man, and to a fierce evolution controversy which lasted for many years.

   It was determined that the Piltdown skull fragment contained 93% of its original carbon-14, which has a half-life of 5730 years. How old was the skull at that time? Incidentally, the jaw was found to be that of a rather recently deceased orangutan.)

CHAPTER SUMMARY

COMPOSITION OF FUNCTIONS	The **composition** $(g\circ f)(x)$ is given by: $$(g\circ f)(x) = g(f(x))$$ **first** apply f and **then** apply g
ONE-TO-ONE FUNCTION	A function $f: X \to Y$ is said to be **one-to-one** if $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$
ONTO FUNCTION	A function f from a set X to a set Y is **onto** (or a surjection) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
BIJECTION	A function f that is both one-to-one and onto is said to be a **bijection.**
INVERSE FUNCTION	The **inverse** of a bijection $f: X \to Y$, is the function $f^{-1}: Y \to X$ given by: $$f^{-1}(y) = x \text{ where } f(x) = y$$ If f is a bijection, then so is f^{-1}. Moreover: $$(f^{-1}\circ f)(x) = x \text{ and } (f\circ f^{-1})(x) = x$$
EXPONENTIAL FUNCTION	Let b be a positive number other than 1. The function f given by: $$f(x) = b^x$$ is said to be the **exponential function with base *b***. Every exponential function is one-to-one.
LOGARITHMIC FUNCTIONS	For any positive number b, other than 1, and any $x > 0$: $$\log_b x = y \text{ if and only if } b^y = x$$ The **common logarithm function:** $\log_{10}x$, or simply $\log x$. The **natural logarithm**: $\log_e x$, or simply $\ln x$
EXPONENTIAL AND LOGARITHMIC EQUATIONS	$b^{\text{expression}_1} = b^{\text{expression}_2}$ if and only if $\text{expression}_1 = \text{expression}_2$ $\log_b(\text{expression}_1) = \log_b(\text{expression}_2)$ if and only if $\text{expression}_1 = \text{expression}_2$

LOGARITHMIC THEOREMS	$\log_b b^x = x \qquad b^{\log_b x} = x$ $\log_b(st) = \log_b s + \log_b t \quad \log_b\left(\frac{s}{t}\right) = \log_b s - \log_b t \quad \log_b s^r = r\log_b s$ Change of base formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$
EXPONENTIAL GROWTH/DECAY FORMULA	If the rate of change of the amount of a substance is proportional to the amount present, then the amount present at a time t before or after an established initial time $(t = 0)$ is given by the formula: $A(t) = A_0(e^k)^t$ where A_0 is the initial amount present, and k is a constant that depends on the particular substance.

CHAPTER 4
Differential Calculus

§1. LIMITS

At the very heart of the calculus is the concept of a limit. Here is one of them:

$$\lim_{x \to 2}(3x+5)$$

It is read: ***The limit as x approaches 2 of the function*** $\mathbf{3x+5}$.

It represents: The number $3x+5$ approaches, as x approaches 2.

Clearly, as x gets closer and closer to 2, $3x$ will get closer and closer to 6, and $3x+5$ will consequently approach 11. We therefore write:

$$\lim_{x \to 2}(3x+5) = 11$$

By the same token,

$$\lim_{x \to 3}\frac{x}{x^2+5} = \frac{3}{14}$$

(as x approaches 3, the numerator approaches 3, and the denominator approaches $3^2+5 = 14$.)

CHECK YOUR UNDERSTANDING 4.1

Determine the given limit.

(a) $\lim_{x \to -1}(4x^2+x)$ (b) $\lim_{x \to 2}\frac{x+3}{x+2}$ (c) $\lim_{x \to 3}[x(3x^2+1)]$

Answers: (a) 3 (b) $\frac{5}{4}$ (c) 84

At this point, you might be wondering what all the fuss is about. Couldn't we simply plug the relevant number into the expression to arrive at the limit. Yes, but consider:

$$\lim_{x \to 2}\frac{x^2+x-6}{x^2-4}$$

Attempting to substitute 2 for x in the above expression brings us to the **meaningless** form "$\frac{0}{0}$." However, if you let the value of x get closer and closer to 2; say $x = 1.99$, $x = 2.001$, $x = 1.9999$, $x = 2.00001$, and so on, you will find that $\frac{x^2+x-6}{x^2-4}$ will indeed approach a particular number. To find that number, we turn to a related algebra problem:

You can use your calculator to see what happens, but at some point, say for $x = 1.99999999$, you may receive an error message, since most calculators think that $1.99999999 = 2$. Poor things.

Simplify: $\dfrac{x^2+x-6}{x^2-4}$.

Solution: $\dfrac{x^2+x-6}{x^2-4} = \dfrac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \dfrac{x+3}{x+2}$, but:

the above is not totally correct, for one should really write:

We remind you that one cannot "cancel a 0."

$$\frac{x^2+x-6}{x^2-4} = \frac{(x+3)(x-2)}{(x+2)(x-2)} = \frac{x+3}{x+2} \text{ if } x \neq 2$$

conditional equality

In the limit process, however, the variable x **approaches 2** — it can get as close to 2 as you wish but it is **<u>never</u> equal to 2**. Thus:

$$\lim_{x \to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x \to 2} \frac{(x+3)(x-2)}{(x+2)(x-2)} = \lim_{x \to 2} \frac{x+3}{x+2} = \frac{5}{4}$$

not conditional

We've encountered two types of limits:

DETERMINED FORM

Those like $\lim\limits_{x \to 3} \dfrac{x}{x^2+5}$ and $\lim\limits_{x \to 2} \dfrac{x+3}{x+2}$, which can be determined by simply plugging in the indicated x-value.
(such limits are of **determined form**)

UNDETERMINED FORM

And a more interesting type, like $\lim\limits_{x \to 2} \dfrac{x^2+x-6}{x^2-4}$,
which cannot be simply evaluated at $x = 2$.
(such limits are of **undetermined form**)

EXAMPLE 4.1

Evaluate:

(a) $\lim\limits_{x \to 2} \dfrac{x^3-2x^2-3x}{x^2+2x-15}$ (b) $\lim\limits_{x \to 3} \dfrac{x^3-2x^2-3x}{x^2+2x-15}$

(c) $\lim\limits_{x \to -2} \dfrac{\frac{1}{2}+\frac{1}{x}}{x+2}$

SOLUTION: (a) Challenging the denominator at $x = 2$ we find that it does **not** turn out to be 0: $2^2+2\cdot 2-15 = -7$. So, being faced with a determined form, we simply plug in 2 in the expression to arrive at the answer:

$$\lim_{x \to 2} \frac{x^3-2x^2-3x}{x^2+2x-15} = \frac{2^3-2\cdot 2^2-3\cdot 2}{2^2+2\cdot 2-15} = \frac{-6}{-7} = \frac{6}{7}$$

(b) We are confronted with an undetermined form, which we now transform into a determined form:

A useful hint: The only way the denominator and numerator can be zero, is if each contains $(x-3)$ as a factor, and you can use this fact to help you factor the polynomials.

$$\lim_{x\to 3}\frac{x^3-2x^2-3x}{x^2+2x-15} = \lim_{x\to 3}\frac{x(x^2-2x-3)}{(x-3)(x+5)} = \lim_{x\to 3}\frac{x\cancel{(x-3)}(x+1)}{\cancel{(x-3)}(x+5)}$$

$$= \underbrace{\lim_{x\to 3}\frac{x(x+1)}{(x+5)}}_{\text{determined form}} = \underbrace{\frac{3(3+1)}{3+5}}_{\text{just plug in}} = \frac{12}{8} = \frac{3}{2}$$

Note that we carry the limit symbol until the limit is performed. An analogous situation:

$3\cdot 2+4 = 6+4 = 10$

you write the "+" until the sum is performed

(c) We cannot simply substitute -2 for x (why not?). So, we do what needs to be done:

$$\lim_{x\to -2}\frac{\frac{1}{2}+\frac{1}{x}}{x+2} = \lim_{x\to -2}\frac{\frac{x+2}{2x}}{x+2} = \lim_{x\to -2}\left(\frac{x+2}{2x}\cdot\frac{1}{x+2}\right)$$

$$= \lim_{x\to -2}\frac{1}{2x} \underset{\text{see margin}}{=} \frac{1}{2(-2)} = -\frac{1}{4}$$

CHECK YOUR UNDERSTANDING 4.2

Determine the given limit.

(a) $\lim_{x\to 1}\frac{x^2+3x-4}{x^2+1}$ (b) $\lim_{x\to 1}\frac{x^2+3x-4}{x^2-1}$ (c) $\lim_{x\to 1}\frac{x^2-1}{\frac{x}{2}-\frac{1}{2}}$

Answers: (a) 0 (b) $\frac{5}{2}$ (c) 4

A LIMIT NEED NOT EXIST

Does $\lim_{x\to 2}\frac{x+2}{x-2}$ exist? We certainly cannot substitute 2 for x in the expression, for that would yield a zero in the denominator. That, in and of itself, is not necessarily a problem (See Example 4.1). The problem is that, as x approaches 2, the denominator of $\frac{x+2}{x-2}$ gets closer and closer to 0 while its numerator tends to 4. The net result is that the quotient will just keep getting bigger and bigger: the limit does not exist (DNE).

Consider the function $f(x) = \begin{cases} 3x+1 & \text{if } x<2 \\ x^2 & \text{if } x>2 \end{cases}$. Does $\lim_{x\to 2} f(x)$ exist? No. Why not? Because:

As x tends to 2 from the left, the top rule is in effect, and $f(x)$ approaches $3\cdot 2+1 = 7$. On the other hand, if x approaches 2 from the right, then the bottom rule is in effect, and $f(x)$ tends to $2^2 = 4$.

The notation

$$f(x) = \begin{cases} 3x+1 & \text{if } x<2 \\ x^2 & \text{if } x>2 \end{cases}$$

is very transparent. For example:

$f(1) = 3\cdot 1+1 = 4$ (since $1<2$)

while

$f(3) = 3^2 = 9$ (since $3>2$)

GEOMETRICAL INTERPRETATION OF THE LIMIT CONCEPT

Consider the functions, f, g, h, and k in Figure 4.1.

The solid dot above 3 in (a) depicts the value of the function at 3: $f(3) = 4$. Similarly: $g(3) = 7$ and $h(3) = 7$.

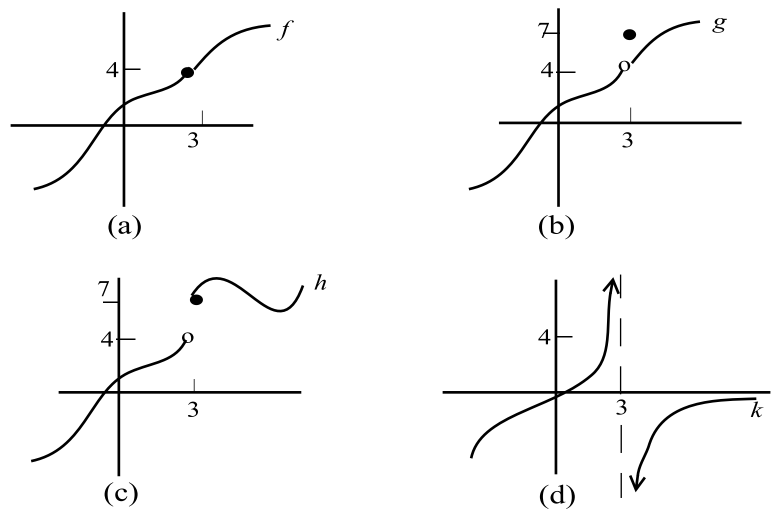


Figure 4.1

Looking at the function f in (a), we see that as x approaches 3, from **either the left or the right**, the function values (y-values) approach the number 4. Thus: $\lim\limits_{x \to 3} f(x) = 4$. The function g in (b) differs from f only at $x = 3$, where it has a "hiccup." But that anomaly has absolutely no effect on the limit, since the limit does not care about what happens at 3 — it only cares about what happens as x **approaches** 3. The function h in (c) does **not** have a limit at $x = 3$, since as x approaches 3 from the left the function values approach 4 while they approach 7 as x approaches from the right. The function in (d) also does not have a limit at 3, as the function values get larger and larger as x approaches 3.

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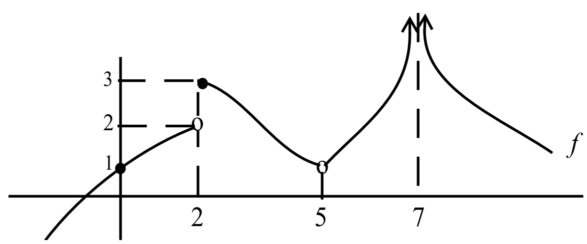
CHECK YOUR UNDERSTANDING 4.3

Referring to the graph of the function f below, determine if the given limit exists, and if it does, indicate its value.



(a) $\lim\limits_{x \to 0} f(x)$ (b) $\lim\limits_{x \to 2} f(x)$ (c) $\lim\limits_{x \to 5} f(x)$ (d) $\lim\limits_{x \to 7} f(x)$

Answers: (a) 1 (b) DNE (c) 1 (d) DNE.
Note: It is also acceptable to write: $\lim\limits_{x \to 7} f(x) = \infty$ since the function values get larger and larger in the positive direction as x approaches 7 from either side.

PROPERTIES OF LIMITS

The following theorem formalizes results that you have been taking for granted all along.

It is certainly "believable," for example, that if $f(x)$ approaches L and if $g(x)$ approaching M as x tends to c, then $f(x)+g(x)$ must surely tend to $L+M$ as x approaches c.
Without a rigorous definition of the limit concept, however, we are not in a position to prove any of these statements.

THEOREM 4.1
LIMIT THEOREMS

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then:

(i) $\lim_{x \to c} [f(x) + g(x)] = L + M$

(ii) $\lim_{x \to c} [f(x) - g(x)] = L - M$

(iii) $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$

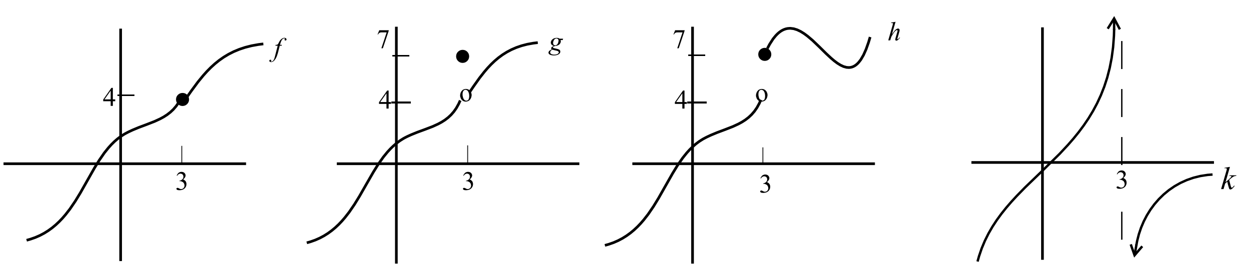
(iv) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

(v) $\lim_{x \to c} [af(x)] = aL$

IN WORDS:	
	(i) The limit of a sum is the sum of the limits.
	(ii) The limit of a difference is the difference of the limits.
	(iii) The limit of a product is the product of the limits.
	(iv) The limit of a quotient is the quotient of the limits (providing the limit of the denominator is not zero).
	(v) The limit of a constant times a function is the constant times the limit of the function.

CONTINUITY

Let's reconsider the functions of Figure 4.2:



$\lim_{x \to 3} f(x) = 4$
(a)

$\lim_{x \to 3} g(x) = 4$
(b)

$\lim_{x \to 3} h(x)$ Does Not Exist
(c)

$\lim_{x \to 3} k(x)$ Does Not Exist
(d)

Figure 4.2

While the functions in (c) and (d) do not have limits as x approaches 3, both the functions in (a) and (b) do. The limit, being oblivious of what happens at 3, cannot tell you that the function g in (b) behaves in a somewhat peculiar fashion at $x = 3$. Another concept, one more sensitive than that of the limit, is called for:

A **continuous function** is a function that is continuous at every point in its domain.

DEFINITION 4.1
CONTINUITY

A function f is **continuous** at c if:

$$\lim_{x \to c} f(x) = f(c)$$

A function that is not continuous at c is said to be **discontinuous** at that point.

In other words, for a function f to be continuous at c, three things must happen:

(1) f must be defined at c
(2) $\lim_{x \to c} f(x)$ must exist
(3) The limit must equal $f(c)$

If a function f has a limit at c and that limit is not equal to $f(c)$ then f is said to have a **removable discontinuity** at c. [The function in Figure 4.2(b) has a removable discontinuity at 3].

If a function f fails even to have a limit at c, because the two one-sided limits exist but are not equal, then f is said to have a **jump discontinuity** at c. [The function in Figure 4.2(c) has a jump discontinuity at 3].

Returning to Figure 4.2, we see that:

The function f in (a) is continuous at 3.
(limit equals 4, and $f(3) = 4$)
The function g in (b) is discontinuous at 3.
(limit equals 4, but $g(3) = 7$)
The functions in (c) and (d) are also discontinuous at 3
(limit does not even exist)

CHECK YOUR UNDERSTANDING 4.4

Indicate whether or not the given function f is continuous at $x = 3$.

(a) $f(x) = \dfrac{x^2 - 9}{x - 3}$ (b) $f(x) = \dfrac{x - 3}{x + 3}$

Answers:
(a) Not continuous.
(b Continuous.

EXERCISES

Exercises 1-27. Evaluate the given limit.

1. $\lim_{x \to 3} (x^2 - 5)$

2. $\lim_{x \to 0} (x^2 - 5)$

3. $\lim_{x \to 3} \frac{x^2 - 5}{x + 3}$

4. $\lim_{x \to 3} \frac{x^2 - 9}{x + 3}$

5. $\lim_{x \to 5} \frac{x - 5}{x + 5}$

6. $\lim_{x \to 5} \frac{x^2 - 5}{x - 5}$

7. $\lim_{x \to 5} \frac{x^3 - 25x}{x - 5}$

8. $\lim_{x \to 5} \frac{x^2 - 5}{x^2 - 3x - 10}$

9. $\lim_{x \to 5} \frac{x^2 - 25}{x^2 - 4x - 5}$

10. $\lim_{x \to 5} \frac{x^2 - 4x - 5}{x^2 - 25}$

11. $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$

12. $\lim_{x \to -2} \frac{x^2 - 4}{3x^2 + 6x + 2}$

13. $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - x - 6}$

14. $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 2x - 3}$

15. $\lim_{x \to -3} \frac{x^2 + 4x + 3}{x^2 + 3x}$

16. $\lim_{x \to 4} \frac{x^2 - 4x - 4}{x^2 - x - 12}$

17. $\lim_{x \to 4} \frac{x^2 - 8x + 16}{x^2 - 4x}$

18. $\lim_{x \to -2} \frac{x^2 + 4x + 4}{x^2 + 3x + 2}$

19. $\lim_{x \to -1} \frac{2x^3 + 5x^2 + 3x}{x^2 - 3x - 4}$

20. $\lim_{x \to 1} \frac{x^4 - 1}{x^2 - 1}$

21. $\lim_{x \to -2} \frac{x^2 + x - 2}{x^4 - 16}$

22. $\lim_{x \to -\sqrt{2}} \frac{x^2 - 2}{x + \sqrt{2}}$

23. $\lim_{x \to 0} \frac{x^3 + 2x^2 - 3x}{x^3 - 2x^2 - 15x}$

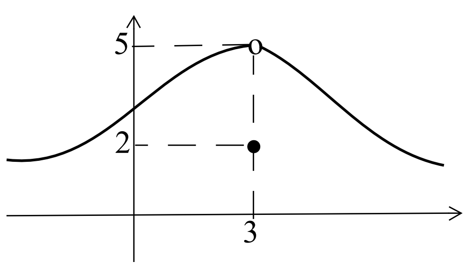
24. $\lim_{x \to -3} \frac{x^3 + 2x^2 - 3x}{x^3 - 2x^2 - 15x}$

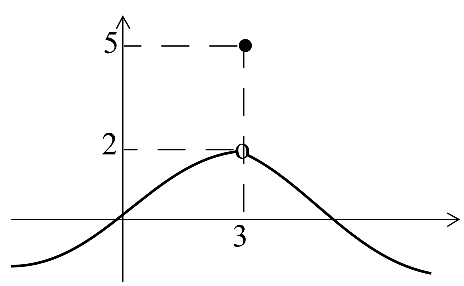
25. $\lim_{x \to -1} \frac{\frac{1}{x} + 1}{x^2 - 1}$

26. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2 + x} \right)$

27. $\lim_{x \to 2} \frac{\frac{x^2}{x - 1} - 4}{\frac{1}{x + 2} - \frac{1}{4}}$

Exercises 28-31. Referring to the graph of the function *f*, determine if $\lim_{x \to 3} f(x)$ exists. If it does, indicate its value. Is the function continuous at 3?

28. 

29. 

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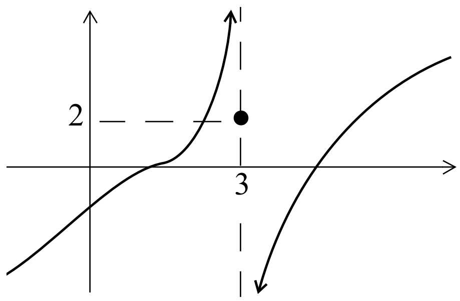

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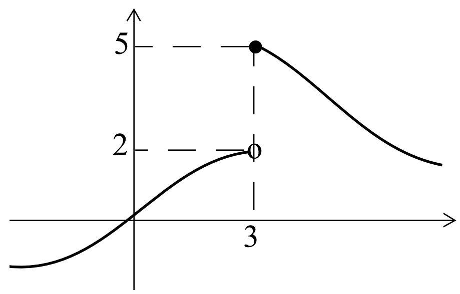


Exercises 32-35. (Piecewise-Defined Functions) Determine if $\lim_{x \to 2} f(x)$ exists. Is the function continuous at 2?

32. $f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$

33. $f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$

34. $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$

35. $f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ 2 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$

§2. Tangent Lines and the Derivative

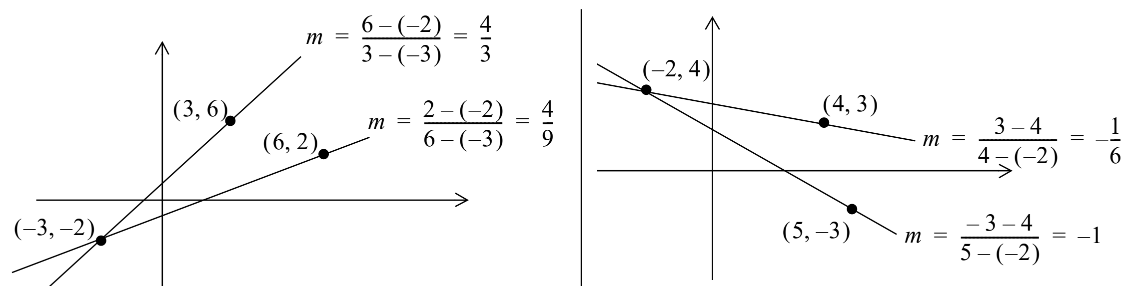
The following definition attributes a measure of "steepness" to any non-vertical line in the plane.

DEFINITION 4.2 Slope of a line

For any nonvertical line L and any two distinct points (x_1,y_1) and (x_2,y_2) on L, we define the **slope** of L to be the number m given by:

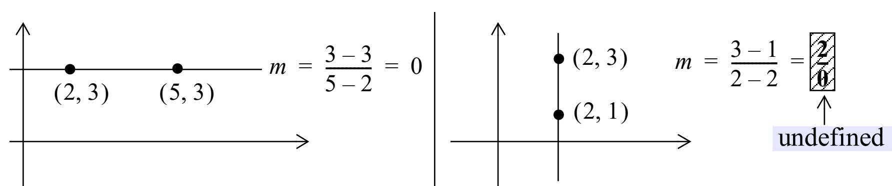
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

For Example:



The steeper the climb, the more positive the slope. **The steeper the fall, the more negative the slope.**

Note:



The slope of a horizontal line is 0 **No slope is associated with a vertical line**

***Y*-INTERCEPT**

Consider the line L of slope m in Figure 4.3. Being nonvertical, it must intersect the y-axis at some point $(0, b)$. The number b, where the line intersects the y-axis, is called the ***y*-intercept** of the line.

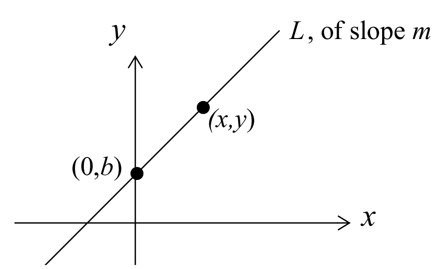


Figure 4.3

Now let (x, y) be any point on L other than $(0, b)$. From the fact that any two distinct points on a line determine its slope, we have:

$$m = \frac{y-b}{x-0}$$

$$m = \frac{y-b}{x}$$

multiply both sides of the equation by x:

$$y - b = mx$$

add b to both sides of equation: $y = mx + b$

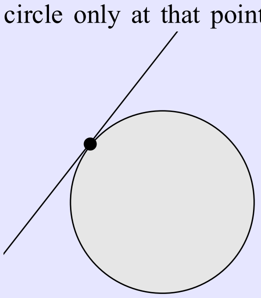
Direct substitution shows that the above equation also holds at the point $(0, b)$; thus:

The Point-Slope form is introduced in the exercises.

THEOREM 4.2 Slope-Intercept Equation of a line A point (x, y) is on the line of slope m and y-intercept b if and only if its coordinates satisfy the equation $y = mx + b$.

TANGENT LINES

The tangent line to a point on the circle is that line which touches the circle only at that point:



This will not do for more general curves. Line 2 in Figure 4.5, for example, touches the curve at more than one point.

Consider the two lines of Figure 4.3. Which do you feel better represents the tangent line to the curve at the point $(c, f(c))$? Chances are that you chose line 2, and might have based that decision on the concept of a tangent line to a circle (see margin). .

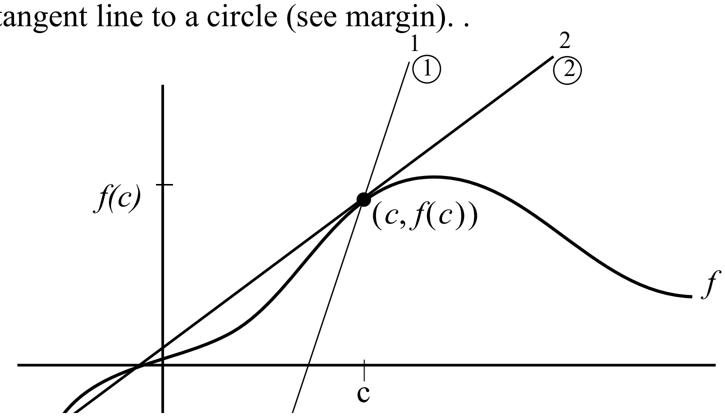
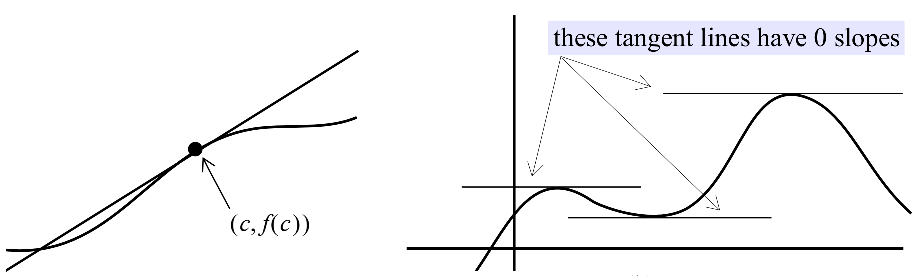


Figure 4.3

Our goal is to define the notion of a tangent line in such a way that it conforms with our predisposed notion of tangency. But why bother? For one thing, near the point of interest, a tangent line offers a nice approximation for the given function [see Figure 4.4(a)]. For another, tangent lines can be used to find where maxima and minima occur [see Figure 4.4(b)] and this enables one to solve a host of practical problems.



(a) (b)

Figure 4.4

To begin with, we demand that the tangent line at $(c, f(c))$ in Figure 4.4(a) passes through that point (a reasonable demand). Consequently, since a (non-vertical) line is determined by its slope and any point on the line, it suffices to define the slope of the tangent line we seek.

But there is a problem. We need 2 points to find the slope of a line, and here we have but one: $(c, f(c))$. So, consider Figure 4.5 where the would-be tangent line T is represented in dotted form (it really doesn't exist, until we define it). A solid line W_h passing through the two points on the curve $(c, f(c))$ and $(c+h, f(c+h))$ also appears.

We called the line W_h, to remind us that we got it by moving h units from c along the x-axis. (If h were negative, then $c+h$ would lie to the left of c.)

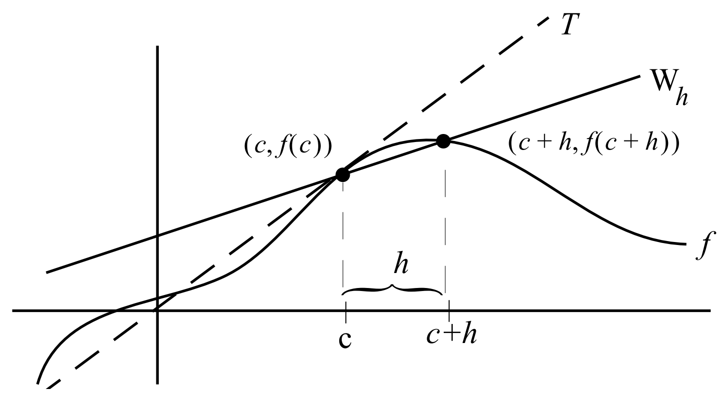


Figure 4.5

The line W_h is wrong — it is not the line we seek. But we can do something with W_h which we were not able to do with our phantom line T; we can calculate its slope:

$$m = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h} \qquad \left(\frac{\text{change in } y}{\text{change in } x} : \frac{\Delta y}{\Delta x}\right)$$

see margin

The Greek letter Δ, pronounced "delta," is often used to denote "***a change in***".

It is easy to see that those lines W_h will pivot closer and closer to T as h gets smaller and smaller! It is therefore totally natural to define:

$$\text{slope of } T = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

The above limit is called the derivative of f at c, and is denoted by $f'(c)$:

DEFINITION 4.3
DERIVATIVE OF A FUNCTION AT A POINT

The **derivative of a function *f* at *c*** is the number $f'(c)$ given by:

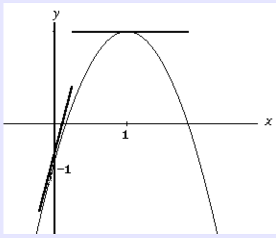
$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

(Providing, of course, that the above limit exists.)

The graph of:

$f(x) = -3x^2 + 6x - 1$

appears below.



From the graph, we can anticipate that $f'(0)$ will be a positive number (tangent line climbs, and rather rapidly), and that $f'(1) = 0$ (why?).

EXAMPLE 4.2

Determine $f'(0)$ and $f'(1)$ for the function:

$$f(x) = -3x^2 + 6x - 1$$

SOLUTION: Using Definition 4.3 with $c = 0$, we have:

$c = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

undetermined form

$$= \lim_{h \to 0} \frac{(-3h^2 + 6h - 1) - (-1)}{h} = \lim_{h \to 0} \frac{-3h^2 + 6h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-3h+6)}{\cancel{h}} = \lim_{h \to 0} (-3h+6) = 6$$

determined form, just plug in 0 for h

Repeating the process with $c = 1$, we have:

It is important that you are able to deal with $f(1+h)$. Once you do that correctly the rest is rather easy: just manipulate the numerator into a form which enables you to cancel that bothersome h in the denominator of the expression [go from an undetermined form to a determined form].

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[-3(1+h)^2 + 6(1+h) - 1] - 2}{h}$$

$$= \lim_{h \to 0} \frac{[-3(1 + 2h + h^2) + 6 + 6h - 1] - 2}{h}$$

$$= \lim_{h \to 0} \frac{-3 - 6h - 3h^2 + 6 + 6h - 1 - 2}{h} = \lim_{h \to 0} \frac{-3h^2}{h}$$

$$= \lim_{h \to 0} -3h = 0$$

CHECK YOUR UNDERSTANDING 4.5

Determine $f'(2)$ for the function $f(x) = -3x^2 + 6x - 1$ of Example 4.2.

Answer: -6

We did some work in Example 4.2 to find $f'(0)$, and repeated the same process to find $f'(1)$. We could have saved some time by finding the derivative function, $f'(x)$, and then evaluating it at 0 and at 1; where:

Note that the $f'(c)$ of Definition 4.2 is a **number**: the slope of the tangent line at $(c, f(c))$. On the other hand, $f'(x)$ is a **function** whose value at x is the slope of the tangent line at the point $(x, f(x))$.

DEFINITION 4.4 DERIVATIVE FUNCTION

The **derivative of a function f** is the function $f'(x)$ given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(providing of course, that the above limit exists)

EXAMPLE 4.3 Find the derivative, $f'(x)$, of the function $f(x) = -3x^2+6x-1$, and then use it to determine $f'(0)$, $f'(1)$, and $f'(-1)$.

Note that we are just duplicating the process of Example 4.2, but with "x" instead of "0" or "1."

SOLUTION: Turning to Definition 4.3, we have:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[-3(x+h)^2+6(x+h)-1]-(-3x^2+6x-1)}{h}$$

$$= \lim_{h\to 0}\frac{[-3(x^2+2xh+h^2)+6x+6h-1]+3x^2-6x+1}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{-3x^2}-6xh-3h^2+\cancel{6x}+6h-\cancel{1}+\cancel{3x^2}-\cancel{6x}+\cancel{1}}{h}$$

$$= \lim_{h\to 0}\frac{-6xh-3h^2+6h}{h} = \lim_{h\to 0}\frac{h(-6x-3h+6)}{h}$$

$$= \lim_{h\to 0}(-6x-3h+6) = -6x+6$$

Derivatives are limits of undetermined form. If the derivative exists, then the h in the denominator **has** to eventually cancel with an h-factor in the numerator. For this to happen, all terms in the numerator that do not "contain an h" must drop out.

We have shown that the derivative of $f(x) = -3x^2+6x-1$ is:

$$f'(x) = -6x+6$$

In particular:
$$f'(0) = -6\cdot 0+6 = 6,$$
$$f'(1) = -6\cdot 1+6 = 0,$$
$$f'(-1) = -6(-1)+6 = 12.$$

CHECK YOUR UNDERSTANDING 4.6

Find the derivative, $f'(x)$, of the function $f(x) = x^2+x+1$, and use it to determine $f'(0)$, $f'(1)$, and $f'(-1)$.

Answer: $f'(x) = 2x+1$
$f'(0) = 1$, $f'(1) = 3$,
$f'(-1) = -1$

EXAMPLE 4.4 Determine the tangent line to the graph of the function $f(x) = -3x^2+6x-1$, at $x = 2$.

SOLUTION: Whenever you see the word "line" you should think of:

$$y = \underbrace{m}_{\text{slope}}x + \underbrace{b}_{y\text{-intercept}}$$

As for the slope of the tangent line at $x = 2$:

$$y = \underbrace{m}_{f'(2)}x + b$$

(slope of tangent line at $x = 2$)

In Example 4.3, we showed that the derivative of the given function is $f'(x) = -6x + 6$. Consequently, the slope of our tangent line is $f'(2) = -6 \cdot 2 + 6 = -6$; bringing us to:

$$y = -6x + b \qquad (*)$$

The procedure for finding b has not changed. We need to know a point on the line, and we do:

> The tangent line must pass through the point on the curve whose x-coordinate is 2, namely, the point:

$$(2, f(2)) = (2, -1)$$

$$f(2) = -3 \cdot 2^2 + 6 \cdot 2 - 1 = -1$$

Since the point $(2, -1)$ satisfies (*):

$$y = -6x + b$$
$$-1 = -6 \cdot 2 + b$$
$$b = 11$$

Tangent line: $\boxed{y = -6x + 11}$

CHECK YOUR UNDERSTANDING 4.7

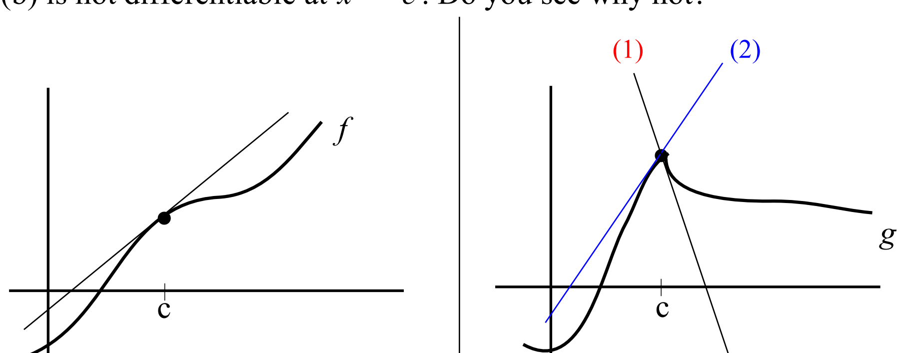
Find the tangent line to the graph of the function $f(x) = x^2 + x + 1$ at $x = 2$.

Answer: $y = 5x - 3$

Geometrical Insights into the Derivative

Consider the two graphs of Figure 4.6. Geometrically speaking, you can see that the function f in (a) has a (positive) derivative at $x = c$ (tangent line exists and has positive slope). In contrast, the function g in (b) is not differentiable at $x = c$. Do you see why not?

If a function is differentiable at c, then the graph has to be "smooth" at that point [as in Figure 4.6(a)]. If the graph "changes direction abruptly" at that point [as in Figure 4.6(b)], then the function will not be differentiable at c.



Differentiable at c
(a)

Not differentiable at c
(b)

Figure 4.6

Because: Lines are notoriously straight, and the graph of g has a sharp bend at c [the "would-be tangent line (1)" is of negative slope, while the "would-be tangent line **(2)**" is of positive slope]. Since no line can approximate the graph of g at c, there cannot be a tangent line at c [in other words: $f'(c)$ does not exist].

CONTINUITY AND THE DERIVATIVE

The following result, which you are invited to establish in the exercises, says that differentiability is a stronger condition than continuity:

To put it another way: if a function is not continuous at c, then it is not differentiable at c.

THEOREM 4.3 If a function f is differentiable at c, then f is continuous at c.

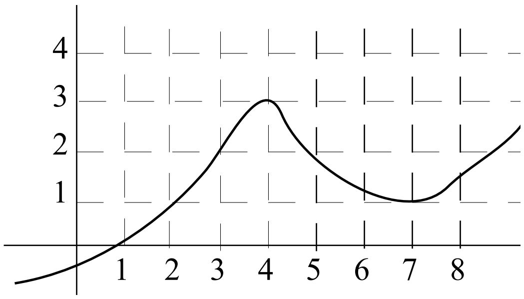
Here is the pecking order of the concepts of limit, continuity, and differentiability:

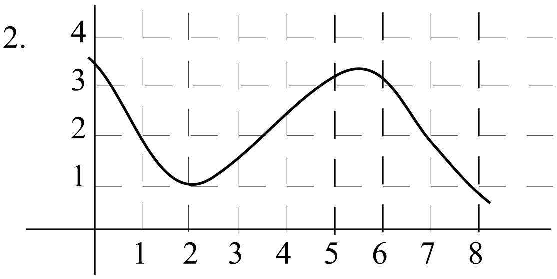
$$\text{Differentiability} \Rightarrow \text{Continuity} \Rightarrow \text{Limit Exists}$$

read: *implies*

EXERCISES

Exercises 1-2. Sketch a tangent line to the graph of the function f, at the indicated point and then estimate the value of $f'(2)$, $f'(4)$, and $f'(7)$.

1. 

2. 

Exercises 3-11. Use Definition 2.5 to determine $f'(2)$ for the given function.

3. $f(x) = 5x + 1$
4. $f(x) = -x + 4$
5. $f(x) = 4x^2$
6. $f(x) = 3x^2 + x$
7. $f(x) = -x^2 + 3x - 1$
8. $f(x) = -3x^2 + 3x - 500$
9. $f(x) = 55$
10. $f(x) = x^3$
11. $f(x) = x^3 + x + 1$

Exercises 12-20. Use Definition 2.6 to determine $f'(x)$ for the given function.

12. $f(x) = x$
13. $f(x) = -5x - 4000$
14. $f(x) = 7x + 500$
15. $f(x) = 3x^2$
16. $f(x) = x^2 + 2x$
17. $f(x) = -2x^2 + x - 2$
18. $f(x) = 5$
19. $f(x) = 101$
20. $f(x) = -x^3 - 3$

Exercises 21-28. Find the equation of the tangent line to the graph of the given function at the indicated point.

21. $f(x) = -3x$ at $x = 5$
22. $f(x) = -3x + 5$ at $x = 500$
23. $f(x) = x^2 + 2x$ at $x = 0$
24. $f(x) = x^2 + 2x$ at $x = 1$
25. $f(x) = x^2 + 2x$ at $x = 2$
26. $f(x) = 2x^2 + x + 1$ at $x = 1$
27. $f(x) = 11$ at $x = -9$
28. $f(x) = 11$ at $x = 99$

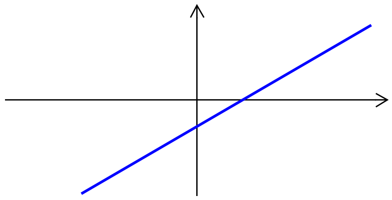
29. Give a geometrical argument involving tangent lines to suggest that the derivative of the function $f(x) = x$ equals 1, and then establish the result via Definition 2.6.

30. Give a geometrical argument involving tangent lines to suggest that the derivative of the function $f(x) = c$, where c is a constant, equals 0; and then establish the result via Definition 2.6.

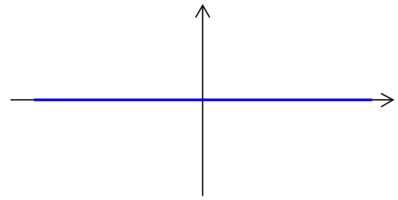
31. Give a geometrical argument involving tangent lines to suggest that the derivative of the linear function $f(x) = mx + b$ equals m, and then establish the result via Definition 2.6.

32. Pair off each function [A] through [F] with its corresponding derivative function [1] through [6]. Suggestion: Think "slope of tangent lines." Where the tangent line is horizontal, for example, the derivative at that point will be zero; where the tangent line has a positive slope, the derivative will be positive (the greater the slope, the greater the value of the derivative), and so on

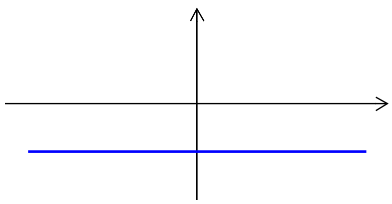
[A]



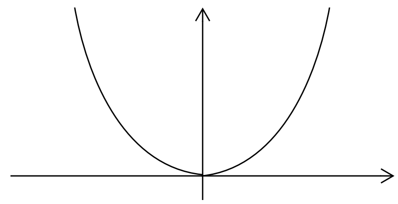
[1]



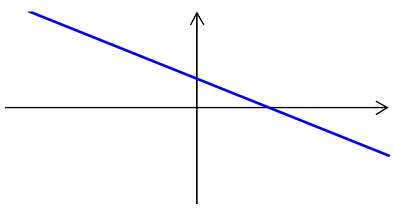
[B]



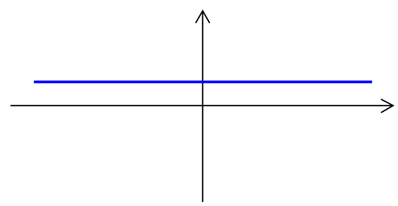
[2]



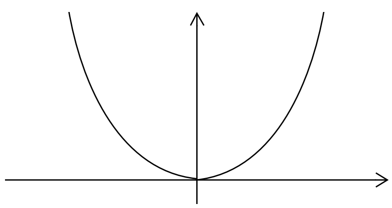
[C]



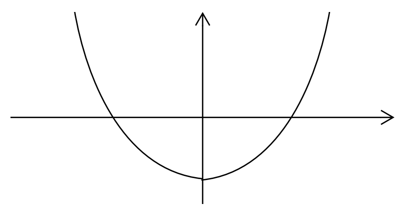
[3]



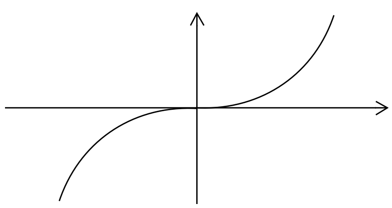
[D]



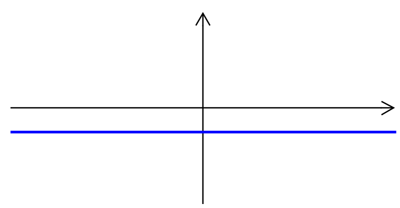
[4]



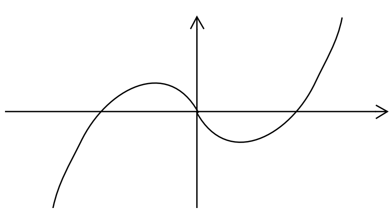
[E]



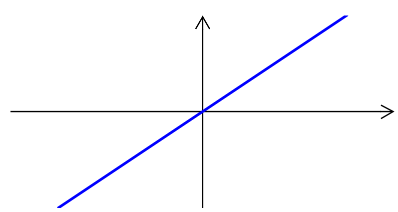
[5]



[F]

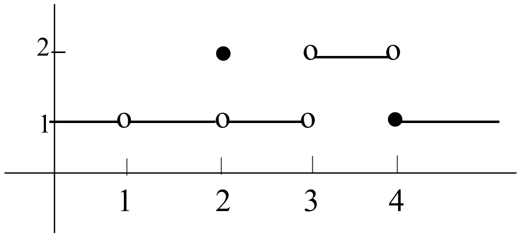


[6]

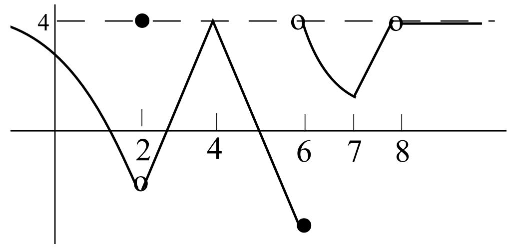


Exercises 33-34. Referring to the given graph of the function f, indicate where the function: fails to have a limit, is discontinuous (not continuous), and where it is not differentiable.

33.

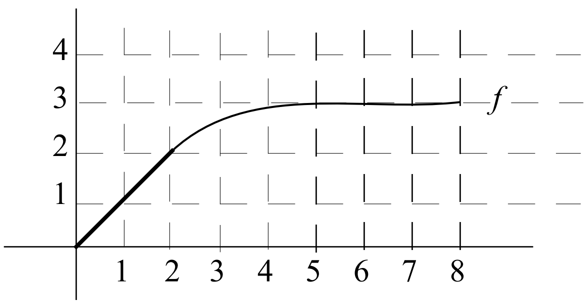


34.

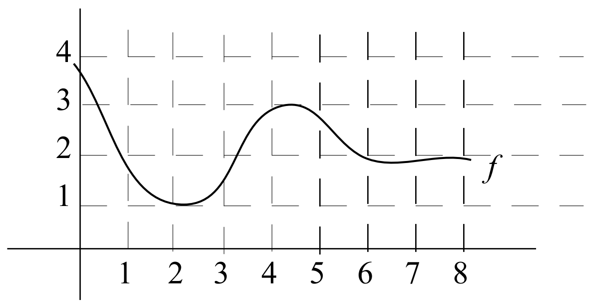


Exercises 37-38. Sketch the graph of $y = f'(x)$ from the given graph of the function $y = f(x)$. Suggestion: Think "slope of tangent lines." Where the tangent line is horizontal, for example, the derivative at that point will be zero; where the tangent line has a positive slope, the derivative will be positive (the greater the slope, the greater the value of the derivative), and so on.

35.



36.



37. **(Theory)** Prove that if a function f is differentiable at c, then $f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$.

Suggestion: let $h = x - c$

38. **(Theory)** Prove that if a function f is differentiable at c, then f is continuous at c.

Suggestion: Begin with $\lim\limits_{x \to c} [f(x) - f(c)] = \lim\limits_{x \to c} \left[\frac{f(x)-f(c)}{x-c} \cdot (x-c)\right]$.

§3. DIFFERENTIATION FORMULAS

In this section we introduce formulas that will enable you to differentiate a function such as

$$f(x) = 4x^5 - 3x^2 - 5x + 2$$

in a matter of seconds. Our main goal is to render you proficient in the use of those formulas. You are, however, invited to establish some of them in the exercises.

For the sake of brevity, we use the expression $(x^n)'$ to represent the derivative of the function $f(x) = x^n$.

Here is the most indispensable formula:

THEOREM 4.4 POWER RULE For any number n:

$$(x^n)' = nx^{n-1}$$

For example:

$$(x^5)' = 5x^4 \leftarrow \text{one power less}$$

$$(x^{-5})' = (-5)x^{-6} \leftarrow \text{one power less: } -5-1 = -6$$

A particularly important case:

$$\boldsymbol{x'} = (x^1)' = 1x^{1-1} = x^0 = 1$$

Here is a geometrical argument for the above derivative formula:

The graph of the function $f(x) = x$ is a line of slope 1. Since the tangent line to a line at any point must be the line itself, $x' = 1$.

While in a geometrical mode, let us also point out that the derivative of any constant function must be 0:

The graph of the function $f(x) = c$ is a horizontal line of slope zero. Since the tangent line to that line at any point must be the line itself:

$$c' = 0$$

Before turning to other differentiation formulas we remind you that for given functions f and g, and any constant c, the functions $f+g$, $f-g$, fg, $\frac{f}{g}$, and cf are defined in a natural fashion:

$$(f+g)(x) = f(x)+g(x) \qquad (f-g)(x) = f(x)-g(x)$$

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (cf)(x) = cf(x)$$

providing $g(x) \neq 0$

These results hold for any finite sum or difference of differentiable functions.

THEOREM 4.5 For any two differentiable functions f and g:

$$(f+g)'(x) = [f(x)+g(x)]' = f'(x)+g'(x)$$

and

$$(f-g)'(x) = [f(x)-g(x)]' = f'(x)-g'(x)$$

IN WORDS: The derivative of a sum (difference) is the sum (difference) of the derivatives.

For example:

$$(x^5+x^3)' = (x^5)'+(x^3)' = 5x^4+3x^2$$

$$(x^4-x^3+x)' = 4x^3-3x^2+1$$

$$(x^7+x^3+x-x^{-2})' = 7x^6+3x^2+1+2x^{-3}$$

$$(x^9+x^2-x^{-4}+325)' = 9x^8+2x+4x^{-5}$$

(what happened to the 325?)

THEOREM 4.6 For any differentiable function f and any constant c:

$$(cf)'(x) = [cf(x)]' = cf'(x)$$

IN WORDS: The derivative of a constant times a function is the constant times the derivative of the function.

For example:

Generally, one omits that middle step and simply multiplies the power **5** of x with the coefficient **4** of x^5, lowering the power by 1: $4(x^5)' = 20x^4$

$4 \cdot 5$

$$(4x^5)' = 4(x^5)' = \boxed{4(5x^4)} = 20x^4$$

see comment in margin

$$(3x^7+6x^2+4x-3)' = 21x^6+12x+4$$

With the above theorems at hand, you are now in a position to quickly differentiate any polynomial function, and beyond.

EXAMPLE 4.5 Differentiate:

(a) $f(x) = 3x^4-2x^3+x-2$

(b) $f(x) = 5x^3+2x^{-3}+7x-1$

(c) $f(x) = (x^2+3)(x^3+x^2)$

(d) $f(x) = \dfrac{5x^6+4x^2-6}{2x^2}$

SOLUTION: The dominant player is the "power rule" of Theorem 2.6:

$$(x^n)' = nx^{n-1}:$$

Since the derivative of a constant is zero, it really doesn't matter if the constant term in the expression for f is -2 or 100. Replacing the -2 with 100 would simply raise the graph of f by 102 units. That would not change the shape of the graph, and cannot therefore alter the derivative of the function.

(a)

$$\text{If: } f(x) = 3x^4 - 2x^3 + x - 2$$

derivative of a constant is 0

one power less

derivative of x is 1

$$\text{Then: } f'(x) = 12x^3 - 6x^2 + 1$$

(b)

$$\text{If: } f(x) = 5x^3 + 2x^{-3} + 7x - 1$$

one power less

$$\text{Then: } f'(x) = 15x^2 - 6x^{-4} + 7 = 15x^2 - \frac{6}{x^4} + 7$$

(c) To differentiate the function $f(x) = (x^2+3)(x^3+x^2)$, we first perform the product to arrive at a sum of **powers-of-*x***:

$$f(x) = (x^2+3)(x^3+x^2) = x^2(x^3+x^2) + 3(x^3+x^2)$$
$$= x^5 + x^4 + 3x^3 + 3x^2$$

and then differentiate:

$$f'(x) = (x^5 + x^4 + 3x^3 + 3x^2)' = 5x^4 + 4x^3 + 9x^2 + 6x$$

(d) To differentiate the function $f(x) = \dfrac{5x^6 + 4x^2 - 6}{2x^2}$, we first express it as a sum of **powers-of-*x***:

$$f(x) = \frac{5x^6 + 4x^2 - 6}{2x^2} = \frac{5x^6}{2x^2} + \frac{4x^2}{2x^2} - \frac{6}{2x^2} = \frac{5}{2}x^4 + 2 - 3x^{-2}$$

and then differentiate:

$$f'(x) = \left(\frac{5}{2}x^4 + 2 - 3x^{-2}\right)' = \frac{5}{2}(4x^3) + 0 - 3(-2x^{-3})$$
$$= 10x^3 + 6x^{-3} = 10x^3 + \frac{6}{x^3}$$

CHECK YOUR UNDERSTANDING 4.8

Differentiate the given function.

(a) $f(x) = -4x^3 + 2x^2 - 3x + 5$ (b) $f(x) = x^2(3x^3 + 2x - 5)$

(c) $f(x) = \dfrac{5x^3 + 2x - 7}{2x^2}$

Answers:
(a) $-12x^2 + 4x - 3$
(b) $15x^4 + 6x^2 - 10x$
(c) $\frac{5}{2} - \frac{1}{x^2} + \frac{7}{x^3}$

EXAMPLE 4.6 Find the tangent line to the graph of the function $f(x) = 2x^3 + x^2 - 5x + 1$ at $x = 1$.

SOLUTION: The word "line" translates to:

$$y = mx + b$$

To find the slope of the tangent line we differentiate:

$$f'(x) = (2x^3 + x^2 - 5x + 1)' = 6x^2 + 2x - 5$$

and evaluate the derivative at $x = 1$:

$$y = \mathbf{m}x + b$$
$$\uparrow$$
$$\boldsymbol{f'(1)}$$

$$m = f'(1) = 6(1)^2 + 2(1) - 5 = 3$$

At this point, we know that the tangent line is of the form:

$$y = 3x + b$$

To find b, we make use of the fact that the tangent line passes through the point:

$$(1, f(1)) = (1, -1)$$

$$2(1)^3 + (1)^2 - 5(1) + 1 = 2 + 1 - 5 + 1 = -1$$

Finding b:

Since the line passes through (1, -1)

$$y = 3x + b$$
$$-1 = 3(1) + b$$
$$b = -4$$

We now have the tangent line: $y = 3x - 4$.

CHECK YOUR UNDERSTANDING 4.9

Find the equation of the tangent line to the graph of the function $f(x) = \dfrac{x^3 - x}{x^2}$ at $x = 2$.

Answer: $y = \frac{5}{4}x - 1$

PRODUCT AND QUOTIENT RULES

It would be nice if the derivative of a product were the product of the derivatives. Alas, however, this cannot be; for:

$$\text{While } x^5 = (x^3)(x^2),\ (x^5)' \neq (x^3)'(x^2)'$$

$$\text{Since: } (x^5)' = 5x^4 \text{ while } (x^3)'(x^2)' = (3x^2)(2x) = 6x^3$$

Niceties aside, here is the truth of the matter:

THEOREM 4.7 If f and g are differentiable, then so is fg, and:
$$(fg)'(x) = [f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$$

Note:
$$\begin{aligned}(x^5)' &= (x^3 \cdot x^2)' \\ &= x^3(x^2)' + x^2(x^3)' \\ &= x^3(2x) + x^2(3x^2) \\ &= 2x^4 + 3x^4 \\ &= 5x^4\end{aligned}$$

IN WORDS: The derivative of a product of two functions is one function times the derivative of the other, plus the other times the derivative of the one.

EXAMPLE 4.7 Differentiate:
$$f(x) = (x^3 + x^2)(x^2 + 3)$$
(a) Using the product rule.
(b) Without using the product rule.

SOLUTION:

(a) $$[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$$

$$\begin{aligned}[(x^3 + x^2)(x^2 + 3)]' &= (x^3 + x^2)(x^2 + 3)' + (x^2 + 3)(x^3 + x^2)' \\ &= (x^3 + x^2)(2x) + (x^2 + 3)(3x^2 + 2x) \\ &= 2x^4 + 2x^3 + 3x^4 + 2x^3 + 9x^2 + 6x \\ &= 5x^4 + 4x^3 + 9x^2 + 6x\end{aligned}$$

The expanding approach is certainly the easier way to go, but there are situation, such as with the function $f(x) = x^2\ln(x + 1)$, in which expanding is not an option.

(b) $$\begin{aligned}[(x^3 + x^2)(x^2 + 3)]' &= (x^5 + x^4 + 3x^3 + 3x^2)' \\ &= 5x^4 + 4x^3 + 9x^2 + 6x\end{aligned}$$

CHECK YOUR UNDERSTANDING 4.10

Determine the derivative of $f(x) = (2x - 1)(x^2 - 3)$ with and without using the product rule.

Answer: $6x^2 - 2x - 6$

Just as the derivative of a product is **not** the product of the derivatives, the derivative of a quotient is **not** the quotient of the derivatives:

THEOREM 4.8 If f and g are differentiable, then so is $\frac{f}{g}$ and:
$$\left(\frac{f}{g}\right)'(x) = \left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$
providing $g(x) \neq 0$

IN WORDS: The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator — all over the denominator squared.

EXAMPLE 4.8 Differentiate:

$$f(x) = \frac{5x^4+4x^2-3}{2x^2}$$

(a) Using the quotient rule.
(b) Without using the quotient rule.

SOLUTION:

(a) $$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$\left[\frac{5x^4+4x^2-3}{2x^2}\right]' = \frac{(2x^2)(5x^4+4x^2-3)'-(5x^4+4x^2-3)(2x^2)'}{(2x^2)^2}$$

$$= \frac{(2x^2)(20x^3+8x)-(5x^4+4x^2-3)4x}{4x^4}$$

$$= \frac{40x^5+16x^3-20x^5-16x^3+12x}{4x^4}$$

$$= \frac{20x^5+12x}{4x^4} = 5x+\frac{3}{x^3}$$

(b) $$\left[\frac{5x^4+4x^2-3}{2x^2}\right]' = \left[\frac{5x^4}{2x^2}+\frac{4x^2}{2x^2}-\frac{3}{2x^2}\right]' = \left[\frac{5}{2}x^2+2-\frac{3}{2}x^{-2}\right]' = 5x+0+3x^{-3} = 5x+\frac{3}{x^3}$$

Option (b) is clearly the better choice in this example, but it is not available in the next example:

EXAMPLE 4.9 Determine the derivative of:

$$f(x) = \frac{5x^4+4x^2-3}{2x^2+1}$$

SOLUTION:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$f'(x) = \frac{(2x^2+1)(5x^4+4x^2-3)'-(5x^4+4x^2-3)(2x^2+1)'}{(2x^2+1)^2}$$

$$= \frac{(2x^2+1)(20x^3+8x)-(5x^4+4x^2-3)(4x)}{(2x^2+1)^2}$$

$$= \frac{(40x^5+36x^3+8x)-(20x^5+16x^3-12x)}{(2x^2+1)^2}$$

$$= \frac{20x^5+20x^3+20x}{(2x^2+1)^2}$$

CHECK YOUR UNDERSTANDING 4.11

Differentiate:

$$f(x) = \frac{2x^2+3x}{2x^2+3}$$

Answer:
$$\frac{-6x^2+12x+9}{(2x^2+3)^2}$$

Higher Order Derivatives

As you might expect, the second derivative of a function f is the derivative of its first derivative $f''(x) = [f'(x)]'$.

For example, if $f(x) = 2x^5 + x^2 - 5x + 2$, then:

$$f'(x) = 10x^4 + 2x - 5$$

$$f''(x) = 40x^3 + 2$$

Answer:

$$f''(x) = 30x + \frac{2}{x^3}$$

$$f'''(x) = 30 - \frac{6}{x^4}$$

CHECK YOUR UNDERSTANDING 4.12

Determine the second and third derivatives of the function.

$$f(x) = 5x^3 - x + \frac{1}{x}$$

EXERCISES

Exercises 1-24. Differentiate the given function.

1. $f(x) = 2x + 7$
2. $f(x) = -3x^2 - 2x + 5$
3. $f(x) = 3x^5 + 4x^3 - 7$
4. $f(x) = 4x^4 + 7x^3 - 3x - 2$
5. $g(x) = 2x^{-3} + 4$
6. $f(x) = x^2 - 1 + 3x^{-3}$
7. $g(x) = 7x^3 + 5x^2 - 4x + x^{-4} + 1$
8. $f(x) = \frac{1}{3}x^3 + \frac{1}{5}x^2 - x - 1$
9. $g(x) = x + \frac{2}{x} + \frac{3}{x^2}$
10. $g(x) = 5 - \frac{3}{x^3}$
11. $k(x) = \frac{x^5 + 3x - 5}{x^3}$
12. $k(x) = \frac{4x^4 + x^3 - 2x^2 + 3x + 7}{x^2}$
13. $K(x) = (x^3 + 2x)(3x^3 + 2x + 3)$
14. $K(x) = (4x^4 + 2x^3 + x^2)(x^3 + x + 1)$
15. $F(x) = \frac{3x^2 + 2x - 5}{x}$
16. $F(x) = \frac{-x^5 + 3x - 4}{x^2}$
17. $F(x) = \frac{3x^2 + 2x - 5}{x + 4}$
18. $F(x) = \frac{-x^5 + 3x - 4}{x^2 + 2x}$
19. $f(x) = \frac{5}{3x^2 + 1}$
20. $g(x) = \frac{1}{(x-2)^2}$
21. $H(x) = \frac{x}{2x + 1} + \frac{x}{3x - 1}$
22. $H(x) = \frac{2x + 2}{2x - 1} - 3x^2 + x + 5$
23. $f(x) = \left(\frac{x}{3x + 1}\right)(x^2 + 2x)$
24. $f(x) = \left(\frac{x}{3x + 1}\right)\left(\frac{x^2 + 2x}{x + 3}\right)$

Exercises 25-30 (Tangent Line) Determine the tangent line to the graph of the given function at the indicated point.

25. $f(x) = 3x^2 - x - 1$ at $x = 1$
26. $f(x) = -x^3 - 2x + 2$ at $x = 0$
27. $f(x) = \frac{x^5 + 2x}{x^4}$ at $x = -1$
28. $f(x) = \frac{3x^3 + x^2 - 2x - 1}{x^2}$ at $x = 1$
29. $f(x) = \frac{x^2 + 2x}{x - 1}$ at $x = -1$
30. $f(x) = \frac{3x^3 + x^2 - 2x - 1}{x^2 + 1}$ at $x = 1$

Exercises 31-34. (Higher Order Derivatives) Determine the second derivatives of the given function.

31. $f(x) = x^3 - 6x^2 + 12x$

32. $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - x + 1$

33. $f(x) = x^3 + 2x^2 - \frac{1}{x}$

34. $f(x) = \frac{x^2+1}{3x}$

Exercises 35-48 Evaluate the given expression at the indicated point, if:

$$f(0) = 1, f(1) = 3, f(2) = 6, f'(0) = 2, f'(1) = 6, f'(2) = 0$$
$$g(0) = 3, g(1) = 2, g(2) = 5, g'(0) = 1, g'(1) = 2, g'(2) = 2$$
$$h(0) = 0, h(1) = 6, h(2) = 2, h'(0) = 3, h'(1) = 1, h'(2) = 1$$

35. $[f(x) + g(x)]'$ at $x = 1$

36. $[f(x)g(x)]'$ at $x = 1$

37. $\left[\frac{f(x)}{g(x)}\right]'$ at $x = 1$

38. $\left[\frac{g(x)}{f(x)}\right]'$ at $x = 1$

39. $[f(x) + g(x) + h(x)]'$ at $x = 2$

40. $[f(x) + g(x)h(x)]'$ at $x = 2$

41. $[f(x)g(x) + h(x)]'$ at $x = 2$

42. $\left[\frac{f(x)+g(x)}{h(x)}\right]'$ at $x = 1$

43. $\left[\frac{f(x)g(x)}{h(x)}\right]'$ at $x = 1$

44. $\left[\frac{f(x)-g(x)}{h(x)+1}\right]'$ at $x = 0$

45. $\left[\frac{g(t)}{h(t)} + g(t)\right]'$ at $t = g(1)$

46. $\frac{f(s)}{g'(s)} + g(2)h'(s)$ at $s = g'(1)$

47. $\left[\frac{f(x)+g(x)+h(x)}{f(x)-g(x)-h(x)}\right]'$ at $x = g[f(0)]$

48. $h'(x) + h(1)\left[\frac{g(x)f(x)}{h(x)}\right]'$ at $x = g'[f(0)]$

§4. OPTIMIZATION PROBLEMS

An optimization problem is one where a maximum or minimum value of a function is to be determined.

The figure below depicts the graph of a function f over the interval $[L, R]$ (left endpoint L, and right endpoint R).

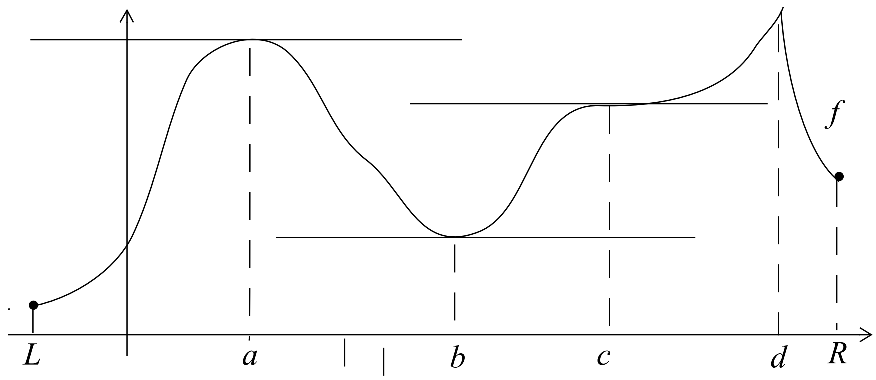


Figure 4.7

Forgetting those endpoints momentarily, we cite three observations that can be drawn from the figure, in order of their importance:

1. Where a (local) maximum or (local) minimum occurs, if the function is differentiable, then its derivative must be zero (see a and b in Figure 4.7 and their associated **horizontal** tangent lines).
2. The derivative of a function may be zero, without either a maximum or minimum occurring at that point (see c).
3. A maximum or minimum can occur without the derivative being zero at that point, but only because the derivative does not exist there (see d, where the function fails to be differentiable).

The story concerning endpoints is easily told:

4. An endpoint maximum or minimum occurs at the endpoints. For example, a minimum occurs at L in Figure 4.7, and also at R: the graph climbs as you move to the right of L, and it falls as you approach R.

A **local maximum** (**local minimum**) point on the graph of a function is a point on the graph which is as high or higher (low or lower) than any other point in its immediate vicinity.
When one says that a maximum (minimum) occurs at $x = a$, one generally means that a **local** maximum (**local** minimum) occurs at that point.
The terms **absolute** maximum (**absolute** minimum) value of a function denotes the largest (smallest) value the function assumes within a specified domain.

Endpoints aside, we have observed that a maximum or minimum **can only** occur where the derivative of the function is zero or where the derivative does not exist. Since one is often concerned with locating optimal values, any x where $f'(x) = 0$, or where $f'(x)$ does not exist is said to be a **critical point** of the function.

Finding the critical points of a function $y = f(x)$ reveals where (local) maximums or (local) minimums might occur. But what exactly occurs at those points — a maximum, a minimum, or neither? The following theorem addresses that question:

THEOREM 4.9

THE FIRST DERIVATIVE TEST

If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, then f has a local minimum at c.

If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then f has a local maximum at c.

THE SECOND DERIVATIVE TEST

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

The second derivative test is inconclusive if $f''(c) = 0$, or if it does not exist.

The second derivative test is generally the better choice—if it works. It sometimes turn out to be inconclusive.

A glance at Figure 4.7 should convince you of the validity of the above claims. Let's start with the First Derivative Test:

Look at the maximum point occurring at a in that figure. Immediately to the left of a the graph is climbing and to the right of a the graph is falling — i.e. the derivative is positive to the left of (a) and negative to its right. Now look at the minimum point at b. To the left of b the graph falls and it climbs to its right — i.e. the derivative is negative to the left of b and positive to its right.

For the Second Derivative Test, consider again the maximum point at a. Since the tangent lines decrease as you move from the left of a to the right of a, the slope of those tangent lines are decreasing—i.e the derivative of the first derivative (the second derivative) must be negative. A similar observation suggests that the second derivative at the minimum point b must be positive.

EXAMPLE 4.10

MAXIMUM PROFIT

A company can produce up to 500 units per month. Its profit, in terms of number of units produced, x, is given by:

$$P(x) = -\frac{x^3}{30} + 9x^2 + 400x - 7500$$

How many units should the company produce to maximize profit?

SOLUTION: Differentiating the profit function we have:

$$P'(x) = -\frac{x^2}{10} + 18x + 400 = -\frac{1}{10}(x^2 - 180x - 4000)$$

$$= -\frac{1}{10}(x - 200)(x + 20)$$

The critical points occur at $x = 200$, and at $x = -20$. We have no interest in the latter point, since production cannot be negative. Applying the second derivative test we see that the maximum does occur at a production level of $x = 200$ units:

$$P''(x) = \left(-\frac{x^2}{10} + 18x + 400\right)' = -\frac{x}{5} + 18 \Rightarrow P''(200) < 0$$

CHECK YOUR UNDERSTANDING 4.13

The total monthly profit (in thousands of dollars) from the sale of x sailboats is given by:

$$P(x) = 5.7x - 0.025x^2 \qquad (x \le 150)$$

Find the maximum profit.

Answer: \$324,900

Example 4.10 was kind of easy in that we were conveniently given the function to be optimized. The real challenge in most optimization problems, however, is to extricate the function to be maximized or minimized from a given word problem. The first three steps in the following procedure can help you meet that challenge:

Step 1. See the problem.

Step 2. Express the quantity to be optimized in terms of any convenient number of variables.

Step 3. In the event that the expression in Step 3 involves more than one variable, use the given information to arrive at an expression involving but one variable.

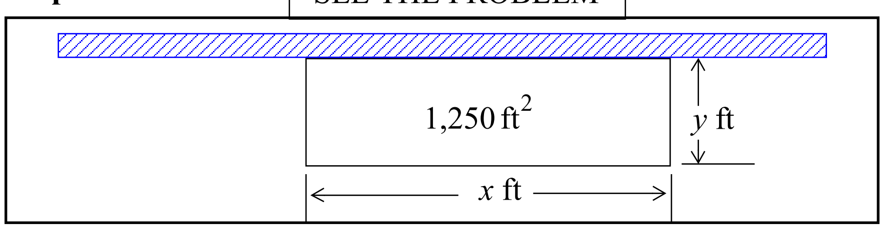
Step 4. Differentiate, set equal to zero, and solve (find the critical points). If necessary, analyze the nature of the critical points, and any endpoint of the domain.

EXAMPLE 4.11

A 1,250 ft^2 rectangular region is to be enclosed. The region is adjacent to an already existing fence and only three of its sides will require new fencing. Determine the dimensions of the region requiring the least amount of new fencing.

SOLUTION:

Step 1: SEE THE PROBLEM



Step 2: You want to minimize **F**encing. To better focus your attention, we suggest that you immediately write down: $\boldsymbol{F} = \quad$, and then fill in the right side of the equation: $\boldsymbol{F} = x + 2y$

Step 3: We need to get rid of one of the two variables on the right side. In order to do so, we look for a relation between those variables and find it in the given area information:

$$xy = 1250$$
$$y = \frac{1250}{x} \qquad (*)$$

We can now express ***F*** as a function of one variable:

$$F(x) = x + 2y = x + 2\left(\frac{1250}{x}\right) = x + \frac{2500}{x} = x + 2500x^{-1}$$

Step 4: Differentiate, set equal to 0, and solve:

$$F'(x) = 1 - 2500x^{-2} = 0$$
$$1 = \frac{2500}{x^2}$$
$$x^2 = 2500$$
$$x = \pm 50$$

At this point, we know that the graph of the volume function has a horizontal tangent line at $x = 50$ and at $x = -50$. You can forget about the -50, for x represents a length. Since a minimum amount of fencing surely exists, it must occur at $x = 50$ (margin).

If you doubt that a minimum occurs at 50, you can employ the second derivative test:
$F''(x) = 5000x^{-3}$
$\Rightarrow F''(50) > 0$

Conclusion: The dimensions of the region requiring the least amount of fencing are:

$$\underset{\substack{\uparrow \\ \text{side opposite the existing fence}}}{x} = 50 \text{ feet, and } y \underset{\substack{\uparrow \\ \text{from } (*)}}{=} \frac{1250}{x} = \frac{1250}{50} = 25 \text{ feet}$$

CHECK YOUR UNDERSTANDING 4.14

Determine the maximum rectangular area that can be enclosed using 1200 feet of fencing.

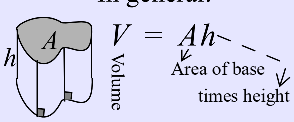
Answer: 90,000 ft^2.

EXAMPLE 4.12

MAXIMUM VOLUME

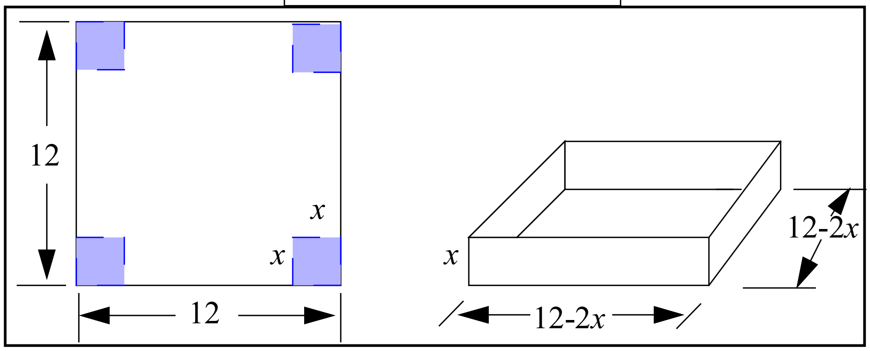
Best Box Company is to manufacture open-top boxes from 12 in. by 12 in. pieces of cardboard. The construction process consists of cutting the same size squares from each corner of the cardboard, and then folding the resulting cross-like configuration into a box. What size square should be cut out, if the resulting box is to have the largest possible volume?

In general:



SOLUTION:

Step 1: SEE THE PROBLEM



Step 2: Volume is to be maximized, and from the above, we see that:

$$V(x) = (12-2x)(12-2x)x = 4x^3 - 48x^2 + 144x$$

Step 3: Not applicable, since volume is already expressed as a function of **one** variable.

Step 4: Differentiate, set equal to 0, and solve:

$$\begin{aligned} V'(x) = 12x^2 - 96x + 144 &= 0 \\ 12(x-6)(x-2) &= 0 \\ x = 6 \quad \text{or} \quad x &= 2 \end{aligned}$$

At this point we know that there are two critical points: $x = 6$ and $x = 2$. We can forget about the 6, for if you cut a square of length 6 from the piece of cardboard, nothing will remain. As for $x = 2$:

Since $V''(x) = (12x^2 - 96x + 144)' = 24x - 96$, $V''(2) < 0$, it follows (by the second derivative test) that a maximum volume will be obtained by cutting out a square of length 2 in.

CHECK YOUR UNDERSTANDING 4.15

An eight cubic foot closed box with square base is to be constructed. Determine the minimum amount of cardboard required to manufacture the box.

Answer: 24 square feet.

EXAMPLE 4.13
MAXIMUM YIELD

When 20 peach trees are planted per acre, each tree will yield 200 peaches. For every additional tree planted per acre, the yield of each tree diminishes by 5 peaches. How many trees per acre should be planted to maximize yield?

SOLUTION:

Step 1:

SEE THE PROBLEM

20 trees/acre $\longrightarrow$	200 per tree
$20+x$ trees/acre $\longrightarrow$	$200-5x$ per tree

Step 2: Letting x denote the number of trees above 20 to be planted per acre, we express the yield per acre as a function of x:

$$Y(x) = \overbrace{(20+x)}^{\text{number of trees per acre}}\overbrace{(200-5x)}^{\text{number of peaches per tree}} = -5x^2 + 100x + 4000$$

Step 3: Not applicable [The function to be maximized, $Y(x)$, is already expressed in terms of one variable].

Step 4: $Y'(x) = -10x + 100 = 0$

$$x = \frac{100}{10} = 10$$

From $Y''(x) = -10 < 0$, we conclude that maximum yield will be attained with the planting of 30 trees.

CHECK YOUR UNDERSTANDING 4.16

A car-rental agency can rent 150 cars per day at a rate of \$24 per day. For each price increase of \$1 per day, 5 less cars are rented. What rate should be charged to maximize the revenue of the agency?

Answer: \$27.00

USING A GRAPHING CALCULATOR TO FIND APPROXIMATE SOLUTIONS TO OPTIMIZATION PROBLEMS.

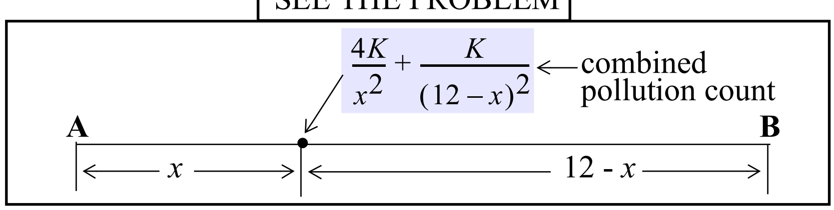
EXAMPLE 4.14

MINIMAL POLLUTION

Two chemical plants are located 12 miles apart. The pollution count from plant A in parts per million, at a distance of x miles from plant A, is given by $(4K)/x^2$ for some positive constant K. The pollution count from the cleaner plant B, at a distance of x miles from plant B, is one quarter that of A. Determine the point on the line joining the two plants where the pollution count is minimal.

SOLUTION:

SEE THE PROBLEM



We see that $\frac{4K}{x^2} + \frac{K}{(12-x)^2}$ is to be minimized, but what are we going to do with the K? We factor it out:

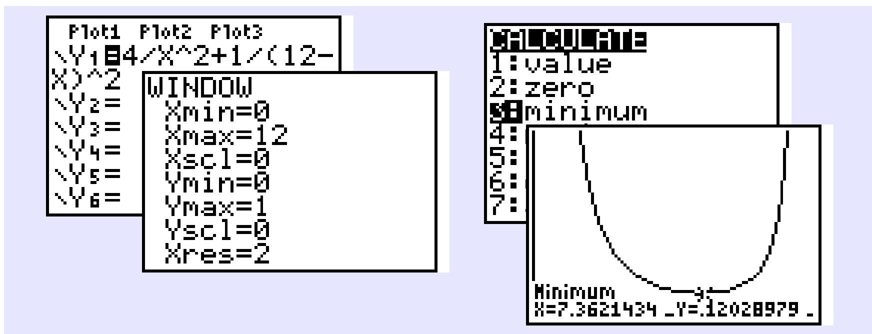
$$\frac{4K}{x^2} + \frac{K}{(12-x)^2} = K\left(\frac{4}{x^2} + \frac{1}{(12-x)^2}\right)$$

Conclusion: Minimum pollution count will occur where the function:

$$\frac{4}{x^2} + \frac{1}{(12-x)^2}$$

is minimum (what about the "K?"). Taking the easy way out:

Actually, it's not so much an "*easy way out*," for when it comes to optimization problems the real challenge is to express the quantity to be optimized as a function of one variable. The rest involves a routine process which can be relegated to machines.



We determined that the minimum pollution count between the two plants occurs at a distance of approximately 7.36 miles from plant A.

CHECK YOUR UNDERSTANDING 4.17

A straight road runs from North to South. Point A is 5 miles due West of point E on the road. If you walk 10 miles south of A and then go 30 miles due East, you will reach point B. If you walk 10 miles due South of B and then go 15 miles due west, then you will reach point C.

Use a graphing utility to determine, to 2 decimal places, the point P on the road whose combined distance from the three points A, B, and C is minimal.

Answer: 8m miles south of E

EXERCISES

Exercises 1-2. (Maximum Profit) Determine the production level that will maximize profit, for the given cost and revenue functions.

1. $C(x) = 5000 + 100x$, $R(x) = 200x - \frac{x^2}{10}$, for $0 \le x \le 2000$.

2. $C(x) = 1200 + 300x + \frac{250x^2}{100 + x}$, $R(x) = 500x$, for $0 \le x \le 2000$.

3. **(Maximize Yield)** The cabbage yield, Y, in bushels per acre, is given by $Y(x) = 60 + 16x - 4x^2$, where x denotes the number of quarts of fertilizer used per acre. How many quarts of fertilizer should be used to maximize yield?

4. **(Minimize Cost)** Determine the minimum monthly cost of a company if its monthly costs for producing x units is given by $C(x) = x^2 - 150x + 500$.

5. **(Smallest Perimeter)** A rectangle has an area of 900 in.^{2} What are the dimensions of the rectangle of smallest perimeter?

6. **(Air Velocity)** When a person coughs, the radius r of the trachea decreases. The velocity of air in the trachea during a cough can be approximated by the function $v(r) = ar^2(r_0 - r)$, where a is a constant, and r_0 is the radius of the trachea in a relaxed state. Determine the radius at which the velocity is greatest.

7. **(Minimum Fencing)** A fenced-in rectangular garden is divided into 2 areas by a fence running parallel to one side of the rectangle. Find the dimensions of the garden that minimizes the amount of fencing needed, if the garden is to have an area of 15,000 square feet.

8. **(Minimum Cost)** A fenced-in rectangular garden is divided into 3 areas by two fences running parallel to one side of the rectangle. The two fences cost \$6 per running foot, and the outside fencing costs \$4 per running foot. Find the dimensions of the garden that minimizes the total cost of fencing, if the garden is to have an area of 8,000 square feet.

9. **(Maximum Revenue)** A car-rental agency can rent 100 cars per day at a rate of \$40 per day. For each price increase of \$1 per day, 2 less cars are rented. What rate should be charged to maximize the revenue of the agency?

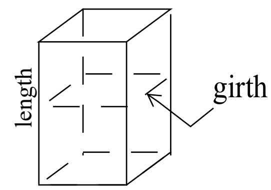
10. **(Minimum Area)** A poster is to surround 1200 in.^{2} of printing material with a top and bottom margin of 4 in. and side margins of 3 in. Find the outside dimensions of the minimum poster area.

11. **(Maximum Revenue)** A chemical company charges \$900 per ton of a certain chemical, for orders up to and including 10 tons. For every ton in access of 10 ordered by a customer, the cost of each ordered ton will be reduced by \$30. Find the maximum revenue for the company.

12. **(Maximum Volume)** A shipping crate with base twice as long as it is wide is to be shipped by freighter. The shipping company requires that the sum of the three dimensions of the crate cannot exceed 288 inches. What are the dimensions of the crate of maximum volume?

13. **(Minimum Material Cost)** A closed crate with square base is to have a volume of 250 cubic feet. The material for the top and bottom of the crate costs \$2 per square foot, and the material for the sides costs \$1 per square foot. Find the minimum cost of material.

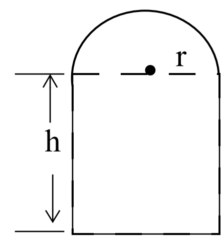
14. **(Maximum Volume)** A box with a square base is to be constructed. For mailing purposes, the perimeter of the base (the girth of the box), plus the length of the box, cannot exceed 108 in. Find the dimensions of the box of greatest volume.



15. **(Minimum Cost)** A cylindrical drum is to hold 65 cubic feet of chemical waste. Metal for the top of the drum costs \$2 per square foot, and \$3 per square foot for the bottom. Metal for the side of the drum costs \$1.00 per square foot. Find the minimal cost of the drum.

16. **(Maximum Light Emission)** A Norman window is a window in the shape of a rectangle surmounted by a semicircle. Find the dimensions of the window that admits the most light if the perimeter of the window (total outside length) is 15 feet. (Assume that the same type of glass is used for both parts of the window.)

    Area of a circle: πr^2; circumference of a circle: $2\pi r$



USE A GRAPHING CALCULATOR TO APPROXIMATE, TO TWO DECIMAL PLACES, THE SOLUTION OF THE OPTIMIZATION PROBLEMS IN EXERCISE 18 THROUGH 25.

17. **(Maximum Profit)** Given the cost and revenue functions:

$$C(x) = 5000 + 100x + \frac{x^2}{\sqrt{x+500}} \quad \text{and} \quad R(x) = 130x - \sqrt{\frac{x^2}{x+100}}$$

    Determine the number of units, x, to be manufactured in order to maximize profit.

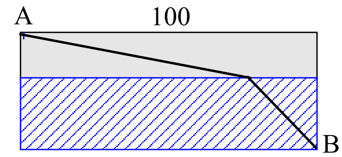
18. **(Maximum Drug Concentration)** The concentration (in milligrams per cubic centimeter) of a particular drug in a patient's bloodstream, t hours after the drug has been administered has been modeled by $C(t) = \frac{0.2t}{0.9t^2 + 5t + 3}$. How many hours after the drug is administered will the concentration be at its maximum? What is the maximum concentration?

19. **(Shortest Distance)** Determine, to two decimal places, the shortest distance between a point on the curve $y = 2x^3 + 3x - 1$ and the point $\left(0, \frac{1}{2}\right)$.

20. **(Minimum Construction Cost)** A cable is to be run from a power plant, on one side of a river that is 600 feet wide, to a tower on the other side, which is 2000 feet downstream. The cost of

laying the cable in the water is \$6 per foot, while the cost on land is \$4 per foot. Find the most economical route for laying the cable.

21. (**Minimum Cost)** Point A is at ground level, and point B is 35 feet below ground level and 100 feet away from A (at ground level). The first 15 feet below ground level is soil, after which there is shale. A pipe is to join the two points. It costs \$76 per foot to lay piping in the soil layer, and \$245 per foot to lay piping in the shale layer. Find the minimum labor cost of the project.



CHAPTER SUMMARY

At the very heart of the calculus is the concept of a **limit.** Here is one of them:

$$\lim_{x \to 2} (3x + 5)$$

It is read: ***The limit as x approaches 2 of the function*** $3x + 5$.

It represents: The number $3x + 5$ approaches, as x approaches 2.

LIMIT THEOREMS	If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then: (i) $\lim_{x \to c} [f(x) + g(x)] = L + M$ (ii) $\lim_{x \to c} [f(x) - g(x)] = L - M$ (iii) $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$ (iv) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$ (v) $\lim_{x \to c} [af(x)] = aL$

CONTINUITY	A function f is **continuous** at c if: $\lim_{x \to c} f(x) = f(c)$

DERIVATIVE OF A FUNCTION AT A POINT	The **derivative of a function *f* at *c*** is the number $f'(c)$ given by: $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ (Providing that the limit exists.)

DERIVATIVE FUNCTION	The **derivative of a function *f*** is the function $f'(x)$ given by: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (Providing that the limit exists.)

Note: Differentiability $\Rightarrow$ Continuity $\Rightarrow$ Limit Exists

SECOND DERIVATIVE	The **second derivative of a function *f*** is the derivative of its first derivative $f''(x) = [f'(x)]'$.

DERIVATIVE FORMULAS	(a) $(x^n)' = nx^{n-1}$ (b) $x' = 1$ and $c' = 0$ for any constant c. (c) $[cf(x)]' = cf'(x)$ for any constant c. (d) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$ (e) $[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$ (f) $\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

MAXIMUM/MINIMUM POINTS	Where a (local) maximum or (local) minimum of a function occurs, the derivative is zero or does not exist.

DERIVATIVE TESTS	**THE FIRST DERIVATIVE TEST** If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, then f has a local minimum at c. If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then f has a local maximum at c. **THE SECOND DERIVATIVE TEST** If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c. The second derivative test is inconclusive if $f''(c) = 0$, or if it does not exist.

CHAPTER 5
PERSONAL FINANCE

§1. INTEREST

Interest is the price you pay to borrow money, or the cost you charge to lend money. Basically there are two types of interest: Simple interest, and compound interest.

SIMPLE INTEREST

Here is the Simple Interest Formula:

$$\boldsymbol{I = P \cdot r \cdot n}$$

where

P is the amount invested
r is the annual percentage investment rate
n is the number of years invested.

Consider the following example:

EXAMPLE 5.1 (a) $5,000 is invested at a simple interest rate of 4% for 5 years. Find the amount of interest earned.

(b) $5,000 is invested at a simple interest rate of 4.7% for 18 months. Find the amount of interest earned.

SOLUTION:

(a) Turning to the simple interest formula, we have:

$$I = \$5,000(.04)(5) = \$1{,}000$$

(b) Before we can apply the formula, we need to express 18 months in terms of years. Since there are 12 months in a year:

$$18 \text{ months} = 1 \text{ year} + 6 \text{ months} = 1.5 \text{ years}$$

Turning to the simple interest formula, we have:

$$I = \$5,000(.047)(1.5) = \$352.50$$

The current value of an asset is said to be its **present value** (***P***). The value of an asset at a future date is said to its **future value** (***A***). In particular, the present values of the investments in both (a) and (b) of Example 5.1 is $5,000. Here are their future values:

(a) $A = \$(5000 + 1000) = \$6{,}000$

(b) $A = \$(5000 + 352.50) = \$5{,}352.50$

CHECK YOUR UNDERSTANDING 5.1

A total of \$5,000 is invested at a simple interest rate of 5.2% for 29 months. Find the amount of interest earned and the future value of the investment.

Answers: $I = \$628.33$
$A = \$5,628.33$

COMPOUND INTEREST FUTURE VALUE

In compound interest, interest is earned on the initial amount invested as well as on the interest that accumulates over time:

THEOREM 5.1
Future Value of Invesment

If an amount $\boldsymbol{P}$ is invested in an account at $\boldsymbol{r}$ compounded $\boldsymbol{n}$ times during each year, then the future value of the amount invested at the end of $\boldsymbol{t}$ years is given by:

$$\boldsymbol{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

PROOF: Since there are n compounding periods in each year, the interest rate during each of the nt periods is $\frac{r}{n}$.

Here is the future value at the end of the first period:

$$A_1 = P + P\left(\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)$$

Here is the future value at the end of the second period:

$$A_2 = \underbrace{\boxed{P\left(1+\frac{r}{n}\right)}}_{A_1} + \underbrace{\boxed{P\left(1+\frac{r}{n}\right)\left(\frac{r}{n}\right)}}_{\text{interest on } A_1} = P\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)^2$$

Here is the future value at the end of the third period:

$$A_3 = \underbrace{\boxed{P\left(1+\frac{r}{n}\right)^2}}_{A_2} + \underbrace{\boxed{P\left(1+\frac{r}{n}\right)^2\left(\frac{r}{n}\right)}}_{\text{interest on } A_2} = P\left(1+\frac{r}{n}\right)^2\left(1+\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)^3$$

Spotting the above pattern, we can reasonably assume that at the end of t years the future value will be:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

EXAMPLE 5.2 $6,000 is invested in a savings account for 7 years. Determine the future value of the investment with:

(a) a simple interest rate of 4%.

(b) an annually compounded interest rate of 4%.

(c) a monthly compounded interest rate of 4%.

SOLUTION:

(a) Using the simple interest formula:

$$A = \$(6{,}000 + I) = \$\left(6{,}000 + 6{,}000 \cdot \frac{4}{100} \cdot 7\right) = \$7{,}680.00$$

(b) From Theorem 5.1, with $n = 1$:

$$A = P(1+r)^t = 6000(1+.04)^7 = \$7{,}895.59$$

(b) From Theorem 5.1 with $n = 12$:

$$A = P\left(1+\frac{r}{12}\right)^{12 \cdot 7} = 6000\left(1+\frac{.04}{12}\right)^{84} = \$7{,}935.08$$

CHECK YOUR UNDERSTANDING 5.2

$10,000 is invested in a savings account for 10 years. Determine the future value of the investment with:

(a) a simple interest rate of 5%.

(b) an annual compounded interest rate of 5%.

(c) a quarterly compounded interest rate of 5%.

Answers: (a) $15,000
(b) $16,288.95
(c) $16,436.19

COMPOUND INTEREST PRESENT VALUE

Starting with the future value formula of Theorem 5.1:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

and solving for the present value, P, we arrive at:

THEOREM 5.2
Present Value of Investment

The present value $\boldsymbol{P}$ required to achieve a future value $\boldsymbol{A}$ after $\boldsymbol{t}$ years at a rate of $\boldsymbol{r}$ compounded $\boldsymbol{n}$ times per year is given by:

$$\boldsymbol{P = A\left(1+\frac{r}{n}\right)^{-nt}}$$

EXAMPLE 5.3

(a) You invest \$100 in an account at 4.3% compounded quarterly. How much will that account be worth in seven years?

(b) How much should you invest in an account at 4.3% compounded monthly so as to have \$25,000 in that account nine years from now?

(c) You can purchase a 10 year government bond at 4% compounded monthly. How much should you invest in order that the bond will mature at \$12,000?

SOLUTION:

(a) This is a future value problem. So:

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 100\left(1+\frac{.043}{4}\right)^{28} = \$134.90$$

(b) This is a present value problem. So:

$$P = A\left(1+\frac{r}{n}\right)^{-nt} = 25000\left(1+\frac{.043}{12}\right)^{-108} = \$16{,}969.03$$

(c) This is a present value problem. So:

$$P = A\left(1+\frac{r}{n}\right)^{-nt} = 12000\left(1+\frac{.04}{12}\right)^{-120} = \$8{,}049.19$$

CHECK YOUR UNDERSTANDING 5.3

(a) How much should you invest in an account at 5% compounded semiannually in order to have \$100,000 in that account 20 years from now?

(b) You invest \$37,243 in an account at 5% compounded semiannually. What will that account be worth 20 years from now?

Answers: (a) \$37,243.06
(b) \$99,999.83

CONTINUOUS COMPONDING

We know that the compounding formula of Theorem 5.1

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

will increase as n increases, but will it converge to some finite number as n increases? Yes:

Continuous compounding is when the years of investment are broken down into an infinite number of periods.

This is not possible. Still, the concept of continuously compounded interest plays an important role in the finance field.

DEFINITION 5.1
Continuous Compounding Future Value

If an amount P is invested at r compounded continuously for t years then:

$$A = Pe^{rt}$$

EXAMPLE 5.4 $10,000 is invested for 10 years at an interest rate of 4% .

Determine the future value of the investment if it is:

(a) compounded daily.

(b) compounded continuously.

SOLUTION:

(a) From Theorem 5.1:

Just about the same

$$A = \$10{,}000\left(1 + \frac{.04}{365}\right)^{365 \cdot 10} = \$14{,}917.92$$

(b) From Definition 5.1:

$$A = \$10{,}000e^{0.04 \cdot 10} = \$10{,}000e^{0.4} = \$14{,}918.25$$

CHECK YOUR UNDERSTANDING 5.4

$25,000 is invested for 20 years at an interest rate of 7%.

Determine the future value of the investment if it is:

(a) compounded monthly.

(b) compounded continuously.

Answers: (a) $100,968.47
(b) $101,380.00

EXAMPLE 5.5 P dollars are invested at 5% compounded continuously. How long will it take for that investment to double?

SOLUTION: We need to solve for t in the following equation:

$$2P = Pe^{.05t}$$

Let's do it:

$$\begin{aligned} 2P &= \cancel{P}e^{.05t} \\ e^{.05t} &= 2 \\ .05t &= \ln 2 \\ t &= \frac{\ln 2}{.05} \approx 13.86 \text{ years} \end{aligned}$$

Answers: (a) 7.70 years
(b) 12.21 years

CHECK YOUR UNDERSTANDING 5.5

P dollars are invested at 9% compounded continuously. How long will it take for that investment to:

(a) Double? (b) Triple?

DEFINITION 5.2 Continuous Compounding Present Value

The present value $\boldsymbol{P}$ required to achieve a future value $\boldsymbol{A}$ after $\boldsymbol{t}$ years at a rate of $\boldsymbol{r}$ compounded continuously is given by:

$$P = Ae^{-rt}$$

EXAMPLE 5.6

How much should you invest in an account at 5% compounded continuously in order to have \$15,000 in ten years?

SOLUTION: This is a continuous compounding present value problem. So:

$$P = Ae^{-rt} = \$150.000e^{-(.05)(10)} = \$9{,}097.96$$

Answer: \$11,156.51

CHECK YOUR UNDERSTANDING 5.6

How much should you invest in an account at 6% compounded continuously in order to have \$50,000 25 years from now?

EXERCISES

1. \$5,000 is invested at a simple interest rate of 3.5% for five years. Find the amount of interest earned and the future value of the investment.
2. \$10,000 is invested at a simple interest rate of 5% for ten years. Find the amount of interest earned and the future value of the investment.
3. \$5,000 is invested at an annually compounded interest rate of 3.5% for five years. Find the amount of interest earned and the future value of the investment.
4. \$10,000 is invested at an annually compounded interest rate of 5% for ten years. Find the amount of interest earned and the future value of the investment.
5. \$9,000 is invested at an interest rate of 4% compounded semiannually for seven years. Find the amount of interest earned and the future value of the investment.
6. \$8,000 is invested at an interest rate of 4% compounded monthly for fifteen years. Find the amount of interest earned and the future value of the investment.
7. \$15,000 is invested at an interest rate of 4% compounded continuously for fifteen years. Find the amount of interest earned and the future value of the investment.
8. \$18,000 is invested at an interest rate of 4% compounded continuously for 40 months. Find the amount of interest earned and the future value of the investment.
9. P dollars are invested at a simple interest rate of 5%. How long will it take for that investment to double?
10. P dollars are invested at an annually compounded interest rate of 5%. How long will it take for that investment to double?
11. P dollars are invested at a continuously compounded interest rate of 3.2%. How long will it take for that investment to double?
12. You can purchase a 10 year government bond at 3.7% compounded annually. How much should you invest in order that the bond will mature at \$18,000?
13. You can purchase a 10 year government bond at 3.7% compounded semiannually. How much should you invest in order that the bond will mature at \$18,000?
14. You can purchase a 10 year government bond at 3.7% compounded monthly. How much should you invest in order that the bond will mature at \$18,000?
15. You can purchase a 10 year government bond at 3.7% compounded continuously. How much should you invest in order that the bond will mature at \$18,000?

§2. ANNUITIES

A **geometric sequence** is a sequence of numbers of the form:

$$a, ar, ar^2, ar^3, ar^4, \ldots, ar^n, \ldots$$

with **initial term** a, and **common ratio** r.

The third term of the geometric sequence: $a, ar, ar^2, ar^3, \ldots$ is ar^2 and not ar^3.

Note that the n^{th} term of the sequence is ar^{n-1} (see margin).

THEOREM 5.3 The sum of the first n terms of the sequence

$$a, ar, ar^2, ar^3, ar^4, \ldots, ar^n, \ldots$$

is given by:

$$\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}$$

Proof: Let S_n denote the sum of the first n terms of the sequence:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

We then have:

$$\begin{array}{rl} S_n: & a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} \\ rS_n: & \quad \cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{n-1}} + ar^n \\ \hline & S_n - rS_n = a - ar^n \\ & S_n(1-r) = a(1-r^n) \\ & S_n = \dfrac{a(1-r^n)}{1-r} \end{array}$$

EXAMPLE 5.7 Find the sum of the first 20 terms of the geometric sequence

$$3, 6, 12, 24, 48, \ldots$$

SOLUTION: We are dealing with the geometric sequence:

$$3, 6, 12, 24, 48, \ldots = 3, 3(2), 3(2^2), 3(2^3), 3(2^4)\ldots$$

with $a = 3$, and $r = 2$.

Turning to Theorem 5.3 we find the sum of the first 20 terms:

$$S_{20} = \frac{3(1-2^{20})}{1-2} = 3{,}145{,}725$$

CHECK YOUR UNDERSTANDING 5.7

Find the sum of the first 10 terms of the geometric sequence

$$2, 10, 50, 250, \ldots$$

Answer: 4,882,812

An **ordinary annuity** is a series of payments (deposits) possessing the following four characteristics:

(1) All payments are of the same amount (such as $1,000).

(2) All payments are made at the same intervals of time (such as once a month).

(3) All payments are made at the end of each period (such as payments on the last day of the month).

(4) The frequency of compounding is the same as the frequency of payments.

THEOREM 5.4
Future Value of an Annuity

The future value of an ordinary annuity account is given by:

$$\boldsymbol{A = d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]}$$

where r is the annual interest rate, d is the periodic payment, t is the time (in years), and n is the number of payments per year.

Proof: At the end of the first period, the amount in the account is

$$S_1 = d$$

At the end of the second period, the amount is

$$S_2 = d + d\cdot\frac{r}{n} = d\left(1+\frac{r}{n}\right)$$

At the end of the third period, the amount is

$$S_3 = d\left(1+\frac{r}{n}\right) + d\left(1+\frac{r}{n}\right)\frac{r}{n}$$

$$= d\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right) = d\left(1+\frac{r}{n}\right)^2$$

At the end of the fourth period, the amount is

$$S_4 = d\left(1+\frac{r}{n}\right)^2 + d\left(1+\frac{r}{n}\right)^2\frac{r}{n}$$

$$= d\left(1+\frac{r}{n}\right)^2\left(1+\frac{r}{n}\right) = d\left(1+\frac{r}{n}\right)^3$$

We are dealing with sums of the elements in the geometric sequence

$$d,\ d\left(1+\frac{r}{n}\right),\ d\left(1+\frac{r}{n}\right)^2,\ d\left(1+\frac{r}{n}\right)^3,\ \ldots$$

with initial term d and common ratio $\left(1+\frac{r}{n}\right)$.

Applying Theorem 5.3, we arrive at the future value of the investment at the end of t years:

$$A = \frac{d\left[1-\left(1+\frac{r}{n}\right)^{nt}\right]}{1-\left(1+\frac{r}{n}\right)} = \frac{d\left[1-\left(1+\frac{r}{n}\right)^{nt}\right]}{-\frac{r}{n}}$$

multiply numerator and denominator by -1: $$= d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

EXAMPLE 5.8 (a) You put \$120 in a shoe box every month. How much will be in the box at the end of eight years?

(b) You invest \$120 each month in an ordinary annuity earning 4% compounded monthly for eight years. What is the future value of that investment?

SOLUTION: (a) $\$(120 \cdot 12 \cdot 8) = \$11{,}520$

(b) Substituting 120 for d, .04 for r, 12 for n, and 8 for t in the formula of Theorem 5.4 we have:

$$A = d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right] = \$120\left[\frac{\left(1+\frac{.04}{12}\right)^{12\cdot 8}-1}{\frac{.04}{12}}\right] = \$49{,}550.22$$

CHECK YOUR UNDERSTANDING 5.8

You invest \$120 each quarter for eight years in an ordinary annuity at a rate of 4% compounded quarterly. What is the future value of that investment?

Answer: \$4499.29

PERIODIC PAYMENT FOR AN ANNUITY

EXAMPLE 5.9 How much has to be invested each month in an ordinary annuity, at a rate of 3.9% compounded monthly, in order to end up with \$20,000 at the end of five years?

SOLUTION: Plugging the known values into the formula:

$$A = d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

brings us to:

$$20{,}000 = d\left[\frac{\left(1+\frac{.039}{12}\right)^{12\cdot 5}-1}{\frac{.039}{12}}\right] = d(66.13)$$

Solving for d: $$d = \frac{20000}{66.13} = \$302.43$$

Let's generalize the above approach:

THEOREM 5.5
Periodic Payment for an Annuity

The required periodic payment, d, to achieve a given future value A of an ordinary annuity is given by:

$$\boldsymbol{d = \frac{A\left(\frac{r}{n}\right)}{\left(1+\frac{r}{n}\right)^{nt}-1}}$$

where $\boldsymbol{r}$ is the annual interest rate, $\boldsymbol{t}$ is the time (in years), and $\boldsymbol{n}$ is the number of payments per year.

Proof: We start with the formula of Theorem 5.3 and solve for d:

$$A = d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

$$d = \frac{A\left(\frac{r}{n}\right)}{\left(1+\frac{r}{n}\right)^{nt}-1}$$

EXAMPLE 5.10 You would like to acquire \$43,000 by making monthly payments for three years at a rate of 5.2% compounded monthly. How much will you have to pay each month?

SOLUTION: (a) From Theorem 5.5:

You can check this result by plugging it into d in the future value formula of Theorem 5.4.

$$d = \frac{A\left(\frac{r}{n}\right)}{\left(1+\frac{r}{n}\right)^{nt}-1} = \frac{43000\left(\frac{.052}{12}\right)}{\left(1+\frac{.052}{12}\right)^{36}-1} = \$1{,}106.28$$

CHECK YOUR UNDERSTANDING 5.9

How much has to be invested each quarter in an ordinary annuity, at a rate of 3.9% compounded quarterly, in order to end up with \$50,000 at the end of ten years?

Answer: \$1,028.06

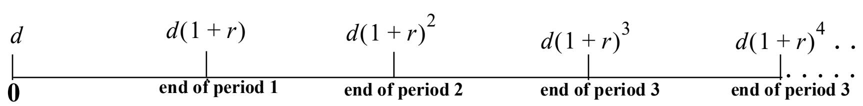
In the previous section, we derived the formula for the future value of an investment P (Theorem 5.1, page 114):

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

And then arrived at the present value formula on page 115:

$$P = A\left(1+\frac{r}{n}\right)^{-nt}$$

Determining the present value of an annuity is considerably more challenging. Why? Because, unlike the situation of the previous section, in an annuity, periodic payments come into play. Consider, for example, an annuity with periodic payments d per year, at an annual interest rate r. Suppose you want to end up with a given amount after t years. You get there via a sequence of cash flows:



To find the present value of each of those individual cash flows, just reverse the process in the above figure:

The present value of the first period is $\dfrac{d}{1+r}$

The present value of the second period is $\left[\dfrac{d}{(1+r)^2}\right]$

Of the third period: $\left[\dfrac{d}{(1+r)^3}\right]$

Of the last period: $\left[\dfrac{d}{(1+r)^t}\right]$

Summing those present values, we arrive at the present value of the annuity:

$$P = \frac{d}{1+r} + \frac{d}{(1+r)^2} + \frac{d}{(1+r)^3} + \cdots + \frac{d}{(1+r)^t}$$

which is the sum of the first t terms of the geometric sequence with leading term $\frac{d}{1+r}$ and common ratio $\frac{1}{1+r}$.

Specifically (see Theorem 5.3):

$$P = \frac{\frac{d}{1+\frac{r}{n}}\left[1-\left(\frac{1}{1+\frac{r}{n}}\right)^{nt}\right]}{1-\frac{1}{1+\frac{r}{n}}}$$

Which can be written in a nicer form:

$$P = \frac{\frac{d}{1+\frac{r}{n}}\left[1-\left(\frac{1}{1+\frac{r}{n}}\right)^{nt}\right]}{1-\frac{1}{1+\frac{r}{n}}} = \frac{\frac{d}{1+\frac{r}{n}}\left[1-\left(\frac{1}{1+\frac{r}{n}}\right)^{nt}\right]}{\frac{1+\frac{r}{n}}{1+\frac{r}{n}}-\frac{1}{1+\frac{r}{n}}} = \frac{\frac{d}{1+\frac{r}{n}}\left[1-\left(\frac{1}{1+\frac{r}{n}}\right)^{nt}\right]}{\frac{r}{n}\left(\frac{1}{1+\frac{r}{n}}\right)}$$

multiply numerator and denominator by (1+r): $= \frac{d\left[1-\frac{1}{\left(1+\frac{r}{n}\right)^{nt}}\right]}{\frac{r}{n}} = d\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right]$

Generalizing, we have:

THEOREM 5.6
Present Value of an Annuity

The present value of an ordinary annuity account is given by:

$$\boldsymbol{P = d\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right]}$$

where $\boldsymbol{r}$ is the annual interest rate, $\boldsymbol{d}$ is the periodic payment, $\boldsymbol{t}$ is the time (in years), and $\boldsymbol{n}$ is the number of payments per year,

EXAMPLE 5.11 The current interest rate for a 30 year mortgage is 6.3%. What price home can you afford, making monthly payments of $2,000?

SOLUTION: Turning to Theorem 5.6, we have:

$$P = d\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right] = \$2{,}000\frac{1-\left(1+\frac{.063}{12}\right)^{-12(30)}}{\frac{.063}{12}}$$

$$= \$323{,}115.98$$

CHECK YOUR UNDERSTANDING 5.10

A bank is offering a 30 year mortgage at 5.7%, following a 15% down payment on the property. You can manage up to \$50,000 for a down payment, and will be able to make monthly payments of \$1,500. What is your upper cost limit?

Answer:
\$304,049.72

The process of making regular periodic partial payments to pay back a debt is called **amortization**. Starting with the Present Value of an Annuity formula of Theorem 5.6 and solving for *d,* we arrive at:

THEOREM 5.7
Amortization

The periodic payments needed in order to pay off a loan is given by:

$$\boldsymbol{d = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}}}$$

EXAMPLE 5.12 You wish to purchase a house valued at \$312,000, and you would like to pay back the loan by making monthly payments. The interest is 0.05, and you intend to repay the loan completely in 25 years.

SOLUTION:

$$d = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}} = \frac{312{,}000\left(\frac{.05}{12}\right)}{1-\left(1+\frac{.05}{12}\right)^{-(12)(25)}} = \$1{,}823.92$$

The amount of interest paid can be calculated by multiplying the monthly payment by the number of months that the payment was made:

$$\$1{,}823.92(12)(25)=\$547{,}176.00$$

and subtracting from that result the present value of the house:

$$\$547{,}176 - 312{,}000 = \$235{,}176.$$

Answer:
Payments: \$2,149.29.
Interest paid: \$140,829.60
Reduced interest cost:
\$128,959.20

CHECK YOUR UNDERSTANDING 5.11

You wish to purchase a house valued at \$375,000, and you would like to pay back the loan by making monthly payments. The interest is 0.06, and you intend to repay the loan completely in 20 years. In order to reduce the interest costs, you decide to make a down payment of 20% of the value of the house. What is your monthly payment, and how much interest are you paying in total? How much less interest are you paying because you made a down payment?

EXERCISES

Exercises 1-3 Find, to two decimal places, the sum of the first 10 terms of the given geometric sequence.

1. 5, 20, 80,...
2. 29, 58, 116,...
3. 100, 50, 25,...

4. You invest $4,000 each year for ten years in an ordinary annuity at a rate of 4% compounded annually. What is the future value of that investment?
5. You invest $5,000 each year for eight years in an ordinary annuity at a rate of 4% compounded annually. What is the future value of that investment?
6. You invest $400 each quarter for eight years in an ordinary annuity at a rate of 0.04% compounded quarterly. What is the future value of that investment?
7. You invest $1,200 each quarter for ten years in an ordinary annuity at a rate of 4% compounded quarterly. What is the future value of that investment?
8. You invest $250 each month for eight years in an ordinary annuity at a rate of 4% compounded monthly. What is the future value of that investment?
9. You invest $450 each month for ten years in an ordinary annuity at a rate of 4% compounded monthly. What is the future value of that investment?
10. You invest $450 each month for ten years in an ordinary annuity at a rate of 4% compounded monthly. What is the future value of that investment?
11. How much has to be invested each quarter in an ordinary annuity, at a rate of 3.9% compounded quarterly, in order to end up with $50,000 at the end of ten years?
12. How much has to be invested each quarter in an ordinary annuity, at a rate of 4% compounded quarterly, in order to end up with $100,000 at the end of ten years?
13. How much has to be invested each quarter in an ordinary annuity, at a rate of 4% compounded quarterly, in order to end up with $100,000 at the end of ten years?
14. How much has to be invested each month in an ordinary annuity, at a rate of 4% compounded monthly, in order to end up with $100,000 at the end of ten years?
15. How much has to be invested each month in an ordinary annuity, at a rate of 4% compounded monthly, in order to end up with $50,000 at the end of five years?
16. How much has to be invested each quarter in an ordinary annuity, at a rate of 4% compounded quarterly, in order to end up with $100,000 at the end of ten years?
17. You can afford to pay $275 per month for a car. What is the most expensive car that you can buy if the interest rate is 8% and you want to pay off the loan in 5 years?
18. You are able to pay $1650 per month to finance the purchase of a house. What is the price of the house that you can afford if the interest rate is 6.5% and you will make the payments for 30 years?
19. You would like to purchase a home that costs $315,000. A financing arrangement is set up for you for monthly payments for 25 years at an interest rate of 7%. What is the monthly payment? What is the total cost of the interest?
20. You would like to reduce the monthly payment of the house discussed in exercise 19. You decide to make a down payment of 15%. Now what is the monthly payment? How much will you save in interest costs?

CHAPTER SUMMARY

SIMPLE INTEREST	$$I = P \cdot r \cdot t$$ Where P is the amount invested r is the annual investment rate n is the number of years invested
COMPOUND INTEREST	If an amount P is invested in an account at r % compounded n times during each year, then the future value of the invested at the end of t years is: $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ In the above, P is the present value of the investment. $$P = A\left(1 + \frac{r}{n}\right)^{-nt}$$
CONTINUOUS COMPOUNDING FUTURE VALUE	If an amount P is invested at r% compounded continuously for t years then: $$A = Pe^{rt}$$
CONTINUOUS COMPOUNDING PRESENT VALUE	The present value P required to achieve a future value A after t years at a rate r compounded continuously is given by: $$P = Ae^{-rt}$$
GEOMETRIC SEQUENCE	The sum of the first n terms of the geometric sequence $$a, ar, ar^2, ar^3, ar^4, ..., ar^n, ...$$ is given by: $$S_n = \frac{a(1 - r^n)}{1 - r}$$

ORDINARY ANNUITY	The future value of an ordinary annuity account is given by: $$A = d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$ where r is the annual interest rate, P is the periodic payment, t is the time (in years), and n is the number of payments per year.

PERIODIC PAYMENT	The required periodic payment, d, to achieve a given future value A of an ordinary Annuity is given by: $$d = \frac{A\left(\frac{r}{n}\right)}{\left(1+\frac{r}{n}\right)^{nt}-1}$$ where r is the annual interest rate, t is the time (in years), and n is the number of payments per year.

PRESENT VALUE OF AN ANNUITY	The present value of an ordinary annuity account is given by: $$P = d\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right]$$ where r is the annual interest rate, d is the periodic payment, t is the time (in years), and n is the number of payments per year.

AMORTIZATION	The periodic payments needed in order to pay off a loan is given by: $$d = \frac{P\left(\frac{r}{n}\right)}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$ where r is the annual interest rate, d is the periodic payment, t is the time (in years), and n is the number of payments per year.

CHAPTER 6
Probability

§1. Definitions and Examples

Roll a die. What is the probability that you roll a 3?

Chances are that you responded with "1 out of 6," or "$1/6$," basing your answer on the fact that there are 6 possibilities: $\{1, 2, 3, 4, 5, 6\}$, with one of them being a 3.

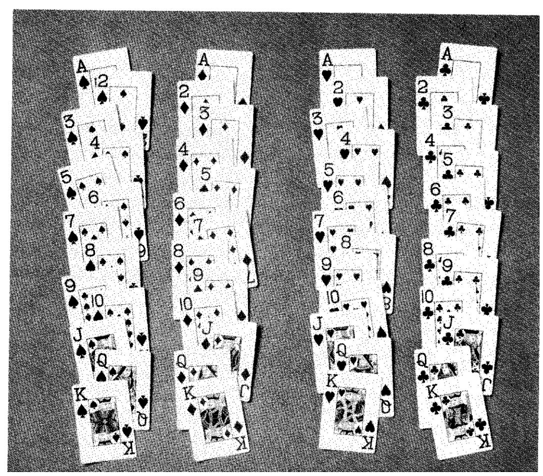
Draw a card from a standard deck.
What is the probability that you draw an ace?

In probability, the form of a number often "tells a story." The form $\frac{4}{52}$ for example, nicely displays the probability of drawing an ace; more so than its reduced form $\frac{1}{13}$. If you want to reduce, go ahead, but you should first display the "story-form:"

$$\mathbf{\frac{4}{52}} = \frac{1}{13} \approx 0.077$$

"story form"

If you are familiar with a deck of cards (Figure 6.1), then you probably answered "4 out of 52," or "$4/52$," for there are 52 cards in a deck, and exactly four of them are aces.



A deck of cards consists of four suits:
Spades ♠, Diamonds ♦, Hearts ♥, and Clubs ♣.
Spades and Clubs are black, Diamonds and Hearts are red.
Each suit consists of 13 cards: Ace, 2, 3,...,10, J, Q, K.
Jacks, Queens and Kings are called face cards
(aces are **not** face cards).

Figure 6.1

In probability, the word **experiment** is used for the activity under consideration (the rolling of a die, or the drawing of a card). The word **success**, or **event**, will be used to denote a specified outcome of the experiment: (rolling a 3, or drawing an ace). In both of the above cases, the probability of an event (or success) turned out to be the number of successes divided by the number of possibilities:

$$Pr(\text{rolling a 3}) = \frac{1}{6} \begin{array}{l} \leftarrow \text{number of successes} \\ \leftarrow \text{number of possibilities} \end{array}$$

$$Pr(\text{drawing an ace}) = \frac{4}{52} \begin{array}{l} \leftarrow \text{number of successes} \\ \leftarrow \text{number of possibilities} \end{array}$$

So far so good. Now roll two dice. What is the probability that you end up with a sum of 7?

Proceeding as above, we consider the 11 possible sums:

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Noting that 7 appears but once in S, we might be tempted to conclude that:

WRONG

$$Pr(\text{rolling a 7}) = \frac{1}{11} \begin{array}{l} \leftarrow \text{number of successes} \\ \leftarrow \text{number of possibilities} \end{array}$$

But 2 also appears exactly one time in the set S, and our chain of thought would lead us to:

$$Pr(\text{rolling a 2}) = \frac{1}{11} \begin{array}{l} \leftarrow \text{number of successes} \\ \leftarrow \text{number of possibilities} \end{array}$$

At this point, you should be feeling a bit uncomfortable with our "findings," since it is certainly more likely to roll a sum of 7 with a pair of dice than it is to roll a sum of two, which can only be done in one way: 1 and 1. The problem stems from the fact that when you roll one die, any of the six faces is as likely to come up as any other. But when you roll two dice, a sum of 7 is more likely to occur than is a sum of 2. The time has come to be a bit more precise:

DEFINITION 6.1
EQUIPROBABLE SAMPLE SPACE

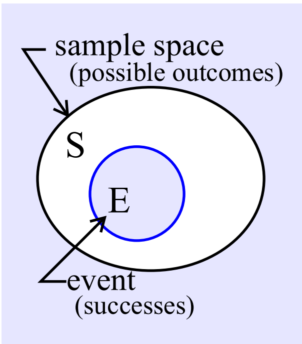
An **equiprobable sample space** for an experiment is a set that represents all of the possible outcomes of the experiment, with each outcome **as likely** to occur as any other.

Note that while $\{1, 2, 3, 4, 5, 6\}$ is an equiprobable sample space for the rolling of a single die (any number is as likely to come up as any other), $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is **not** an equiprobable sample space representing the sum of the roll of two dice (a sum of 2, for example, is less likely to occur than a sum of 7).

Our previous approach to probability:

$$\frac{\textbf{Number of Successes}}{\textbf{Number of Possibilities}}$$

is fine, providing counting takes place in a setting where every outcome of the experiment is as likely to occur as every other:



DEFINITION 6.2

PROBABILITY (When the Sample Space contains finitely many elements.)

Let S be an <u>equiprobable</u> sample space for a given experiment, and let E denote the subset of S (called an **event**) representing the successes of the experiment. The **probability of a success occurring**, or the **probability of the event E**, is given by:

$$Pr(E) = \frac{\#(E)}{\#(S)}$$

where $\#(E)$ and $\#(S)$ denote the number of elements in the sets E and S, respectively.

EXAMPLE 6.1

ROLLING A PAIR OF DICE

Write down an equiprobable sample space for the rolling of a pair of dice. With that sample space at hand, determine the probability of:

(a) Rolling a sum of 7 (b) Rolling a sum of 2

SOLUTION: The two white dice may look alike, but they are not the same. Let's paint one of them red. On rolling the dice, the red die will show a face of 1 through 6, with any of the six faces as likely to show as any other; as will the white die. Denoting, for example, the outcome "1 on the red die and 4 on the white die" in the compact form "$(1, 4)$," we find ourselves face to face with the following equiprobable sample space:

1 on the red and 4 on the white

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & \mathbf{(1,4)} & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ \mathbf{(6,1)} & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}$$

6 on the red and 1 on the white

Figure 6.2

(a) Defining a success to be that of rolling a sum of 7, and noting that there are six such successes in the 36 possibilities in Figure 6.2, we have:

$$Pr(\text{rolling a sum of } 7) = \frac{6}{36} \begin{matrix} \longleftarrow \text{successes} \\ \longleftarrow \text{possibilities} \end{matrix}$$

(b) Since there is but one sum of 2 in the equiprobable sample space (the "(1, 1)"), we have:

$$Pr(\text{rolling a sum of } 2) = \frac{1}{36} \begin{matrix} \leftarrow \text{successes} \\ \leftarrow \text{possibilities} \end{matrix}$$

EXAMPLE 6.2
FLIPPING A COIN

Flip a coin three times. What is the probability that exactly 2 heads are flipped?

SOLUTION: Once you have an equiprobable sample space in front of you:

H H H

H H T **H T H** **T H H** ← successes

T T H T H T H T T

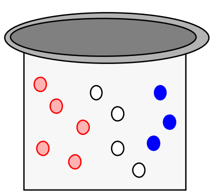
T T T

you need but count to arrive at the answer:

$$Pr(\text{exactly } 2 \text{ H}) = \frac{3}{8} \begin{matrix} \leftarrow \text{successes} \\ \leftarrow \text{possibilities} \end{matrix}$$

EXAMPLE 6.3
MARBLE FROM HAT

A marble is chosen at random from a hat containing 5 red marbles, 4 white marbles, and 3 blue marbles. What is the probability that the marble chosen is white?



SOLUTION: Since the hat contains a total of 12 marbles (the sample space), of which 4 are white (the successes), we have:

$$Pr(W) = \frac{4}{12} \begin{matrix} \leftarrow \text{successes (4 white marbles)} \\ \leftarrow \text{possibilities} \end{matrix}$$

CHECK YOUR UNDERSTANDING 6.1

Five slips of paper, marked 1 through 5, are placed in a hat. Two slips are drawn without replacement (so you can't draw the same number twice). Write down an equiprobable sample space for the experiment (similar to the two-die sample space of Example 6.1), and then use it to determine the probability that:

(a) The first number drawn is odd.

(b) The second number drawn is greater than the first.

Answers: (a) $\frac{12}{20}$ (b) $\frac{10}{20}$

EMPIRICAL PROBABILITY

We began this section by agreeing that the probability of rolling a 3 with a fair die is $Pr(3) = \frac{1}{6}$. We never rolled any die, but took the **theoretical** approach: the die has six faces, and only one of them shows a 3. An **empirical** (or experimental, or statistical approach) would be to make no assumption about the die whatsoever, but to simply roll it a "large" number of times, counting the number of successes along the way. If, for example, a 3 comes up exactly 103 times out of 600 rolls, we would then say that the (empirical) probability of rolling a 3 is $\frac{103}{600} \approx 0.17$.

One should certainly question the fairness of the die, if a 3 came up, say, 200 out of 600 times.

It stands to reason that the more times an experiment is conducted, the more confidence one can place on the resulting empirical probability. Suppose, for example, that you open a carton of twelve eggs and start dropping them, one at a time, from a height of two inches, and that exactly 5 of the eggs break. From this you may conclude that the probability of an egg breaking under such an activity is approximately $\frac{5}{12} \approx 0.42$. A "stronger conclusion" would ensue were you to drop 144 eggs instead of 12, but that would be a gross waste of eggs.

Since empirical probabilities are, by their very nature, approximations, they will be expressed in decimal form.

EXAMPLE 6.4

GRADE IN COURSE

There are 26 students currently enrolled in Professor Chalk's Math 101 course. Professor Chalk has taught that course four times before and here is the grade distribution of those four sections:

Section	A's	B's	C's	D's	F's
1	5	3	16	3	1
2	4	2	18	5	0
3	2	5	12	1	2
4	1	4	13	2	2
Sum:	12	14	59	11	5

Based on the above data, determine the (empirical) probability that a student chosen at random, from the current class, receives an A in the course.

SOLUTION: The number of students in the current class plays no role whatsoever in determining the probability of a randomly chosen student receiving an A in the class, as that probability is based on the data of the instructor's previous four sections.

We calculate the total number of students in those previous sections (the sample space), by summing the total number of grades:

$$12 + 14 + 59 + 11 + 5 = 101$$

In that sample space of 101 students, there were a total of 12 A's (the successes); bringing us to:

$$Pr(\text{A}) = \frac{12}{101} \approx 0.12$$

EXAMPLE 6.5
EXIT POLL

In an exit poll, 620 people indicated that they voted for Proposition 9, and 329 said that they voted against the proposition. Based only on that survey, determine, to two decimal places, the empirical probability that the next person to vote will vote in favor of Proposition 9.

SOLUTION: A total of $620 + 329 = 949$ individuals voted (the sample space), of which 620 voted "Yes" (the success). Consequently:

$$\underset{\substack{\uparrow \\ \text{empirical probability}}}{Pr}(\text{Yes}) = \frac{620}{949} \approx 0.65$$

CHECK YOUR UNDERSTANDING 6.2

A firecracker manufacturer tested 900 lady-fingers and found that all but 32 of them exploded. Determine, to two decimal places, the empirical probability that a randomly chosen lady-finger from the production line will function properly.

Answer: $\frac{868}{900} \approx 0.96$

EXERCISES

Exercises 1-10. Determine an equiprobable sample space for the given experiment.

1. Selecting a month of the year.
2. Flipping 2 coins.
3. Choosing one of three ice cream flavors: chocolate, vanilla, and strawberry, along with one of two toppings: fudge, whipped cream.
4. Answering a three question, True or False test.
5. Flipping a coin and rolling a die.
6. Stacking three blocks, (block A, block B, and block C), one on top of the other.
7. Answering a two-question multiple choice test, with three choices for each question.
8. Mixing the contents of two of four test tubes together.
9. Lining up three ducks, one behind the other.
10. Taking three of four courses: Math, Art, English, Biology.

Exercises 11-14. (**Cards**) Draw a card from a standard deck (Figure 3.1, page 65). What is the probability that the card is:

11. Red?
12. A face card?
13. A red face card?
14. Not a face card?

Exercises 15-18. (**Dice**) You roll two dice. What is the probability that:

15. You roll doubles (same number on both dice)?
16. The first die is a five?
17. The second die is a five?
18. At least one die is a five?

Exercises 19-22. (**Slips of paper**) You draw two slips of paper from a hat containing five slips, marked 1 through 5, **without replacement**. What is the probability that:

19. The number five is chosen?
20. The number five is not chosen?
21. Exactly one of the numbers is a five?
22. Both numbers are fives?

Exercises 23-26. (**Slips of paper**) You draw two slips of paper from a hat containing five slips, marked 1 through 5, **with replacement** (the first number drawn is put back into the hat before drawing the second). What is the probability that:

23. The number five is chosen?
24. The number five is not chosen?
25. Exactly one of the numbers is a five?
26. Both numbers are fives?

Exercises 27-34. (**Coins**) You have a penny, a nickel, a dime, and a quarter in your pocket, and grab two of the coins at random (without replacement). What is the probability that you are holding:

27. Thirty five cents?
28. More than 35 cents?
29. At least 15 cents?
30. More than 15 cents?
31. Less than 10 cents?
32. Less than 25 cents?
33. More than 10 cents?
34. Exactly 10 cents?

Exercises 35-46. (**Coins**) You have 2 pennies, a nickel, a dime, and a quarter in your pocket, and grab two of the coins at random, without replacement. (In developing your sample space, make sure you distinguish between the two different pennies). What is the probability that you are holding:

35. More than 25 cents?
36. More than 26 cents?
37. More than 27 cents?
38. Less than 25 cents?
39. Two cents?
40. 6 cents?
41. Exactly one of the two pennies?
42. At least one of the two pennies?
43. Neither of the pennies?
44. Less than 10 cents?
45. More than 10 cents?
46. Exactly 10 cents?

Exercises 47-50. (**Blocks**) Little Louie has 2 identical blocks, with faces colored red, white, blue, green, yellow, and black. He randomly stacks one block upon the other. What is the probability that:

47. The top of the two-block stack is red?
48. The bottom of the two-block stack is not red?
49. The bottom of the two-block stack is of the same color as the top of the stack?
50. The bottom color of the two-block stack differs from the color at the top of the stack?

51. (**Matching**) You pick a number from 1 to 5, +and your friend does the same. What is the probability that the two of you pick the same number?

Exercises 52-56. (**Lining Up**) Johnny, Mary, and Billy are randomly lined up, one behind the other. What is the probability that:

52. Johnny and Mary are next to each other?
53. Johnny is immediately behind Mary?
54. Johnny and Mary are not next to each other?
55. Billy is first?
56. Billy is in the middle?

Exercises 57-59. (**Baseball**) In the first five games of the season, Bobbie Baseball hit safely 6 of 20 times at bat.

57. What is the (empirical) probability that Bobbie will hit safely the next time at bat?

58. Suppose Bobbie gets a hit in Exercise 57. Calculate the new empirical probability that he will hit safely the next time at bat.

59. Suppose Bobbie makes an out in Exercise 57. Calculate the new empirical probability that he will hit safely the next time at bat.

§2 Unions and Complements of Events

We begin by introducing a bit of set notation:

A set can be described by listing its elements (within brackets), as is done with the set A below (this is called the **roster method**).

$$A = \{-2, 5, 11, 99\}$$

The **descriptive method** can also be used to describe a set. In this method, a statement or condition is used to specify the elements of the set, as is done with the set O below:

$$O = \{x \mid x \text{ is an odd positive integer}\}$$

Read: O is the set of all x **such that** x is an odd positive integer

For a given set A, $x \in A$, is read: ***x* is an element of *A*** (or x is contained in A), and $x \notin A$ is read: x is **not** an element of A. For example:

$$5 \in \{-2, 5, 11, 99\} \quad \text{while} \quad 9 \notin \{-2, 5, 11, 99\}$$

Just as you can add or multiply numbers to obtain other numbers, you can also combine sets to arrive at other sets:

> If someone asks you if you want tea or coffee, you are being offered one or the other, but not both (the **exclusive-or** is used). In mathematics, however, the **inclusive-or** is generally used. In particular, "x is in A or B;" is true even if x is in both A and B.

DEFINITION 6.3
INTERSECTION AND UNION OF SETS

For sets A and B, the **intersection** of A and B, written $A \cap B$, is the set consisting of the elements common to both A and B. That is:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

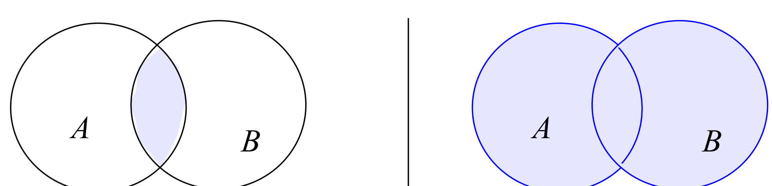
[see Figure 6.3(a)]

The **union** of A and B, written $A \cup B$, is the set consisting of the elements that are in A or in B. That is:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

[see Figure 6.3(b)]

> The adjacent visual representations of sets are called **Venn diagrams**. [John Venn, English logician (1834-1923)].



$A \cap B = \{x: x \in A \text{ and } x \in B\}$

A* intersect *B

(a)

$A \cup B = \{x: x \in A \text{ or } x \in B\}$

A* union *B

(b)

Figure 6.3

For example, if $A = \{3, 5, 9, 11\}$ and $B = \{1, 2, 5, 6, 10, 11\}$, then:

$$A \cap B = \{3, \mathbf{5}, 9, \mathbf{11}\} \cap \{1, 2, \mathbf{5}, 6, 10, \mathbf{11}\} = \{\mathbf{5}, \mathbf{11}\}$$

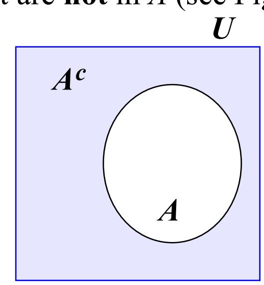
and:

$$A \cup B = \{3, 5, 9, 11\} \cup \{\mathbf{1}, \mathbf{2}, 5, \mathbf{6}, \mathbf{10}, 11\} = \{3, 5, 9, 11, \mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{10}\}$$

What is the intersection of the set $\{1, 2, 3\}$ with the set $\{4, 5\}$? We need a symbol to denote a set which contains nothing, and here it is: $\varnothing$. It is appropriately called the **empty** or **null set**. In particular:

$$\{1, 2, 3\} \cap \{4, 5\} = \varnothing$$

When dealing with sets, one typically has a **universal set** U in mind: a set consisting of all elements under consideration (usually represented as a rectangular region). If A is a **subset** of U (if every element of A is contained in U), then A^c is used to denote the **complement** of A: those elements in U that are **not** in A (see Figue below).



$$A^c = \{x | x \in U \text{ and } x \notin A\}$$

(or simply $\{x | x \notin A\}$ when U is understood)

The complement of *A*

Figure 6.4

For example, if the universal set is $U = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$, then $\{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{8}\}^c = \{\mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{9}\}$.

THE COMPLEMENT THEOREM

Draw a card from a standard deck. What is the probability that it is not a King? That's right: $Pr(\text{not a King}) = \frac{48}{52}$

In answering the above problem, you did not add all of the cards that are not Kings. Rather, you simply subtracted the 4 Kings from the 52 cards:

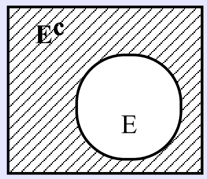
$$Pr(\text{not a King}) = \frac{52 - 4}{52}$$

Breaking the above quotient into two parts, we have:

$$\boldsymbol{Pr}(\textbf{not K}) = \frac{52-4}{52} = \frac{52}{52} - \frac{4}{52} = 1 - \frac{4}{52} = \mathbf{1} - \boldsymbol{Pr}(\mathbf{K})$$

Generalizing:

THEOREM 6.1



For any event E,

$$Pr(\text{not E}) = 1 - Pr(\text{E})$$

More formally:

$$Pr(\text{E}^c) = 1 - Pr(\text{E})$$

Or:

$$Pr(\text{E}) + Pr(\text{E}^c) = 1$$

IN WORDS: The probability that something does not occur is one minus the probability that it does occur.

AND: The probability that something does occur is one minus the probability that it does not occur.

EXAMPLE 6.6
NEGATIVE REACTION

The probability that an individual will have a negative reaction to a medication is 0.07. What is the probability that an individual will not have a negative reaction?

SOLUTION: We simply apply Theorem 6.1:

$$Pr(\text{no neg. reaction}) = 1 - Pr(\text{neg. reaction}) = 1 - 0.07 = 0.93$$

CHECK YOUR UNDERSTANDING 6.3

Draw a card from a standard deck. What is the probability that it is not a face card?

Answer: $\frac{40}{52}$

THE UNION THEOREM

The following example will serve to usher in our next theorem.

EXAMPLE 6.7
DRAW A CARD

Draw a card from a standard deck. What is the probability that it is a Club or a King?

SOLUTION: There are 13 Clubs and there are 4 Kings, but the King of Clubs is **not to be counted twice**:

$$Pr(\text{Club or King}) = \frac{16}{52} \longleftarrow \text{(13 Clubs + 4 Kings – the King of Clubs)}$$

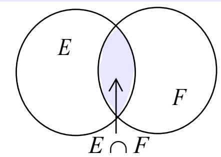
Breaking down the above quotient we find that:

$$\boldsymbol{Pr}(\textbf{C or K}) = \frac{16}{52} = \frac{13+4-1}{52} = \frac{13}{52}+\frac{4}{52}-\frac{1}{52} = \boldsymbol{Pr}(\textbf{C})+\boldsymbol{P}(\textbf{K})-\boldsymbol{Pr}(\textbf{C and K})$$

Generalizing, we have:

PROBABILITY OF THE UNION OF TWO EVENTS.

THEOREM 6.2



For events E and F,

$$Pr(\text{E or F}) = Pr(\text{E}) + Pr(\text{F}) - Pr(\text{E and F})$$

More formally:

$$Pr(\text{E} \cup \text{F}) = Pr(\text{E}) + Pr(\text{F}) - Pr(\text{E} \cap \text{F})$$

In words: The probability that one or another thing happens (possibly both), is the probability that one happens, plus the probability that the other happens, minus the probability that they both happen.

EXAMPLE 6.8

DEFECTIVE LIGHTS AND BRAKES

The probability that a car will have at least one defective light is 0.031, and the probability that a car will have defective brakes is 0.020. Assuming that there is a 0.005 probability that a car will have both a defective light and defective brakes, determine the probability that a car will have either a defective light or defective brakes.

SOLUTION: Let L and B denote the events that a car has a defective light or brakes, respectively. Applying Theorem 6.2, we have:

$$\begin{aligned} Pr(L \overset{\textbf{or}}{\cup} B) &= Pr(L) + Pr(B) - Pr(L \overset{\text{and}}{\cap} B) \\ &= 0.031 + 0.020 - 0.005 = 0.046 \end{aligned}$$

CHECK YOUR UNDERSTANDING 6.4

Draw a card from a standard deck. What is the probability that it is red or a face card?

Answer: $\frac{32}{52}$

DEFINITION 6.4

MUTUALLY EXCLUSIVE EVENTS

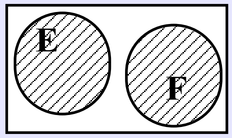
Two events E and F are **mutually exclusive** if they have no element in common:

$$\text{E} \cap \text{F} = \varnothing$$

the empty set

The following result is a special case of Theorem 6.2.

THEOREM 6.3



For **mutually exclusive events** E and F,

$$Pr(\mathrm{E \text{ or } F}) = Pr(\mathrm{E}) + Pr(\mathrm{F})$$

More formally:

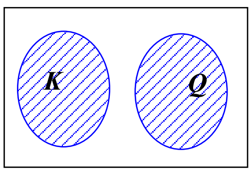
$$Pr(\mathrm{E} \cup \mathrm{F}) = Pr(\mathrm{E}) + Pr(\mathrm{F})$$

PROOF: Since $\mathrm{E} \cap \mathrm{F} = \varnothing$, $Pr(E \cap F) = 0$. Applying Theorem 6.2, we have:

$$\begin{aligned} Pr(\mathrm{E} \cup \mathrm{F}) &= Pr(\mathrm{E}) + Pr(\mathrm{F}) - Pr(E \cap F) \\ &= Pr(E) + Pr(F) - 0 = Pr(E) + Pr(F) \end{aligned}$$

EXAMPLE 6.9
DRAW A CARD

Draw a card from a standard deck. What is the probability that it is a King or a Queen?



SOLUTION: There are 4 Kings and 4 Queens, and if you add them up, nothing is counted twice (so nothing has to be subtracted from the sum):

$$Pr(K \cup Q) = Pr(K) + Pr(Q) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$$

CHECK YOUR UNDERSTANDING 6.5

Draw a card from a standard deck. What is the probability that it is a black face card or a red ace?

Answer: $\frac{8}{52}$

EXERCISES

Exercises 1-6. (**Card**) Draw a card from a standard deck. What is the probability that you draw:

1. A face card?
2. Not a face card?
3. Neither a face card nor an ace?
4. Either a face card or an ace?
5. A face card or a club?
6. Neither a face card nor a club?

Exercises 7-12. (**Dice**) A pair of dice is tossed. What is the probability that their sum is:

7. Odd or greater than 5?
8. Odd and greater than 5?
9. Divisible by 2 or 3?
10. Divisible by 2 and 3?
11. Not divisible by 2 and not divisible by 3?
12. Not divisible by 2 or not divisible by 3?

Exercises 13-18. (**Blocks**) Becky-Boo has three letter-blocks: block-A, block-B, and block-C. She stacks the three blocks, one on top of the other. What is the probability that:

13. The A-block is in the middle?
14. The A-block is not in the middle?
15. The A- or B-block is in the middle?
16. Neither the A- nor the B-block is in the middle?
17. The A-block is either at the bottom or on the top?
18. The letters read ABC in either direction (reading from the bottom up, or from the top down)?

Exercises 19-27. (**Urn**) A marble is drawn at random from an urn containing 4 red marbles numbered 1 through 4, five white marbles numbered 1 through 5, and two blue marbles numbered 5 and 10. Determine the probability that the marble drawn:

19. Is red or displays an even number.
20. Is red and displays an even number.
21. Is not red and displays an even number.
22. Is not red or does not display an even number.
23. Displays the number 10 and is blue.
24. Is not blue and displays the number 10.
25. Is blue or displays the number 10.
26. Is blue and does not display the number 10.
27. Is not blue and does not display the number 10.

§3. Conditional Probability and Independent Events

You get a glimpse of a card as it is dealt to you from a standard deck, and see that it is a Face card. What is the probability that the card is a King? Not $\frac{4}{52}$, since the added information of it being a Face card restricts the 52 card sample space to the 12 Face cards. In that restricted sample space there remain four successes (the four Kings). Hence:

$$Pr(\text{King given that it is a Face Card}) = \frac{4}{12}$$

Notation: $Pr(\text{King}|\text{Face Card}) = \frac{4}{12}$

read: **given**

The above is an example of a **conditional probability** problem, in which a stated condition serves to restrict the possible outcomes of an experiment. The first step in solving such a problem is to determine the **restricted sample space**, and then look for the successes within that restricted sample space.

EXAMPLE 6.10 What is the probability of rolling a pair of fives with two dice, given that at least one die shows a 5.

SOLUTION: First, find the restricted sample space:

$$\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (\mathbf{1},\mathbf{5}) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (\mathbf{2},\mathbf{5}) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (\mathbf{3},\mathbf{5}) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (\mathbf{4},\mathbf{5}) & (4,6) \\ (\mathbf{5},\mathbf{1}) & (\mathbf{5},\mathbf{2}) & (\mathbf{5},\mathbf{3}) & (\mathbf{5},\mathbf{4}) & (5,5) & (\mathbf{5},\mathbf{6}) \\ (6,1) & (6,2) & (6,3) & (6,4) & (\mathbf{6},\mathbf{5}) & (6,6) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} & & & & (\mathbf{1},\mathbf{5}) & \\ & & & & (\mathbf{2},\mathbf{5}) & \\ & & & & (\mathbf{3},\mathbf{5}) & \\ & & & & (\mathbf{4},\mathbf{5}) & \\ (\mathbf{5},\mathbf{1}) & (\mathbf{5},\mathbf{2}) & (\mathbf{5},\mathbf{3}) & (\mathbf{5},\mathbf{4}) & (\mathbf{5},\mathbf{5}) & (\mathbf{5},\mathbf{6}) \\ & & & & (\mathbf{6},\mathbf{5}) & \end{matrix}\right\}$$

Unrestricted Sample Space (see Figure 6.2, page 95)

Restricted Sample Space

Looking at the restricted space of 11 possibilities at the right, we see that there is but one success [the **(5,5)**]. Thus:

$$Pr(\text{Two fives} \mid \text{at least 1 five}) = \frac{1}{11}$$

EXAMPLE 6.11
A CON GAME

Mr. Shill, standing on a corner, displays three cards. One of the cards is red on both sides, one is blue on both sides, and the remaining card is red on one side and blue on the other. Handling the three cards briskly, Mr. Shill slaps one of them on the table, and the side facing you is red.

"Well," says Mr. Shill, "one thing is certain, the card on the table is not the blue-blue card. It is therefore either the red-red card, or the red-blue card, and it is therefore just as likely that the other side of the card is blue as it is that it is red. Tell you what I'm going to do," says our friend, "I'll give you $5 if the card on the other side is blue, if you give me $4 if it is red." Should you accept the wager?

SOLUTION: No. There are three cards, each with two sides. There is no problem distinguishing the two sides of the blue-red card, for they are of different color. To distinguish the two sides of the blue-blue card, we will label one side B_1, and the other side B_2. Similarly the two sides of the red card will be labeled R_1 and R_2. The six possible ways that the three cards can be slapped down on the table is depicted in Figure 6.5(a).

up face **down face** **red face up** **down face**

$$\left\{\begin{array}{cc} B & \mathrm{R} \\ R & \mathrm{B} \\ B_1 & \mathrm{B}_2 \\ B_2 & \mathrm{B}_1 \\ R_1 & \mathrm{R}_2 \\ R_2 & \mathrm{R}_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{cc} \mathbf{R} & \boldsymbol{B} \\ R_1 & \mathrm{R}_2 \\ R_2 & \mathrm{R}_1 \end{array}\right\}$$

Unrestricted Sample Space (a) **Restricted Sample Space** (b) **e**

Figure 6.5

You observe that the face-up card is red, and this restricts the sample space to that of Figure 6.5(b). That restricted sample space has 3 elements, only one of which ($\mathbf{R}_{\text{up}}$- $\boldsymbol{B}_{\text{down}}$) is in your favor. We conclude that the probability that you will win the bet is:

$$Pr(\text{blue down} \mid \text{red up}) = \frac{1}{3}$$

CHECK YOUR UNDERSTANDING 6.6

Roll a pair of dice. (See Figure 6.2, page 95)

(a) What is the probability that their sum is 7, given that at least one die is a 5?

(b) What is the probability that at least one die is a 5, given that their sum is 7?

Answers: (a) $\frac{2}{11}$ (b) $\frac{2}{6}$

PROBABILITY OF TWO EVENTS OCCURRING

By now, you should have little difficulty in answering the following question:

Draw a card from a standard deck. What is the probability that the card is a King, given that the card is Red?

Answer: $Pr(\mathrm{K} \mid \mathrm{R}) = \frac{2}{26}$ ← there are 2 red kings ← there are 26 red cards

Dividing numerator and denominator by 52, we arrive at a representation for the conditional probability as a quotient of "regular" probabilities:

$$Pr(K|R) = \frac{\frac{2}{52}}{\frac{26}{52}}$$

← probability of drawing a red king

← probability of drawing a red card

Which is to say:

$$Pr(\mathrm{K} \mid \mathrm{R}) = \frac{Pr(R \text{ and } \mathrm{K})}{Pr(\mathrm{R})} = \frac{Pr(R \cap K)}{Pr(R)}$$

Generalizing, we have:

THEOREM 6.4 If E and F are events, with $Pr(F) \neq 0$, then:

$$Pr(E|F) = \frac{Pr(F \cap E)}{Pr(F)}$$

A more useful form of the above theorem is obtained by multiplying both sides of the abov4 equation, thereby expressing $Pr(E \cap F)$ in terms of $Pr(E|F)$ and $Pr(F)$:

THEOREM 6.5 Let E and F be two events. Then:

$$Pr(F \cap E) = Pr(F) \cdot Pr(E|F)$$

IN WORDS: The probability of two events occurring, is the probability that the first event occurs, times the probability that the second event occurs **given** that the first event has already occurred.

As is illustrated in the following example, the above theorem can sometimes be used to break down an experiment into two parts:

EXAMPLE 6.12
DRAWING TWO CARDS

Two cards are drawn from a standard deck without replacement. What is the probability that:

(a) You end up with two Kings?

(b) You end up with a King and a Queen?

To solve this problem using Definition 6.2 (page 135) we would need to determine the number of all possible two-card combinations (a big number), along with the number of two-King combinations.

Note how Theorem 6.5 enables us to focus on much smaller sample spaces: the first consisting of a full deck (52 cards) and the second consisting of 51 cards.

SOLUTION:

(a) To end up with two Kings, you must be dealt a King on the first card (K_{1st}) followed by another King on the second card (K_{2nd}). Turning to Theorem 6.5, we have:

$$Pr(\text{K}_{1st} \text{ and } \text{K}_{2nd}) = Pr(\text{K}_{1st}) \cdot Pr(\text{K}_{2nd}|K_{1st})$$
$$= \frac{4}{52} \cdot \frac{3}{51} \approx 0.00452$$

at this stage, there are 51 cards left, with 3 of them Kings.

(b) It is important to note that you can end up with a King and a Queen by being dealt a King first and then a Queen, **or** by being dealt a Queen first and then a King. Both of these possibilities must be taken into account:

$$Pr(\text{K and Q}) = Pr(\text{K}_{1st} \text{ and } \text{Q}_{2nd}) \text{ or } Pr(\text{Q}_{1st} \text{ and } \text{K}_{2nd})$$
$$= \frac{4}{52} \cdot \frac{4}{51} + \frac{4}{52} \cdot \frac{4}{51} \approx 0.012066$$

Taking the ratio of the probability of drawing a King and a Queen with the probability of drawing two Kings:

$$\frac{0.012066}{0.004525} \approx 2.67$$

we find that it is more than twice as likely to be dealt a King and a Queen than it is to be dealt a pair of Kings.

CHECK YOUR UNDERSTANDING 6.7

Two marbles are drawn, without replacement, from a box containing 5 red, 4 blue, and 7 white marbles. Determine the probability of ending up with

(a) Two red marbles. (b) A red and a blue marble.

Answers: (a) $\frac{1}{12}$ (b) $\frac{1}{6}$

INDEPENDENT EVENTS

Roll a die 7 times. Clearly, what happens on, say, the first four rolls of the die will not influence what happens on the next roll. Each roll is independent of the others, in that the outcome of any one of the rolls has no effect on the outcome of any of the others.

In general, to say that two activities are independent, is to say that the occurrence of either of the activities is not dependent on the occurrence of the other. To put it more precisely:

DEFINITION 6.5
INDEPENDENT EVENTS

Two events E and F are **independent** if:

$$Pr(E|F) = Pr(E) \text{ and } Pr(F|E) = Pr(F)$$

The probability of E given F, is simply the probability of E, since E "doesn't care" about F; and ditto for the probability of F given E.

We recall Theorem 6.5 which gives the probability of two events occurring:

$$Pr(E \cap F) = Pr(E)P(F|E)$$

When E and F are independent events, $Pr(F|E) = Pr(F)$, and the theorem takes the following simpler form:

PROBABILITY OF TWO EVENTS OCCURRING

THEOREM 6.6

For **independent** events E and F:

$$Pr(E \cap F) = Pr(E) \cdot Pr(F)$$

IN WORDS: The probability of two (or more) **independent** events occurring is simply the product of their probabilities.

EXAMPLE 6.13
DIE AND COIN

Roll a die and flip a coin. What is the probability that you roll a 3 and flip a Head?

SOLUTION: One approach is to consider a sample space for the experiment of rolling a die and flipping a coin:

success

Sample Space:	1H	2H	**3H**	4H	5H	6H
	1T	2T	3T	4T	5T	6T

Conclusion: $Pr(3 \text{ and H}) = \dfrac{1}{12}$ successes / possibilities

We are still considering sample spaces, but now they are smaller: 6 possible outcomes for the roll of a die, and 2 possible outcomes for the flip of a coin.

Another approach is to think in terms of two independent events, and apply Theorem 6.6:

$$Pr(3 \text{ and H}) = Pr(3) \cdot Pr(\text{H}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

EXAMPLE 6.14
SECURITY CHECK

The probability that a metal object will not be detected at an airport scanning station is 0.01. To improve security, each passenger must pass through three such stations. What is the probability that a metal object will not be detected at any of the three stations?

SOLUTION: All three (independent) stations must "fail" (each with probability 0.01). Applying Theorem 6.6, we conclude that:

$$Pr(\text{All fail}) = (\text{1st fails})(\text{2nd fails})(\text{3rd fails}) = (0.01)^3 = 0.000001$$

We see that the probability of a security breach drops from 1 out of 100 when only one machine is used, to 1 out of a million when three scanning tests are made.

CHECK YOUR UNDERSTANDING 6.8

Roll a die five times. What is the probability that you roll a 4 each time?

Answer: $\left(\frac{1}{6}\right)^5$

EXAMPLE 6.15
HITTING STREAK

Determine the probability that a baseball player with a batting average of .333 (hits safely once out of every three times at bat) will get at least one hit in each of his next 56 consecutive games. Assume that the player comes to bat four times per game.

In 1941, Joe Di Maggio of the New York Yankees did hit safely in 56 consecutive games.

SOLUTION: Before we can calculate the probability that the player will hit safely in 56 consecutive games, we first have to determine the probability that he will hit safely during any given game. Using the complement theorem (Theorem 6.1, page 143) we have:

$$Pr(\text{hits safely in } \mathbf{1} \text{ game}) = 1 - \underbrace{Pr(\text{No hit in } \mathbf{4})}_{*}$$

To find (*), we use the complement theorem again, and first determine the probability that the player will **not** get a hit when he comes to the plate:

$$Pr(\text{does Not get a hit in } \mathbf{1} \text{ time at bat}) = 1 - Pr(\text{hit}) = 1 - \frac{1}{3} = \frac{2}{3}$$

Assuming that whether or not he gets a hit in one time at bat will not influence his next time at bat (independent events), we employ Theorem 6.6, and find the value of (*):

$$\underbrace{Pr(\text{No hit in } \mathbf{4})}_{*} = Pr(\text{Not}) \cdot Pr(\text{Not}) \cdot Pr(\text{Not}) \cdot Pr(\text{Not}) = \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{4}}$$

Thus:

$$Pr(\text{hits safely in } \mathbf{1} \text{ game}) = 1 - \left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\mathbf{4}} \approx 0.8025$$

Alright then, there is a good chance that the player will get a hit in any given game; but to do it 56 games in a row, that's pretty tough. Indeed, assuming independence once more, we find that:

$$Pr(\text{hits safely in } \mathbf{56} \text{ games}) = \overbrace{Pr(\text{safe in } \mathbf{1}) \cdot Pr(\text{safe in } \mathbf{1}) \cdots Pr(\text{safe in } \mathbf{1})}^{56 \text{ times}}$$
$$= Pr(\text{safe in } \mathbf{1})^{56} = \left[1-\left(\frac{2}{3}\right)^4\right]^{56} \approx 0.00000445$$

CHECK YOUR UNDERSTANDING 6.9

At a carnival game you get two shots at a basket. If you play the game five times and if you get at least one basket each of the five times, then you win a large cuddly bear. Assuming that your probability of making a basket is 0.4, determine the probability that you will win the large cuddly bear.

Answer: $(0.4)^5 \approx 0.01$

EXERCISES

1. (**Card**) Draw a card from a standard deck. Determine the probability that:
   (a) It is a spade. (b) It is a spade, given that it is black.
   (c) It is a spade, given that it is not a diamond.
2. (**Die**) Roll a die. Determine the probability that:
   (a) You roll a 5. (b) You roll a 5, given that the number rolled is odd.
   (c) You roll a 5, given that the number rolled is even.
3. (**Urn**) A marble is drawn at random from an urn containing 4 red marbles, five white marbles, and two blue marbles. Determine the probability that the marble drawn:
   (a) Is red. (b) Is red, given that the marble is not white.
4. (**Urn**) A marble is drawn at random from an urn containing 4 red marbles numbered 1 through 4; five white marbles numbered 1 through 5; and two blue marbles numbered 5 and 10. Determine the probability that the marble drawn:
   (a) Is red. (b) Is red, given that the marble is not white.
   (c) Is red, given that the marble displays the number 4.
   (d) Displays the number 4, given that the marble is red.
   (e) Is not red, given that the marble does not display the number 4.
5. (**Dice**) Roll a pair of dice. Determine the probability that the sum is:
   (a) Odd. (b) Odd, given that at least one die is odd.
   (c) Odd, given that exactly one die is odd.
6. (**Dice**) Roll a pair of dice. Determine the probability that the sum is:
   (a) Divisible by 3. (b) Divisible by 3, given that it is divisible by 4.
   (c) Divisible by 5, given that it is divisible by 3.
7. (**Tax Audit**) Agent Eager will audit one of 500 tax forms. Fifty of those forms list a charitable deduction of up to \$500; 250 list a deduction of more than \$500 but not more than \$1000; and the rest, including that of George Generous, list a deduction of more than \$1000. What is the probability that Mr. Generous will be audited if agent Eager chooses the form:
   (a) At random?
   (b) From the forms that list charitable contributions in excess of \$500?
   (c) From the forms that list charitable contributions in excess of \$1000?
8. (**Urn**) Two marbles are drawn, without replacement, from an urn containing 4 red marbles, five white marbles, and two blue marbles. Determine the probability that:
   (a) Both are red.
   (b) The first marble drawn is red, and the second is blue.
   (c) The first marble drawn is blue, and the second is red.
   (d) One of the marbles is red and the other is blue.

9. (**Urn**) Three marbles are drawn, without replacement, from an urn containing 4 red marbles, five white marbles, and two blue marbles. Determine the probability that:
   (a) All are red.
   (b) None is red.
   (c) The first marble drawn is red, and the other two are not.
   (d) Exactly one of the marbles drawn is red.

10. (**Two Cards**) You draw two cards from a standard deck without replacement. Determine the probability that:
    (a) Both are Diamonds.
    (b) Neither is a Diamond.
    (c) Both are of the same suit.
    (d) Both are of the same suit, given that the first card drawn is a Club.
    (e) They are not of the same suit.
    (f) They are not of the same suit, given that the first card drawn is a Club.

11. (**Flags**) There are six different colored signal flags which can be hoisted onto a mast, including a red and a green flag. Two of the six flags are randomly selected and hoisted. Determine the probability that:
    (a) The red flag is hoisted first.
    (b) The red flag is hoisted first, given that the green flag is not hoisted.
    (c) The red flag is hoisted last, given that the green flag is not hoisted.
    (d) The red flag is hoisted last, given that the green flag is hoisted first.

12. (**Die, Coin, and Card**) You roll a die, flip a coin, and draw a card. What is the probability that you roll a 5, flip a Head, and draw an Ace.

13. (**Cards**) Five people draw a card from five different decks. What is the probability that all draw:
    (a) An Ace? (b) An Ace or a King?

14. (**Die**) You roll a die 5 times. What is the probability that:
    (a) The number 1 is rolled each time? (b) An odd number is rolled each time?

15. (**Picking a Number**) Five individuals pick a number from 1 to 10, inclusive. What is the probability that:
    (a) All choose the number 7? (b) All choose the same number?

16. (**Baseball**) Of the 355 batters he faced, pitcher Every Whichway retired all but 75. Determine, to four decimal places, the probability that Every will:
    (a) Retire the next 2 batters he faces.
    (b) Not retire either of the next two players he faces.
    (c) Rethink the above answers, in light of the fact that Every's retirement average will change slightly after he faces the next batter.

17. (**Roll a Die**) Roll a die until you roll a 5, at which time you stop. Determine the probability that you roll the die:
    (a) Exactly three times. (b) At most three times. (c) At least three times.

18. (**Draw a Card**) You draw a card from a deck. If it is an Ace, you stop. If not, you then replace the card, shuffle the deck, and try again. What is the probability that the game stops on the:
    (a) First draw. (b) Second draw. (c) Third draw. (d) Fourth draw.
    (e) Can the game go on "forever?" **If not**, what is the maximum number of draws before the game will end?

19. (**Tournament**) In a single-elimination handball tournament, 16 players are arbitrarily assigned a number from 1 through 16. The winner of the tournament will be awarded the prestigious golden gidget, and the second-place winner will receive the silver gidget. In the first round, player 1 plays player 2, player 3 plays 4, and so on. In the second round the winner of 1-versus-2 plays the winner of 3-versus-4, and so on. Assuming that in each game the better player will win, determine the probability that:
    (a) The golden gidget will be awarded to the best player in the group.
    (b) The silver gidget will be awarded to the second best player in the group.

20. (**Keys and Doors**) To get into her office, Catherina has to unlock the door to her building, and then the door to her office. She has 7 keys in her pocket, all of which look alike. One of the seven keys is for the building, and another is for her office. Determine, to three decimal places, the probability that she will enter her office:
    (a) With the second key (the first key chosen opens the building, and the second key, chosen from the remaining 6, opens her office).
    (b) With the third key.

§4. The Fundamental Counting Principle

By now you worked your way through a number of problems, counting your way toward their solution:

$$Pr(\text{success}) = \frac{\text{number of successes}}{\text{number of possibilities}}$$

Fine, but what if the counting gets out of hand? What is the probability, for example, of being dealt four of a kind in a five card hand, or of winning the state lottery, or that at least two students in your class have the same birthday? The definition of probability will not change, but when dealing with large numbers, more fingers are called for. Those fingers are provided by the Fundamental Counting Theorem, which, for all of its awesome power, is rather apparent:

Suppose you are going to take a journey. You start off by choosing one of two paths (A or B); following which you can choose any of three paths (see Figure 6.6). It is clear that a total of $2 \cdot 3 = 6$ journeys are available to you: **Aa**, **Ab**, **Ac**, **Bd**, **Be**, and **Bf.**

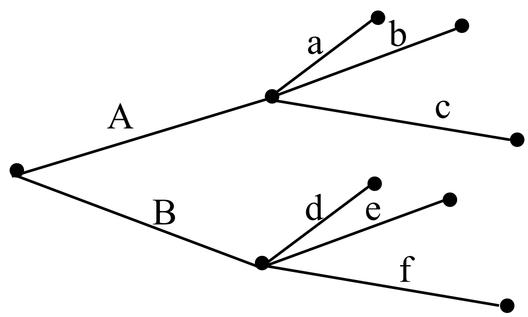


Figure 6.6

Generalizing, we have:

FUNDAMENTAL COUNTING PRINCIPLE

THEOREM 6.7 If there are n choices, each followed by m choices, then there is a total of $n \cdot m$ choices.

EXAMPLE 6.16 Choose a letter and follow it by a digit. How many different possibilities are there?

SOLUTION: Let's write down one of the things we are trying to count:

B 7

We chose the letter **B**, but could of chosen **any of 26** letters. We then chose the digit **7**, but could have chosen **any of 10** digits: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. Applying the Fundamental Counting Principle, we conclude that:

$$\text{Total number of choices} = \mathbf{26} \cdot \mathbf{10} = 260$$

As is illustrated in the following examples, the Fundamental Counting Principle can be used in “longer journeys:”

EXAMPLE 6.17
LICENSE PLATES

How many different license plates are possible if each is to consist of two capital letters, followed by four digits?

SOLUTION: When counting something, you may find it helpful to write down one of the things you are trying to count, say the license plate:

ML 3423

Then step back and ask yourself how many choices you had along the way:

letter choices — digit choices

26 26 10 10 10 10

The Fundamental Counting Principle tells us that there are:

$$26^2 \cdot 10^4 = 6{,}760{,}000$$

different license plates.

CHECK YOUR UNDERSTANDING 6.10

How many different license plates consisting of 2 letters followed by 4 digits are possible, if the two letters must be the same, and the 4 digits must be different.

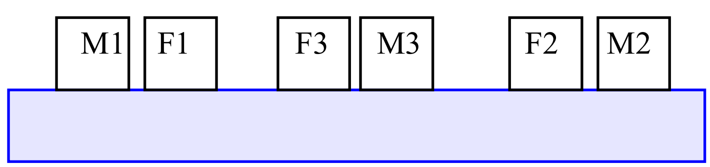
Suggestion: Write down one of the things you are trying to count, and then step back and ask yourself how many choices you had along the way.

Answer: 131,040

EXAMPLE 6.18
SEATING ARRANGEMENT

Three couples (male and female) are to be randomly seated on one side of a rectangular table containing exactly six chairs. In how many ways can this be done if the male and female of each couple are to be seated next to each other?

SOLUTION: Here is a possible seating arrangement:



It is important to note that answers to these kind of problems don't "just happen." You really have to get personally involved; in this case: sit those bodies down!

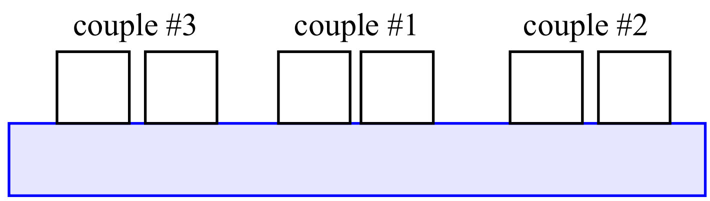
We count the choices along the way:

Anyone can sit in the first chair:	A CHOICE OF 6
Followed by the other half of that couple:	A CHOICE OF 1
Any of the remaining 4 can then be seated in the third chair:	A CHOICE OF 4
Followed by the other half of that couple:	A CHOICE OF 1
Any of the remaining 2 can then be seated in the third chair:	A CHOICE OF 2
Followed by the other half of that last couple:	A CHOICE OF 1

Applying the Fundamental Principle of Counting, we conclude that the number of possible seating arrangements where couples are seated together is: $6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 48$.

AN ALTERNATE SOLUTION:

In seating the 3 couples, we might have written down a specific seating arrangement involving the couples as units, say:



Stepping back, we note that we had a choice of 3 for the couple seated at the left, followed by a choice of 2 for the middle couple, and a choice of 1 for the remaining couple, for a total of $3 \cdot 2 \cdot 1 = 6$ choices. Having decided where the couples sit (6 choices), we still have to choose how each couple is to be seated in their two assigned seats: a choice of 2 for couple 1 (man at left, or woman at left); a choice of 2 for couple 2, and a choice of 2 for couple 3. Putting all of this together we have:

choices for seating the 3 couples

$$3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 2 = 48$$

choices for seating the two individuals in the two chairs

CHECK YOUR UNDERSTANDING 6.11

Referring to the previous example, how many seating arrangements are possible, if the only condition is that all of the men are to be seated next to each other at the table?

Answer: 144

Back to Probability

Actually, we're not going back too far, since probability involves counting two things: possibilities and successes.

EXAMPLE 6.19
Vanity Plate

Assume that license plates consist of two letters followed by four digits. You purchase plates for your car. What is the probability that the plates begin with the initial of your first name, followed by the initial of your family name?

We can't stress this enough:
JOT DOWN ONE OF THE THINGS YOU ARE TRYING TO COUNT
This will help you to better focus on the problem at hand.

Solution: Thee are $26^2 \cdot 10^4$ possible license plates (see Example 6.17)
You now have to count successes, and it is important that you begin by writing down one of the things you are trying to count. If you happen to be John Smith, for example, then you might write down:

JS 3414

Now stand back and count the number of choices along the way:

Number of Successes

J S 3 4 1 4

one choice you **had** to write down your initials

1 1 10 10 10 10

Total: $1^2 \cdot 10^4 = 10^4$

Knowing the number of possibilities (sample space), and the number of successes, we arrive at the answer:

$$Pr(\text{initials}) = \frac{10^4}{26^2 \cdot 10^4} = \frac{1}{26^2} = \frac{1}{676} \approx 0.0015$$

CHECK YOUR UNDERSTANDING 6.12

License plates consist of 2 letters followed by 4 digits. What is the probability that the two letters in a randomly chosen plate are the same, and that the first and fourth digit of that plate are also the same?

Answer: $\frac{1}{260}$

EXAMPLE 6.20
Pizza Palace

Suppose you remember the first three digits and the last two digits of the (7 digit) phone number of the Pizza Palace. You also recall that none of the last four digits of their phone number are the same. Taking this into consideration, you hungrily cross you fingers and dial a number. What is the probability that you reach the Pizza Palace on your first try?

SOLUTION: We first determine the number of possible phone numbers (sample space). To better focus, we assume that the first three (known) digits of the phone number are 242; and that the last two (known) digits are 79 (they could not be 77, for example, since we are told that the last 4 digits are all distinct). With this in mind, we write down one of the numbers that might be dialed:

242- **62**79

Note that we picked these two numbers in such a way that none of the last four digits are the same (given information). We chose to write down a 6 and a 2, but know that our 6 could have been **any of 8** digits (any digit other than the last two chosen digits), and that our 2 could then have been **any of the remaining 7** digits. This leads us to:

Number of Possibilities

	2	4	2	-	**6**	**2**	7	9
	↓	↓	↓		↓	↓	↓	↓
choices:	1	1	1		**8**	**7**	1	1

fixed--no choice

Total: $1 \cdot 1 \cdot 1 \cdot 8 \cdot 7 \cdot 1 \cdot 1 = 56$

Now we have to count the successes—all **one** of them, right? Thus:

$$Pr(\text{dial correctly}) = \frac{1}{56} \approx 0.018$$

CHECK YOUR UNDERSTANDING 6.13

There are four letter blocks: Block A, Block B, Block C, and Block D. Three of the blocks are randomly chosen and laid side by side. What is the probability that the blocks spell CAD, BAD, or DAB?

Answer: $\frac{3}{24}$

A WORD OF CAUTION

We have a question for you:

A true/false quiz consists of 10 questions. In how many different ways can exactly two of the 10 questions be answered incorrectly?

Please try to answer the above before reading on.

How can it be both reasonable and incorrect? Easy, one can certainly analyze a problem in a rational manner, and simply forget to take something or other into account. This happens often in probability, and beyond.

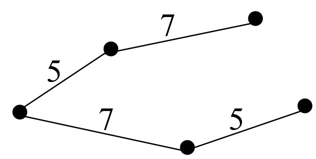
Here is a "reasonable," **but incorrect**, approach:

Two of the 10 questions are going to be answered incorrectly; say *Question 5* and *Question 7*. We said "*Question 5*," but could have chosen any of the 10 questions (choice of 10); and then we said "*Question 7*," but could have chosen any of the remaining 9 questions. "*Choices followed by choices*," and we are lead to the following **INCORRECT** conclusion:

> There are $10 \cdot 9 = 90$ different ways of answering 2 of the 10 questions incorrectly.

WHAT'S WRONG WITH THE ABOVE ARGUMENT?

The point is that questions 5 and 7 could be marked incorrectly in two ways: *take the "5-road" before taking the "7-road,"* or *"take the 7-road before the "5 road"*:



The Fundamental Principle of Counting counts these as two different journeys. So, our initial answer of $10 \cdot 9 = 90$ is incorrect by a factor of 2. The correct answer is $\frac{10 \cdot 9}{2} = 45$.

Exercises

Exercises 1-11. (License Plates) How many different license plates consisting of two letters followed by four digits are possible if:

1. The two letters are different, and the four digits are all the same?
2. The two letters are the same, and the four digits are all odd?
3. The letter A appears first?
4. The letter A appears last?
5. The letter A appears exactly one time?
6. The letter A appears?
7. All of the digits are greater than 5, and the two letters are different?
8. The two letters are different and the first digit is a 0?
9. The two letters are the same and the last digit is a 0?
10. The two letters are the same, all the numbers are different, and the last digit is a 0?
11. The last digit is a four, and the digits increase as you read from left to right?

Exercises 12-18. (Phone Numbers) How many different 10-digit phone numbers are possible if:

12. The number's prefix (first 3 digits) is 201?
13. The number's prefix is either 201 or 914?
14. None of the first three digits is a 7?
15. Exactly two of the first three digits are 7?
16. At least two of the first three digits are 7?
17. None of the first three digits is even, and all of the remaining digits are divisible by 4?
18. The sum of the first three digits is odd?

19. **(Meal)** Each meal at Moe's restaurant comes with one of two soups, one of three salads with one of four salad dressings, one of nine entrées, and one of five desserts. How many different meals are available?
20. **(Barbara Dolly)**. Dolly has three pairs of shoes, four sweaters (one red), two skirts (one orange), and five hats. Determine the number of possible outfits consisting of:
    (a) A sweater, a skirt, a pair of shoes, and a hat.
    (b) A sweater, a skirt, a pair of shoes, and a hat or no hat at all.
    (c) A sweater, a skirt, a pair of shoes, and a hat; but not the red sweater with the orange skirt.
21. **(Social Security Number)** A typical social security number is of the form 986-87-9832. Assuming all such numbers are possible, and that one is drawn at random; what is the probability that:
    (a) The first three digits are all the same?
    (b) The digit 0 does not appear, and the last four digits are the same?
    (c) The digit 0 does not appear, and the last four digits are all different?
    (d) No digit is repeated, and the last digit is a 0?

22. **(Lottery)** In the pick-4 lottery, 4 digits are drawn randomly in order (drawing 1355 is not the same as drawing 1535). You buy one ticket. Determine the probability that:
    (a) You win the jackpot.
    (b) All but your first digit matches the winning four digit number.
    (c) Exactly three of your four digits matches those of the winning four digits number.
    (d) At least three of your four digits match those of the winning four digit number.

23. **(Molecules)** The "sub-molecules" adenine (A), cytosine (C), guanine (G), and thymine (T), link together to form larger molecules in genes. The same sub-molecule can appear more than once, and the order of linkage is a distinguishing factor (the three molecule link C-A-A, for example, is different than A-C-A). Assume that any linking is as likely to occur as any other. What is the probability that:
    (a) A 4-link molecule contains but one of the four sub-molecule types.
    (b) None of the four sub-molecules occurs more than once in a 4-link molecule.
    (c) One of the four sub-molecules occurs at least three times in a 4-link molecule.
    (d) The adenine sub-molecule occurs exactly twice in a 3-link molecule.
    (e) One of the four sub-molecules occurs at least twice in a 3-link molecule.

24. **(Disk Player)** Your friend has a disk collection consisting of 15 Jazz disks, 20 Blues disks, 18 Soft-Rock disks, and 22 Hard-Rock disks. At a party, he randomly chooses 5 of the disks and loads them in a 5-disk player. What is the probability that:
    (a) The first disk is Jazz?
    (b) None of the Hard-Rock disks is loaded?
    (c) All five disks are either Jazz or Blues
    (d) The first disk is a Blues disk, and the fifth disk is not a Blues disk?
    (e) The middle three disks are Soft-Rock?
    (f) Only the middle three disks are Soft-Rock?

25. **(Three Digit Integer)** A three digit positive integer is randomly chosen. (Don't forget that a three digit integer cannot start with a 0.) Determine the probability that:
    (a) All three digits are the same.
    (b) The integer is odd.
    (c) The digit 0 does not appear.
    (d) The digit 0 appears.
    (e) The digit 0 appears exactly one time.

26. **(Seating Arrangement)** Three couples (male and female) are randomly seated at a round table containing exactly six chairs. What is the probability that:
    (a) The male and female of each couple are seated next to each other?
    (b) The three males are seated next to each other?
    (c) No two males are seated next to each other?

§5. Permutations and Combinations

Some of the versatility of the Fundamental Counting Principle were featured in the previous section. In this section, we focus on three important consequences of that principle: (1) Ordering n objects, (2) Selecting r objects from a collection of n objects, when order of selection plays a distinguishing role, and (3) Taking r objects from a collection of n objects, when order of selection does **not** matter.

ORDERING *n* OBJECTS (PERMUTATIONS)

EXAMPLE 6.21
Batting order

A little league baseball coach is to submit a batting order for the 9 children on her team. How many different batting orders are possible?

SOLUTION:

SEE THE ANSWER

choice of 9 for first batter

choice of 8 for 2nd batter

and so on

$\underline{\mathbf{9}}\ \underline{\mathbf{8}}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

Applying the Fundamental Counting Principle, we find that:

$$\text{Number of batting orders} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$$

The above answer turned out to be the product of the first 9 positive integers, leading us to the following useful notation:

Note: Graphing calculators have a "!- button."

A justification for $0! = 1$ is offered at the end of the section.

DEFINITION 6.6
FACTORIAL

For any positive integer n, the symbol $n!$, (read ***n*-factorial)** denotes the product of the integers from 1 to n, inclusive:

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

In addition: $0! = 1$

Generalizing, we have:

Any such ordering is said to be a **permutation of those *n* objects**.

THEOREM 6.8 There are $n!$ different ways of ordering n objects.

For example:

There are 12! different ways of ordering 12 books on a shelf.
$(12! = 479{,}001{,}600)$
There are 35! different ways of ranking 35 individuals.
$(35! \approx 1.03 \times 10^{40})$

An important point of view: In probability, when you see a factorial expression, such as 6!, 15!, or 21!, you should not so much see it as a number, but rather as an abbreviated statement:

6!: The number of ways of ordering 6 objects.
15!: The number of ways of ordering 15 objects.
21!: The number of ways of ordering 21 objects.

Answer: 5,040

CHECK YOUR UNDERSTANDING 6.14

In how many different ways can you arrange 7 books on a shelf?

EXAMPLE 6.22

GIRLS BAT FIRST

A little league baseball coach is to submit a batting order for the 9 children on her team, exactly 4 of whom are girls. She decides to randomly order the players. What is the probability that all the girls will bat first?

SOLUTION:

SEE THE PROBLEM

Success: $\overbrace{\underset{\text{1st}}{\_}\ \_\ \_\ \underset{\text{4th}}{\_}}^{\text{the 4 girls}}\ \_\ \_\ \_\ \_\ \_$

From Example 6.21 we know that there are 9! possible outcomes of the experiment (as many ways as you can order the 9 children). We now have to figure out the number of successes (number of batting orders with the 4 girls batting first). The answer is a blending of Theorem 6.8 and the Fundamental Counting Principle (Theorem 6.7, page 118):

there are 4! ways of ordering the 4 girls

Success: $\overbrace{\_\ \_\ \_\ \_}\ \underbrace{\_\ \_\ \_\ \_\ \_}$

followed by 5! ways of ordering the 5 boys

4! choices followed by 5! **Total choices: 4! times 5!**

Conclusion: $Pr(\text{4 girls bat first}) = \dfrac{4!5!}{9!}$ ← successes / ← possibilities

$$= \frac{(1\cdot 2\cdot 3\cdot 4)(1\cdot 2\cdot 3\cdot 4\cdot 5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot \mathbf{7}\cdot 8\cdot \mathbf{9}}$$

"cancel like crazy:" $= \dfrac{1}{126} \approx 0.008$

CHECK YOUR UNDERSTANDING 6.15

Fifteen children are randomly lined up, one after the other. What is the probability that the tallest child is first and the shortest child is last?

Answer: $\frac{1}{210}$

SELECTING r OBJECTS FROM n, WHEN ORDER COUNTS (PERMUTATIONS OF n OBJECTS TAKEN r AT A TIME)

EXAMPLE 6.23
BATTING ORDER

The baseball coach of Example 6.21 is again to submit a 9-player batting order. This time, however, she is to select the 9 players from 13 available players. In how many ways can this be done?

SOLUTION:

SEE THE ANSWER

choice of 13 for the first batter

choice of 12 for the second batter

and so on

$\underline{\mathbf{13}}\ \ \underline{\mathbf{12}}\ \ \underline{11}\ \ \underline{10}\ \ \underline{9}\ \ \underline{8}\ \ \underline{7}\ \ \underline{6}\ \ \underline{5}$

Applying the Fundamental Counting Principle, we have our answer:

$$\text{batting orders} = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 259{,}459{,}200$$

It behooves us to write the above product in a different form, as it will lead us to a useful generalization:

$$13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{13!}{4!} = \frac{13!}{(13-9)!} \qquad \boxed{\frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 1}$$

Here is some notation that ties in with the above expression:

DEFINITION 6.6
PERMUTATIONS OF n OBJECTS TAKEN r AT A TIME. ORDER COUNTS.

For integers r and n, with $0 \le r \le n$, the symbol $P(n, r)$ is that number given by:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Returning to our result of Example 6.23, we can now say that there are $P(13, 9) = \frac{13!}{(13-9)!}$ ways of picking a 9-player batting order from 15.

Generalizing, we have:

Any such selection is said to be a **permutation of *n* objects taken *r at a time***. With this terminology, one says that there are $P(n, r)$ permutations of n objects taken r at a time.

THEOREM 6.8 There are $P(n, r)$ ways of selecting, without replacement, r objects from a collection of n objects when order is a distinguishing factor.

For example:

There are $P(21, 3)$ different ways of awarding a 1st, 2nd, and 3rd prize to 3 individuals from a group of 21:

$$P(21, 3) = \frac{21!}{(21-3)!} = \frac{21!}{18!} = \frac{\cancel{18!} \cdot 19 \cdot 20 \cdot 21}{\cancel{18!}} = 19 \cdot 20 \cdot 21 = 7980$$

CHECK YOUR UNDERSTANDING 6.16

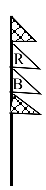
Fifteen individuals are competing in a figure skating competition. In how many different ways can a Gold, Silver, and Bronze medal be awarded to the participants?

Answer: 2,730

EXAMPLE 6.24
FLAGS

There are seven different colored signal flags which can be hoisted onto a mast, two of which are red and blue. Four of the flags are randomly selected and hoisted. What is the probability that both the red and the blue flags are hoisted and that the red is immediately above the blue?

SOLUTION: Applying Theorem 6.8 we easily find the number of possible ways four of the flags can be hoisted: $\boldsymbol{P(7, 4)}$



We now count successes and begin by focusing on one of them (see margin):

The "success journey"
We hoisted the red flag second, but had a **CHOICE OF 3** positions (could not have hoisted last since the blue flag must be below it).
Had to hoist the blue flag right after the red **CHOICE OF 1**.
We also had to hoist two **other** flags. Pick 2 of the remaining 5 flags (order counts): $P(5, 2)$

Applying the Fundamental Counting Principle we multiply the choices to arrive at the number of successes: $\boldsymbol{3 \cdot 1 \cdot P(5, 2)}$:

$$\Pr(\text{R above B}) = \frac{\mathbf{3 \cdot 1 \cdot P(5, 2)}}{\mathbf{P(7, 4)}} = \frac{3 \cdot \frac{5!}{3!}}{\frac{7!}{3!}} = 3 \cdot \frac{5!}{3!} \cdot \frac{3!}{7!}$$
$$= 3 \cdot \frac{5!}{7!} = 3 \cdot \frac{5!}{5! \cdot 6 \cdot 7}$$
$$= \frac{3}{6 \cdot 7} = \frac{1}{14}$$

Note: Graphing calculators can perform the $P(n, r)$ calculation.

CHECK YOUR UNDERSTANDING 6.17

Referring to Example 6.24, what is the probability that red is above blue, but not immediately above blue?

Answer: $\frac{1}{14}$

SELECTING *r* OBJECTS FROM *n*, WHEN ORDER DOES NOT COUNT (COMBINATIONS OF n OBJECTS TAKEN r AT A TIME)

At this point we know that there are $P(15, 3)$ ways of electing a President, Vice President, and Secretary from a group of 15 (picking 3 objects from 15 **when order counts**). We now turn our attention to a closely related question:

In how many ways can a committee of 3 be chosen from a group of 15? (Here, **order does not count**)

To arrive at the answer, we reason as follows:

order counts

In the President-VP-Secretary situation, choosing Mary as president, Johnny as vice-president, and Billy as secretary is not the same as choosing Johnny as president, Mary as vice-president, and Billy as secretary (order counts). Indeed, Mary, Johnny, and Billy can end up on the executive board in as many ways as those 3 individuals can be ordered: **3! ways**. The same can be said for any 3 of the individuals.

order does not count

In the committee situation, where order does not count, Mary, Johnny, and Billy (or any other group of 3) should be counted **once— not 3! times**. In other words, the "order counts" situation is **3! bigger** than the "order does not count situation." Consequently, to arrive at the number when order does not count, we simply take $P(15, 3)$, and divide it by 3! (the "repetition factor"):

$$\text{committees of 3 from 15} = \frac{\mathbf{P(15, 3)}}{\mathbf{3!}} = \frac{\mathbf{15!}}{\mathbf{(15-3)!3!}}$$
$$= \frac{12! \cdot 13 \cdot 14 \cdot 15}{12! \cdot 1 \cdot 2 \cdot 3} = \frac{13 \cdot 14 \cdot 15}{6} = 455$$

'Some additional notation:

DEFINITION 6.7

COMBINATIONS OF n OBJECTS TAKEN r AT A TIME. ORDER DOES NOT COUNT.

For integers r and n, with $0 \le r \le n$, the symbol $C(n, r)$ is that number given by:

$$\boldsymbol{C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}}$$

In particular, as we have seen: There are $C(15, 3)$ committees of 3 that can be chosen from a group of 15.

Generalizing we have:

Any such selection is said to be a **combination of n objects taken r at a time**. With this terminology, one says that there are $C(n, r)$ combinations of n objects taken r at a time.

THEOREM 6.9

There are $C(n, r)$ ways of selecting, without replacement, r objects from a collection of n objects when order is **<u>not</u>** a distinguishing factor.

Here is a statement: $C(14, 5)$. It reads:

The number of ways of selecting 5 objects from 14, when order does not count.

And here is how you can calculate that statement:

Note: Graphing calculators can perform the $C(14, 5)$ calculation,

$$\boxed{14! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14}$$

$$C(14, 5) = \frac{14!}{(14-5)!5!} = \frac{14!}{9!5!} = \frac{\cancel{9!} \cdot \cancel{10} \cdot \mathbf{11} \cdot \cancel{12} \cdot \mathbf{13} \cdot \mathbf{14}}{\cancel{9!} \cdot \mathbf{1} \cdot \cancel{\mathbf{2}} \cdot \cancel{\mathbf{3}} \cdot \cancel{\mathbf{4}} \cdot \cancel{\mathbf{5}}} = \mathbf{11} \cdot \mathbf{13} \cdot \mathbf{14} = 2002$$

CHECK YOUR UNDERSTANDING 6.18

In draw-poker, you are dealt 5 cards (from 52) face down (so order does not count). How many different poker hands are possible?

Answer: 2,598,960

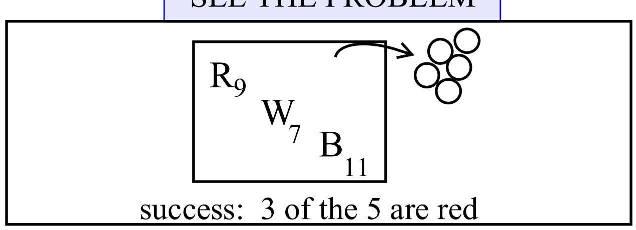
EXAMPLE 6.25

URN

An urn contains 9 red marbles, 7 white marbles, and 11 blue marbles. You reach in and grab 5 of the marbles. What is the probability that exactly 3 of the marbles you are holding are red?

SOLUTION:

SEE THE PROBLEM



There are as many possible outcomes of the experiment as there are ways of grabbing 5 objects from 27: $C(27, 5)$.

There are as many successes as there are ways of grabbing 3 of the 9 red marbles: $C(9, 3)$, and 2 of the 18 non-red marbles: $C(18, 2)$; for a total of $C(9, 3) \cdot C(18, 2)$ successes. Conclusion:

$$Pr(3 \text{ red}) = \frac{C(9, 3)C(18, 2)}{C(27, 5)} \approx 0.159$$

EXAMPLE 6.26
FOUR OF A KIND

You are dealt 5 cards from a standard deck. What is the probability that you are dealt four of a kind (4 Kings, or 4 Aces, etc.)?

SOLUTION: There are as many possible hands as there are ways of grabbing 5 cards from the deck of 52 cards: $C(52, 5)$.

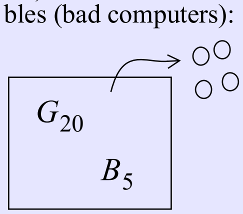
Time to count successes, and here is one of them:

The four Kings and the 3 of spades.

We chose 4 Kings, but could have chosen the four from any of the 13 types (Ace through Kings). **Choice of 13.**
The fifth card can be any of the non-king cards: **Choice of 48.**
Using the Fundamental Counting Principle**,** we arrive at the number of successes: $13 \cdot 48$; and at our answer:

$$Pr(4 \text{ of a kind}) = \frac{13 \cdot 48}{C(52, 5)} \approx 0.00024$$

This is an urn problem consisting of 20 Green marbles (good computers) and 5 Black marbles (bad computers):



EXAMPLE 6.27
DEFECTIVE COMPUTERS

Computer Boutique receives a shipment of 25 computers, exactly five of which are defective. The manager will test four of the units, and return the entire shipment if more than one of the four units malfunctions. What is the probability that the shipment will be returned?

SOLUTION: The formula tells the story; ple

You can use Definition 6.7 to see that $C(5, 0) = 1$, or simply observe that there is but 1 way of grabbing no objects from 5 (don't take any).

$$\begin{aligned} Pr(\text{returned}) &= 1 - [\mathbf{Pr(0\ bad)} + \mathbf{Pr(1\ bad)}] \\ &= 1 - \left[\frac{\mathbf{C(5, 0) \cdot C(20, 4)}}{\mathbf{C(25, 4)}} + \frac{\mathbf{C(5, 1) \cdot C(20, 3)}}{\mathbf{C(25, 4)}}\right] \approx 0.166 \end{aligned}$$

SOLUTION: The formula tells the story; please read it carefully:

CHECK YOUR UNDERSTANDING 6.19

A bag contains 25 red jelly-beans, 7 black jelly-beans, 4 purple jelly-beans, and 9 white jelly-beans. You reach in and grab 8 of the jelly-beans. What is the probability that you end up with 2 of each color?

Answer: ≈ 0.0063

In Definition 6.5, we defined 0! to be 1. Here is one reason in support of that decision:

In how many ways can you grab 5 marbles from a bag containing exactly 5 marbles? Clearly, **1** way.

Plugging into the expression of Theorem 6.9, with $n = r = 5$ we have:

$$\mathbf{1} = C(5, 5) = \frac{5!}{(5-5)!5!} = \frac{5!}{\mathbf{0}!5!}$$

The only way the above can pan out is if $\mathbf{0}! = \mathbf{1}$.

Figure 6.7 summarizes the Fundamental Counting Principle, along with some of its consequences. It is important that you see the first two columns as being "one and the same thing," and to be able to toggle back and forth between the words in column 2 and the mathematical expressions in column 1. The last column just tells you how to calculate the expression in column 1.

Symbol	Represents the number of:	Evaluate
$n \cdot m$	n-choices followed by m-choices	multiply
$n!$	ways of ordering of n objects	$n! = 1 \cdot 2 \cdot 3 \cdots n$
$P(n, r)$	ways of picking r from n, when order counts	$P(n, r) = \frac{n!}{(n-r)!}$
$C(n, r)$	ways of grabbing r from n, when order does not count	$C(n, r) = \frac{n!}{(n-r)!r!}$

Figure 6.7

EXERCISES

Exercises 1-3. Perform the indicated operation.

1. $4!$
2. $P(10, 7)$
3. $C(10, 7)$

Exercises 4-5. (Ordering n objects)

4. In how many different ways can 9 objects be ordered?
5. In how many different ways can 5 children be seated at 5 desks?

Exercises 6-7. (Picking r objects from n when order counts)

6. In how many different ways can 5 children be seated at 11 desks?
7. In how many different ways can 5 of 7 different colored flags be hoisted on a mast?

Exercises 8-10. (Signals) There are six different colored signal flags which can be hoisted (order counts) onto a mast. One of the flags is red and another is yellow.

8. What is the total number of possible signals consisting of four flags, if the red flag is to be hoisted?
9. What is the total number of possible signals consisting of four flags, if the red flag is not to be hoisted?
10. What is the total number of possible signals consisting of four flags, if neiher the red flag nor the yellow flag is to be hoisted?

Exercises 11-14 (Signals) There are six different colored signal flags which can be hoisted onto a mast (red, white, blue, yellow, green, and black). The six flags are randomly selected and hoisted. What is the probability that:

11. The red flag is at the top or at the bottom?
12. The red flag is at the top?
13. The red, white and blue flags are at the top, in any order?
14. The red is at the top, followed by the white, and then the blue flag?

Exercises 15-18. (Prizes) A first, second, third, fourth, and fifth prize are to be randomly awarded to 5 randomly selected children from a group consisting of 9 girls and 6 boys. What is the probability that:

15. Exactly 3 of the girls receive an award?
16. None of the boys receives an award?
17. Girls receive the top three prizes, and the other two prizes go to two of the boys?
18. Girls receive the top two and the fifth prize, and the other two prizes go to two of the boys?

Exercises 19-20. Grabbing r objects from n when order does not count)

19. A bridge hand consists of 13 cards (order not important). How many different bridge hands are there?
20. In how many different ways can you grab three coins from a box containing 12 coins?

21. (Pizzas) Pizza Shack offers 5 toppings for their regular cheese pizza. How many different pizzas are there which contain:

    (a) 3 toppings? (b) Less than 3 toppings (but at least one)?

22. (Drawing three cards) In how many different ways can you draw 3 cards from a standard deck if the order in which they are drawn:

    (a) Matters? (b) Does not matter?

    (c) The answer in (a) is how many times larger than that in (b)?

Exercises 23-27. (Urn) An urn contains 9 red marbles, 7 white marbles, and 5 blue marbles. You grab 4 of the marbles. What is the probability that:

23. None is red?
24. Two of the marbles are white and the other two are blue?
25. All are of the same color?
26. They are not all of the same color?
27. Each of the three colors is drawn?

Exercises 28-31. (Committee) A committee of 5 is randomly selected from a group consisting of 9 women and 6 men. What is the probability that:

28. None of the men are chosen?
29. Exactly 3 women are chosen?
30. At least 3 women are chosen?
31. At most 3 women are chosen?

Exercises 32-34. (Two Cards) Two cards are dealt from a standard deck. What is the probability that:

32. Both cards are Kings?
33. Neither card is a King?
34. Exactly one of the two cards is a King?

Exercises 35-39. (Poker) You are dealt a 5 card poker hand. What is the probability that:

35. All are clubs?
36. All are of the same suit (all hearts, all clubs, all spades, or all diamonds)?
37. You are dealt a full house (3 of one kind and 2 of another—like 3 Kings and 2 Queens)?
38. You are dealt three-of-a-kind (three (only) of one kind and the remaining two cards are not a pair; for example: three Kings, a Jack and an Ace)?
39. You are dealt a royal flush (the 10, Jack, Queen, King, and Ace of the same suit)?

Exercises 40-44. (Bridge Hand) You are dealt a bridge hand (13 cards). What is the probability that:

40. Every card is a club?
41. All of the cards are of the same suit (all hearts, or all clubs, or all spades, or all diamonds)?
42. You are holding all of the face cards?
43. You are holding nothing but face cards and Aces?
44. No card is lower than a seven? (Ace and face cards are higher than seven.)

45. **(Quality Inspection)** Your company ships boxes of calculators containing 24 units. Five calculators from each box are tested. If all 5 pass inspection the box is shipped. If 4 of the 5 pass inspection the box is again shipped (after replacing the defective unit). If more than 1 of the tested units fails inspection, then the entire box is sent back to production. What is the probability that a boxe is shipped if:
    (a) The box contains exactly 1 defective calculator?
    (b) The box contains exactly 5 defective calculators?
    (c) The box contains exactly 5 non-defective units?

46. **(Birthday Problem)** What is the probability that in a class of 25 students at least 2 have the same birthday? (Assume that each year has 365 days.)
    Suggestion: $P(\text{at least 2 have same birthday}) = 1 - P(\text{all have different birthdays})$

47. **(Lottery)** In a pick-5 lottery, you are to pick 5 numbers from a card containing the numbers 1 through 50 (order is of no consequence). If the 5 numbers you selected match the 5 lottery numbers drawn, you win a million dollars. If exactly 4 of your numbers match, you win \$10,000. You buy one card. What is the probability that you will win:
    (a) A million dollars? (b) \$10,000? (c) Nothing?

§6. Expected Value

We recall the equiprobable sample space for the rolling of two dice:

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

Yes, the sample space contains 36 elements, but if you are hoping to roll a seven, then you really don't care if you get it by rolling a 2 and a 5, or by rolling a 4 and a 3. Your concern is with the sum of the two dice — you are interested in the function, let us call it X, which assigns the number 7 to $(4, 3)$, the number 5 to $(3, 2)$, and so on:

The letter X, rather then f, is typically used in this probability-setting.

In general:

DEFINITION 6.8 Random Variable

A **random variable**, X, is a function that assigns a numerical value to each element of a sample space.

In spite of its name, a random variable is neither random nor is it a variable—it is what it is: **a function**.

EXPECTED VALUE OF A RANDOM VARIABLE

The following table gives the number of children per family, for 1000 families surveyed:

Number of Families	Number of Children
150	0
350	1
325	2
125	3
50	4

The concept of mathematical expectation first appears in Gerolamo Cardano's (1501-1576) book *Liber de Ludo Aleas* (Book on Games and Chance), which many consider to be the first book on probability. One can say that Cardano was truly dedicated to the discipline: Having predicted the date of his death, and realizing on that fatal day that he must have erred somewhere along the line, he resolved the problem and saved his reputation by committing suicide.

To determine the average number of children per family you would simply divide the total number of children by the number of families:

$$\text{Average \# of Children} = \frac{0 \cdot 150 + 1 \cdot 350 + 2 \cdot 325 + 3 \cdot 125 + 4 \cdot 50}{1000}$$

Which can be written in the form:

$$\text{Average \# of Children} = 0 \cdot \frac{150}{1000} + 1 \cdot \frac{350}{1000} + \mathbf{2} \cdot \mathbf{\frac{325}{1000}} + 3 \cdot \frac{125}{1000} + 4 \cdot \frac{50}{1000}$$

Let's look closely at a piece of the above expression, say the term:

$$\mathbf{2} \cdot \mathbf{\frac{325}{1000}}$$

The 2 is a possible number of children, and the $\mathbf{\frac{325}{1000}}$ is the probability that a randomly chosen family has 2 children:

(325 of the 1000 families have 2 children)

In probability, the term **expected value** is used instead of average (or mean). Replacing the word "Average" with "Expected," we find that:

possible number of children

$$\text{Expected no. of Children} = 0\cdot\frac{150}{1000}+1\cdot\frac{350}{1000}+2\cdot\frac{325}{1000}+3\cdot\frac{125}{1000}+4\cdot\frac{50}{1000}$$

probability of occurrence

Generalizing:

DEFINITION 6.9
EXPECTED VALUE

Let the random variable X assume values x_1 through x_n. The **expected value of X**, denoted by $E(X)$, is given by:

$$E(X) = x_1Pr(X=x_1)+\ldots+x_nPr(X=x_n)$$

IN WORDS: The expected value is the sum of the possible values times their probabilities of occurrence.

EXAMPLE 6.28
URN

Three marbles are drawn, without replacement, from an urn containing 3 red marbles, 4 white marbles, and 6 blue marbles. What is the expected number of white marbles drawn?

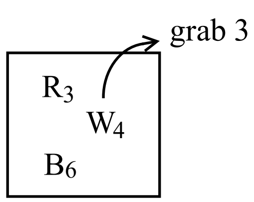
SOLUTION: We suggest that you begin by replacing the word "expected" with the word "**possible**," and ask yourself "what are the **possible** number of white marbles drawn?" This brings you to:

possible number of whites

$$E(\text{Whites}) = 0\,\boxed{}\;+\;1\,\boxed{}\;+\;2\,\boxed{*}\;+\;3\,\boxed{}$$

probability of occurrence

It remains to fill in the four probability boxes. In particular, the $*$-box is to contain the probability of drawing 2 white:



grab 2 of the 4 white and 1 of the 9 non-white

$$Pr(2\text{ White}) = \frac{C(4,2)\cdot C(9,1)}{C(13,3)}$$

grab any 3 of the 13 marbles

The other boxes are filled in the same way, leading us to:

$$E(\text{Whites}) = 0\cdot\frac{C(4,0)\cdot C(9,3)}{C(13,3)}+1\cdot\frac{C(4,1)\cdot C(9,2)}{C(13,3)}$$
$$+2\cdot\frac{C(4,2)\cdot C(9,1)}{C(13,3)}+3\cdot\frac{C(4,3)\cdot C(9,0)}{C(13,3)}\approx 0.92$$

While on any given draw you cannot end up with a fraction of a white marble, on the **average** you can expect to draw 0.92 white marbles. For example, if you conduct the experiment 1000 times, then you can expect to end up with about 920 white marbles

CHECK YOUR UNDERSTANDING 6.20

You are dealt 5 cards from a standard deck. What is the expected number of aces drawn?

Answer: ≈ 0.385

EXAMPLE 6.29 CHECK OUT TIME

The following table records the time, rounded to the nearest minute, that it took for a number of customers on a grocery line to check out. (In particular, from that table we see that 33 customers had a check-out-time of 4 minutes.)

Time	1	2	3	4	5	6	7
Frequency	12	21	35	33	18	5	1

Find the expected waiting time.

A table which displays the number of times each possible value occurs is said to be a **frequency distribution table**.

SOLUTION: One can easily go from the above frequency distribution table to a probability distribution table. The first step is to determine the total number of customers processed (total frequency):

$$12+21+35+33+18+5+1 = 125$$

Dividing the frequency of a value by the total frequency, yields the probability of occurrence of that value:

values

probability of occurrence

Time	1	2	3	4	5	6	7	Sum
Frequency	12	21	35	33	18	5	1	125
Probability	$\frac{12}{125}$	$\frac{21}{125}$	$\frac{35}{125}$	$\frac{33}{125}$	$\frac{18}{125}$	$\frac{5}{125}$	$\frac{1}{125}$	1

Note: being a probability distribution the sum of the probabilities must be 1

For example, since 33 of the 125 individuals are assigned the value of 4 (minutes), the probability of occurrence of that value is: $Pr(4\text{ minutes}) = \frac{33}{125}$

With the probability distribution at hand, we can calculate the expected value of checkout time (average amount of minutes one has to wait at the cash register):

possible values

$$E(\text{Time}) = \mathbf{1}\cdot\frac{\mathbf{12}}{\mathbf{125}} + \mathbf{2}\cdot\frac{\mathbf{21}}{\mathbf{125}} + \mathbf{3}\cdot\frac{\mathbf{35}}{\mathbf{125}} + \mathbf{4}\cdot\frac{\mathbf{33}}{\mathbf{125}} + \mathbf{5}\cdot\frac{\mathbf{18}}{\mathbf{125}} + \mathbf{6}\cdot\frac{\mathbf{5}}{\mathbf{125}} + \mathbf{7}\cdot\frac{\mathbf{1}}{\mathbf{125}} \approx 3.34$$

probability of occurrence

CHECK YOUR UNDERSTANDING 6.21

The manager of Example 3.29 tried out a different scanning machine at the checkout line, and tabulated the following data:

Time	1	2	3	4	5	6	7	8	9
Frequency	15	32	25	19	16	9	3	0	1

Did the checkout time situation improve from that of Example 6.29?

Answer: Yes.

EXPECTED WINNINGS

Monetary expected values in games of chance are often called expected winnings. Consider the following examples.

EXAMPLE 6.30
DRAW A CARD

Maverick is to draw a card from a standard deck. If it is an ace, he wins \$5; if it is a face card, he wins \$2; if he doesn't draw either a face card or an ace, then he loses \$1. Should he play the game?

SOLUTION: The possible values (winnings) are \$5, \$2, and $-\$1$, with probability of occurrence:

$$P(\$5) = \frac{\overset{\text{no. of aces}}{\mathbf{4}}}{52} \qquad P(\$2) = \frac{\overset{\text{no. of face cards}}{\mathbf{12}}}{52} \qquad P(-\$1) = \frac{52-4-12}{52} = \frac{\overset{\text{no. of non-aces and non-face cards}}{\mathbf{36}}}{52}$$

Putting this together, we have:

$$E(\text{Winnings}) = \overbrace{\mathbf{\$5} \cdot \frac{\mathbf{4}}{\mathbf{52}} + \mathbf{\$2} \cdot \frac{\mathbf{12}}{\mathbf{52}} + (\mathbf{-\$1}) \cdot \frac{\mathbf{36}}{\mathbf{52}}}^{\text{possible winnings}} \approx \$0.15$$

Maverick should play the game since, **on the average**, he will win fifteen cents per game.

EXAMPLE 6.31
RAFFLE

A church sells 1000 raffle tickets at a dollar each. You buy one ticket. Two tickets are drawn without replacement. If yours is drawn first, you win \$350; if it is drawn second, you win \$200. What are your expected winnings?

SOLUTION: Since the church is not going to give you back the \$1 you paid for the ticket if you win, the actual possible winnings are \$349, \$199, and $-\$1$.

It is easy to see that the probability of winning the first prize is $\frac{1}{1000}$, and that the probability of not winning any prize is $\frac{998}{1000}$. Please try to figure out the probability that you will win the second prize before reading on.

$$E(\text{Winnings}) = \$349 \cdot \frac{1}{1000} + \$199 \cdot \boxed{?} + (-\$1) \cdot \frac{998}{1000}$$

Here is a "common sense" approach to the problem: The church takes in \$1000 and gives back \$550. It follows that the 1000 ticket holders end up losing \$450—suggesting that, on the average, each ticket holder (including you) will end up losing \$0.45. This, as you will see, turns out to be the correct answer.

Note also that the number in the above box must be $1/1000$ since the sum of the probabilities must be 1.

If you said $\frac{1}{1000}$, you are correct. The probability that you are holding the ticket for any given number (including the number that will be called for the second prize) is $\frac{1}{1000}$.

Conclusion:

$$E(\text{Winnings}) = \$349 \cdot \frac{1}{1000} + \$199 \cdot \frac{1}{1000} + (-\$1) \cdot \frac{998}{1000} = -\$0.45$$

CHECK YOUR UNDERSTANDING 6.22

In a moment of unbridled benevolence, in addition to the \$350 first prize and the \$200 second prize, the church of Example 3.32 decides to award two third prizes of \$50. You purchase one ticket. What are your expected winnings?

Answer: -\$0.35.

EXERCISES

1. **(Coins)** Flip three coins. What is the expected number of Tails tossed?
2. **(Dice)** Roll a pair of dice. Show that the expected sum rolled is 7.
3. **(Delegation)** A delegation of three is to be randomly selected from a group of 5 men and 7 women. Determine, to two decimal places, the expected number of women on the delegation.
4. **(Five Cards)** Draw, without replacement, five cards from a standard deck. Determine, to two decimal places, the expected number of aces drawn.
5. **(Dice)** You roll a pair of dice. If the same number shows on both dice (doubles), you win \$10, otherwise you lose \$1.50. What are your expected winnings?
6. **(Transistors)** A box of 100 transistors contains (exactly) 3 faulty units. Ten of the transistors are randomly chosen. Determine, to two decimal places, the expected number of faulty units chosen.
7. **(Urn)** You draw a marble from an urn containing 5 Red marbles, 3 Blue marbles, and 2 White marbles. If you draw a white marble, you win \$5; if you draw a blue marble, you win \$2; if you draw a red marble you lose \$3.20. What are your expected winnings?
8. **(Urn)** You draw two marbles from an urn containing 5 Red marbles, 3 Blue marbles, and 2 White marbles. If both are white, you win \$5; if both are blue, you win \$2; otherwise you lose \$3.20. What are your expected winnings?
9. **(Cards)** You draw two cards without replacement from a standard deck. If you draw a pair, you win \$10; if both cards are of the same suit, you win \$1; otherwise, you lose \$1. What are your expected winnings?
10. **(Die)** You roll a die. If you roll an odd number, you will receive as many dollars as the number rolled. If you roll an even number, you will lose as many dollars as the number rolled. What are your expected winnings?
11. **(Church Fair)** You pay \$2 for one of 1000 tickets at a church fair. There is a first prize of \$500, two second prizes of \$250 each, and three third prizes of \$100 each. What are your expected winnings?
12. **(Church Fair)** You pay \$1 for one of 1000 tickets at a church fair. Four numbers are drawn. The holder of the first number drawn will win \$500, the holders of the remaining three numbers will each receive \$100. What are your expected winnings?
13. **(Lottery)** In a lottery, you are to pick 5 numbers from a card containing the numbers 1 through 50 (order is of no consequence). If your 5 numbers match the 5 lottery numbers drawn you win a million dollars. If exactly 4 of your numbers match you win \$10,000. You purchase a ticket for \$2. What are your expected winnings?
14. **(Lottery)** In a lottery, you are to pick 5 numbers from a card containing the numbers 1 through 50 (order is of no consequence). To win, your 5 numbers must match the 5 lottery numbers drawn. You purchase a ticket for \$1. What are your expected winnings, if the winning prize is \$1,000,000?
15. **(Coins)** Flip two coins 40 times. What is the expected number of times you will flip two heads?

16. **(Dice)** You roll a pair of dice. If the same number shows on both dice (doubles), you win \$10, otherwise you lose a certain amount. Determine the amount in order for the game to be fair: 0 expected winnings.

CHAPTER SUMMARY

EQUIPROBABLE SAMPLE SPACE

An **equiprobable sample space** for an experiment is a set that represents all of the possible outcomes of the experiment, with each outcome **as likely** to occur as any other.

PROBABILITY OF AN EVENT

Let S be an equiprobable sample space for a given experiment, and let E denote the subset of S (called an **event**) representing the successes of the experiment. The **probability of a success occurring**, or the **probability of the event *E***, is given by:

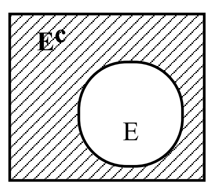
$$Pr(E) = \frac{\#(E)}{\#(S)}$$

EMPIRICAL PROBABILITY

Empirical probability is based on Statistical Evidence

THEOREMS:

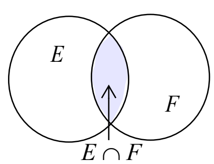
COMPLEMENT



For an event E:

$$Pr(E^c) = 1 - Pr(E)$$

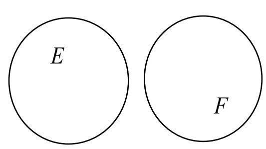
UNION



For events E and F:

$$Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$$

MUTUALLY EXCLUSIVE EVENTS



For **mutually exclusive events** E and F:

$$Pr(E \cup F) = Pr(E) + Pr(F)$$

CONDITIONAL PROBABILITY

Written: $Pr(E|F)$. Pronounced: the probability that the event E occurs, **given** that the event F has occurred.

THEOREM

Let E and F be two events. Then:

$$Pr(F \cap E) = Pr(F) \cdot Pr(E|F)$$

INDEPENDENT EVENTS	Two events E and F are **independent** if: $Pr(E\|F) = Pr(E)$ and $Pr(F\|E) = Pr(F)$ For **independent** events E and F: $$Pr(E \cap F) = Pr(E) \cdot Pr(F)$$

FUNDAMENTAL COUNTING PRINCIPLE	If there are n choices, each followed by m choices, then there is a total of $n \cdot m$ choices.

DEFINITIONS NOTATION AND FACTS	For any positive integer n, the symbol $n!$, (read ***n*-factorial)** denotes the product of the integers from 1 to n, inclusive: $$n! = 1 \cdot 2 \cdot 3 \cdots n \quad \text{In addition:} \quad 0! = 1$$ Fact: There are $n!$ different ways of ordering n objects. **PERMUTATIONS OF *n* OBJECTS TAKEN r AT A TIME** For integers r and n, with $0 \le r \le n$, the symbol $P(n, r)$ is that number given by: $$\boldsymbol{P(n, r) = \frac{n!}{(n-r)!}}$$ Fact: There are $P(n, r)$ ways of selecting, without replacement, r objects from a collection of n objects when order is a distinguishing factor. **COMBINATIONS OF n OBJECTS TAKEN r AT A TIME)** For integers r and n, with $0 \le r \le n$, the symbol $C(n, r)$ is that number given by: $$\boldsymbol{C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}}$$ Fact: There are $C(n, r)$ ways of selecting, without replacement, r objects from a collection of n objects when order is **not** a distinguishing factor.

EXPECTED VALUE OF A RANDOM VARIABLE	Let the random variable X assume values x_1 through x_n. The **expected value of *X***, denoted by $E(X)$, is given by: $$E(X) = x_1 Pr(X = x_1) + \ldots + x_n Pr(X = x_n)$$

CHAPTER 7

METHODS OF PROOF

The following global notation is utilized in the development of this chapter:

$\exists$ read: ***There exists,*** $\ni$ read: ***Such that,*** $\in$ read: ***Is an element of.***

§1. DIRECT PROOF AND PROOF BY CONTRADICTION

The **direct method** of proving $p \Rightarrow q$ (for given propositions p and q) is to assume that p is True and then apply mathematical reasoning to deduce that q is True.

THROUGHOUT THIS SECTION WE WILL BE DEALING EXCLUSIVELY WITH INTEGERS. (A TOUCH OF NUMBER THEORY)

DEFINITION 7.1
EVEN AND ODD

$n \in Z$ is **even** if $\exists k \in Z \ni n = 2k$.

$n \in Z$ is **odd** if $\exists k \in Z \ni n = 2k + 1$.

For example:

$$n = 14 \text{ is even since: } 14 = 2 \cdot 7$$

$$n = 15 \text{ is odd since: } 15 = 2 \cdot 7 + 1$$

EXAMPLE 7.1 Prove that the sum of any two even integers is even.

Note that $n = 2k$ and $m = 2k$ would imply that the two even integers are equal.

SOLUTION: Let n and m be any two even integers, say:

$$n = 2k \quad \text{and} \quad m = 2h$$

Then:

$$n + m = 2k + 2h = 2(k + h) \longleftarrow \text{even}$$

Note how Definition 7.1 was is used in both directions in the above proof. It was used in one direction to accommodate the given information that n and m are even integers, and was then used in the other direction to conclude that $n + m$ is even.

CHECK YOUR UNDERSTANDING 7.1

(a) Prove that the sum of any two odd integers is even.

(b) Formulate a conjecture concerning the sum of an even integer with an odd integer, and then establish the validity of your conjecture.

Answer: See page A-21.

EXAMPLE 7.2 Prove that $2m + n$ is odd **if and only if** n is odd.

SOLUTION: We need to establish validity in both directions; namely:

(1) If $2m + n$ is odd, then n is odd.

(2) If n is odd, then $2m + n$ is odd.

Beginning with (1):

If $2m + n$ is odd, then $2m + n = 2k + 1$ for some k.

Solving for n we have: $n = 2k + 1 - 2m = 2(k - m) + 1$.

Conclusion: n is odd.

Now for (2):

If n is odd, then $n = 2k + 1$ for some k.

Consequently: $2m + n = 2m + 2k + 1 = 2(m + k) + 1$.

Conclusion: $2m + n$ is odd.

CHECK YOUR UNDERSTANDING 7.2

Prove that $2m + n$ is even **if and only if** n is even.

Answer: See page A-21.

PROOF BY CONTRADICTION

This method of proof is called: **proof by contradiction** or, if you prefer Latin: ***reductio ad absurdum***.

One can establish that a proposition p is True by demonstrating that the assumption that p is False leads to a False conclusion. Invoking the "logical commandment" that from *Truth only Truth can follow*, one can then conclude that the assumption that p is False must itself be False, and that therefore p has to be True. Consider the following example:

EXAMPLE 7.3 (a) Prove that if $\mathbf{3n + 2}$ **is even**, then n is even.

(b) Prove that if $\boldsymbol{n^2}$ **is even**, then n is even.

SOLUTION: (a) Let's try to prove that $3n + 2$ even $\Rightarrow n$ even using a direct approach:

$3n + 2 = 2k \Rightarrow 3n = 2k - 2$ — Now what? Trying something like $n = \frac{2k-2}{3}$, takes us out of the realm of integers.

The direct approach does not work. And so we offer a proof by contradiction:

Assume that n is odd, say $\boldsymbol{n = 2k + 1}$. Then:

$$\begin{aligned}\mathbf{3n + 2} &= 3(\mathbf{2k + 1}) + 2\\ &= 6k + 5\\ &= 2(3k + 2) + 1 \longleftarrow \textbf{ odd}\end{aligned}$$

Contradicting the given condition that $\mathbf{3}\boldsymbol{n} + \mathbf{2}$ **is even**.

(b) Once more we find that a direct approach does not work:

$$n^2 = 2k \Rightarrow n = \sqrt{2k} \text{ — Now what?}$$

We turn to a proof by contradiction, that's what:

Assume that n is odd, say $\boldsymbol{n} = \mathbf{2}\boldsymbol{k} + \mathbf{1}$. Then:

$$\begin{aligned} \boldsymbol{n}^{\mathbf{2}} &= (\mathbf{2}\boldsymbol{k} + \mathbf{1})^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \longleftarrow \textbf{odd} \end{aligned}$$

Contradicting the given condition that $\boldsymbol{n}^{\mathbf{2}}$ **is even**.

CHECK YOUR UNDERSTANDING 7.3

Prove that $3n + 2$ is odd if and only if n is odd.

Answer: See page A-22.

TYPICALLY:

A GENERAL ARGUMENT is needed to establish the validity of a True statement.

A (<u>SPECIFIC</u>) COUNTEREXAMPLE is needed to establish that a statement is False.

Consider the following example.

EXAMPLE 7.4

Prove or Give a Counterexample.

(a) If n is odd and m is even then $n - m$ is odd.

(b) The sum of three odd numbers is always divisible by 3.

Solution:

(a) If you don't lean one way or another, you should challenge the statement several times (subtracting an even number from an odd number):

$7 - 2 = 5$ (odd) $\quad -11 - 4 = -15$ (odd) $\quad 3 - (-8) = 11$ (odd)

Each time we ended up with an odd number— **suggesting** that (a) is a True statement. Suggesting, yes—but **not** a proof. Here is a proof:

Let n be odd: $n = 2k + 1$; and let m be even: $m = 2h$. Then:

$$n - m = (2k + 1) - 2h = 2(k - h) + 1, \text{ which is odd!}$$

(b) Yes: $1 + 3 + 5 = 9$ is divisible by 3, as is $1 + 5 + 9 = 15$, but the claim is False. Here is a counterexample:

$$1 + 1 + 5 = 7$$

CHECK YOUR UNDERSTANDING 7.4

Prove or Give a Counterexample.

(a) If $2m + n$ is even then both m and n are even.

(b) $n^2 + 3n + 5$ is odd for all n.

Answer: See page A-22

DEFINITION 7.2
DIVISIBILITY

We say that a nonzero integer a **divides** an integer b, written $a|b$, if $b = ak$ for some integer k.

In the event that $a|b$, we say that b is **divisible** by ***a*** and that b is a **multiple** of a.

Here are two statements for your consideration:

(a) If $a|b$ and $b|c$, then $a|c$.

(b) If $a|bc$, then $a|b$ or $a|c$.

Let's challenge (a) with some specific integers:

Challenge 1: 3 divides 9 and 9 divides 18; does 3 divide 18? Yes.

Challenge 2: 2 divides 4 and 4 divides 24; does 2 divide 24? Yes.

Challenge 3: 5 divides 55 and 55 divides 110; does 5 divide 110? Yes.

The above "Yeses" may certainly suggest that (a) does indeed hold for all $a, b, c \in Z$ — suggest, yes, but **NOT PROVE**. A proof is provided in Theorem 7.1(a), below.

Statement (b): If $a|bc$, then $a|b$ or $a|c$ is False.

A **counterexample:** $6|(2 \cdot 3)$ but $6\nmid 2$ and $6\nmid 3$.

THEOREM 7.1

Let b and c be nonzero integers. Then:

(a) If $a|b$ and $b|c$, then $a|c$.

(b) If $a|b$ and $a|c$, then $a|(b+c)$.

(c) If $a|b$, then $a|bc$ for every c.

PROOF: (a) If $\boldsymbol{a|b}$ **and** $\boldsymbol{b|c}$, then, by Definition 7.2:

$$b = ak \text{ and } c = bh \text{ for some } h \text{ and } k.$$

Consequently:

$$c = bh = (ak)h = a(kh) = at \text{ (where } t = kh\text{)}.$$

It follows from Definition 7.2 that $\boldsymbol{a|c}$.

(b) If $a|b$ and $a|c$, then $b = ah$ and $c = ak$ for some integers h and k. Consequently:

$$b + c = ah + ak = a(h + k) = at \text{ (where } t = h + k\text{)}.$$

It follows that $a|(b+c)$.

(c) If $a|b$, then $b = ak$ for some k. Consequently, for any c:

$$bc = (ak)c = a(kc) = at \text{ (where } t = kc\text{)}.$$

Hence: $a|bc$.

EXAMPLE 7.5 Prove or give a counterexample.
(a) If $a|(b+c)$, then $a|b$ or $a|c$.
(b) If $a|b$ and $a|(b+c)$, then $a|c$.

SOLUTION: (a) Unless you are fairly convinced that a given statement is True, you may want to start off by challenging it:

Challenge 1. $4|(8+16)$, and 4 certainly divides 8 or 16 (in fact, it divides both 8 and 16). Inconclusive.

Challenge 2. $4|(3+1)$, and 4 divides neither 3 nor 1 — a **counterexample**! The statement is False.

(b) Challenging the statement "If $a|b$ and $a|(b+c)$, then $a|c$" will not yield a counterexample. It can't, since the statement is True. To establish its validity, a **general argument** is called for:

Since $a|b$, there exists h such that: (1) $b = ah$.

Since $a|(b+c)$, there exists k such that: (2) $b + c = ak$.

(Now we have to go ahead and show that $c = at$ for some t)

From (2): $c = ak - b$.

From (1): $c = ak - ah = a(k - h)$.

Since $c = at$ (where $t = k - h$): $a|c$.

CHECK YOUR UNDERSTANDING 7.5

(a) Prove:

(a-i) If $a|n$ and $a|m$ then $a|(n+m)$.

(a-ii) If $n|a$ then $n|ca$ for every $c \in Z$.

(b) Prove or give a counterexample:

(b-i) If $a|b$ and $a|c$, then $a|(b+c)$.

(b-ii) If a and b are even and if $a|(b+c)$, then c must be even.

Answer: See page A-22.

As you worked your way through this section, you must have observed that:

DEFINITIONS RULE!

They are the physical objects in the mathematical universe.

EXERCISES

Exercises 1-24. Establish the validity of the given statement**.**

1. 0 is even and 1 is odd.
2. The product of any two even integers is even.
3. The sum of any two odd integers is even.
4. The product of any two even integers is divisible by 4.
5. If $5n - 7$ is even then n is odd.
6. $11n - 7$ is even if and only if n is odd.
7. $3n + 1$ is even if and only if n is odd.
8. n^3 is even if and only if n is even.
9. If n^4 is even then so is $3n$.
10. $3n^3$ is even if and only if $5n^2$ is even.
11. $5n - 11$ is even if and only if $3n - 11$ is even.
12. If $3n + 5m$ is even, then either n and m are both even or they are both odd.
13. The square of every odd integer is of the form $4n + 1$ for some $n \in Z$.
14. Let $b = aq + r$. Prove that if $c|a$ and $c|b$, then $c|r$.
15. If $a|b$ then $a^2|b^2$.
16. $3|n$ if and only if $3|(n + 9)$.
17. If $a|b$, and $a|c$ then $a|(bn + cm)$ for every n and m.
18. If $a|c$, and $b|d$ then $ab|cd$.
19. If a and b are odd positive integers and if $c|(a + b)$, then c is even.
20. If a is odd and b is even and if $c|(a + b)$, then c is odd.

Exercises 25-31. Disprove the given statement:

21. If $2m + n$ is odd then both m and n are odd.
22. The product of any two even integers is divisible by 6.
23. The sum of any two even integers is divisible by 4.
24. The sum of any three odd integers is divisible by 3.
25. If $a|b$ and $b|a$ then $a = b$.

Exercises 26-46. Prove or give a counterexample.

26. If n is odd and m is even then $n - m$ is odd.
27. If $n + m$ is odd then neither n nor m can be even.
28. If $n + m$ is even then neither n nor m can be odd.
29. If $n + m$ is odd then n or m must be odd.
30. If $n + m$ is even then n or m must be even.
31. If $n + 1$ is even then so is $n^3 - 1$.
32. If $n + 1$ is even then so is $n^2 - 1$.
33. If $n + 1$ is even then so is $n^3 + 1$.
34. If $n + 1$ is even then so is $n^2 + 1$.
35. If $n + 1$ is odd then so is $n^3 - 1$.
36. If $n + 1$ is odd then so is $n^2 - 1$.
37. If $n + 1$ is odd then so is $n^3 + 1$.
38. If $n + 1$ is odd then so is $n^2 + 1$.
39. If a is even and b is odd then $a^2 + 2b$ is even.
40. If a is even and b is odd then $a^2 + 3b$ is odd.
41. If a is odd then so is $a^2 + 2a$.
42. If a is odd then so is $a^2 + 3a$.
43. If a is even and b is odd then $(a + 2)^2 + (b - 1)^2$ is even.
44. If $9|(n + 3)$ then $3|n$
45. If $(a + 2)^2 + (b - 1)^2$ is even then a or b has to be even.
46. If $a|b$, $b|c$, and $c|a$, then $a = b = c$.

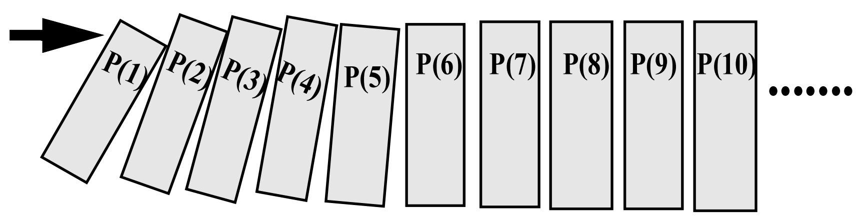
§2. Mathematical Induction

A form of the Principle of Mathematical Induction is actually one of Peano's axioms. [Giuseppe Peano (1858-1932).]

This section introduces a most powerful mathematical tool, the Principle of Mathematical Induction (*PMI*). Here is how it works:

Principle of Mathematical Induction Let $P(n)$ denote a proposition that is either true or false, depending on the value of the integer n.	
If:	I. $P(1)$ is True.
And if, from the **assumption** that:	II. $P(k)$ is True
one can show that:	III. $P(k+1)$ is also True.
then the proposition $P(n)$ is valid **for all integers** $n \geq 1$	

Step II of the induction procedure may strike you as being a bit strange. After all, if one can assume that the proposition is valid at $n = k$, why not just assume that it is valid at $n = k+1$ and save a step! Well, you can assume whatever you want in Step II, but if the proposition is not valid for all n you simply are not going to be able to demonstrate, in Step III, that the proposition holds at the **next** value of n. It's sort of like the domino theory. Just imagine that the propositions $P(1), P(2), P(3), \ldots, P(k), P(k+1), \ldots$ are lined up, as if they were an infinite set of dominoes:



If you knock over the first domino (Step I), and if when a domino falls (Step II) it knocks down the next one (Step III), then all of the dominoes will surely fall. But if the falling k^{th} domino fails to knock over the next one, then all the dominoes need not fall.

The *Principle of Mathematical Induction* might have been better labeled the *Principle of Mathematical Deduction,* for inductive reasoning is used to formulate a hypothesis or conjecture, while deductive reasoning is used to rigorously establish whether or not the conjecture is valid.

To illustrate how the process works, we ask you to consider the sum of the first n odd integers, for $n = 1$ through $n = 5$:

n	Sum of the first n odd integers	Sum
1	1	1
2	1 + 3	4
3	1 + 3 + 5	9
4	1 + 3 + 5 + 7	16
5	1 + 3 + 5 + 7 + 9	25

n	Sum
1	1
2	4
3	9
4	16
5	25
6	?

Figure 7.1

Looking at the pattern of the table on the right in Figure 6.2, you can probably anticipate that the sum of the first 6 odd integers will turn out to be $6^2 = 36$, which is indeed the case. Indeed, the pattern suggests that: The sum of the first n odd integers is n^2

Using the Principle of Mathematical Induction, we now establish the validity of the above conjecture:

Let $P(n)$ be the proposition that the sum of the first n odd integers equals n^2.

I. Since the sum of the first 1 odd integers is 1^2, $P(1)$ is true.

II. **Assume** $P(k)$ is true; that is:

$$\mathbf{1+3+5+\cdots+(2k-1)} = \boldsymbol{k^2}$$

see margin

The sum of the first **3** odd integers is:

$1 + 3 + \boxed{5} \leftarrow \boxed{2 \cdot \mathbf{3} - 1}$

The sum of the first **4** odd integers is:

$1 + 3 + 5 + \boxed{7} \leftarrow \boxed{2 \cdot \mathbf{4} - 1}$

Suggesting that the sum of the first ***k*** odd integers is:

$1 + 3 + \ldots + \boxed{(2k-1)}$

III. We now show that $P(k+1)$ is true, thereby completing the proof:

$$\overbrace{\underbrace{[\mathbf{1+3+5+\cdots+(2k-1)}]}_{\text{induction hypothesis: Step II}} + (\mathbf{2k+1})}^{\text{the sum of the first } k+1 \text{ odd integers}} = \boldsymbol{k^2} + (\mathbf{2k+1}) = (k+1)^2$$

EXAMPLE 7.5 Use the Principle of Mathematical Induction to establish the following formula for the sum of the first n positive integers:

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

SOLUTION: Let $P(n)$ be the proposition:

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} \quad (*)$$

I. $P(1)$ is true: $1 = \frac{1(1+1)}{2}$ Check!

II. **Assume** $P(k)$ is true: $\mathbf{1+2+3+\ldots+k} = \boldsymbol{\frac{k(k+1)}{2}}$

III. We are to show that $P(k+1)$ is true; which is to say, that (*) holds when $n = k+1$. In other words, that:

$$1+2+3+\ldots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

Let's do it:

$$1+2+3+\ldots+k+(k+1) = [\mathbf{1}+\mathbf{2}+\mathbf{3}+\cdots+\boldsymbol{k}]+(\boldsymbol{k}+\mathbf{1})$$

induction hypothesis: $$= \frac{\boldsymbol{k}(\boldsymbol{k}+\mathbf{1})}{\mathbf{2}}+(\boldsymbol{k}+\mathbf{1})$$

$$= \frac{k(\boldsymbol{k}+\mathbf{1})+2(\boldsymbol{k}+\mathbf{1})}{2} = \frac{(\boldsymbol{k}+\mathbf{1})(k+2)}{2}$$

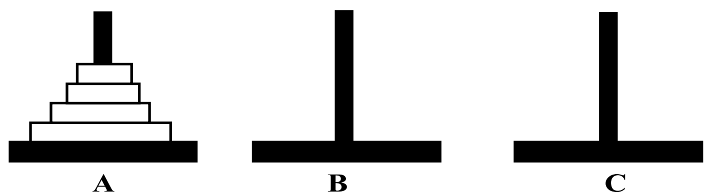
CHECK YOUR UNDERSTANDING 7.6

Prove that the sum of the first n positive even integers is n^2+n

Answer: See page A-22.

Our next application of the Principle of Mathematical Induction involves the following **Tower of Hanoi** puzzle:

Start with a number of washers on spindle A, as is depicted below:



The objective of the game is to transfer the arrangement currently on spindle A to one of the other two spindles. The rules are that you may only move one washer at a time, without ever placing a larger one on top of a smaller one.

Edouard Lucas formalized the puzzle in 1883, basing it on the following legend: *In a temple at Benares, there are 64 golden disks mounted on one of three diamond needles. At the beginning of the world, all the disks were stacked on the first needle. The priests attending the temple have the sacred obligation to move all the disks to the last needle without ever placing a larger disk on top of a smaller one. The priests work day and night at this task. If and when they finally complete the job, the world will end.*

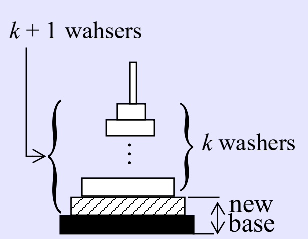
EXAMPLE 7.6 Show that the tower of Hanoi game is winnable for any number n of washers.

SOLUTION: If spindle A contains one washer, then simply move that washer to spindle B to win the game (Step I).

Assume that the game can be won if spindle A contains k washers (Step II —the induction hypothesis).

We now show that the game can be won if spindle A contains $k+1$ washers (Step III):

Just imagine that the largest bottom washer is part of the base of spindle A. With this sleight of hand, we are looking at a situation consisting of k washers on a modified spindle A (see margin). By the induction hypothesis, we can move those k washers onto spindle B. We now take the only washer remaining on spindle A (the largest of the original $k+1$ washers), and move it to spindle C, and then think of it as being part of the base of that spindle. Applying the induction hypothesis one more time, we move the k washers from spindle B onto the modified spindle C, thereby winning the game.



The "domino effect" of the Principle of Mathematical Induction need not start by knocking down the first domino $P(1)$. Consider the following example where domino $P(0)$ is the first to fall.

EXAMPLE 7.7 Use the Principle of Mathematical Induction to establish the inequality $n < 2^n$ for all $n \geq 0$.

SOLUTION: Let $P(n)$ be the proposition $n < 2^n$.

I. $P(0)$ is true: $0 < 2^0$, since $2^0 = 1$.

II. **Assume** $P(k)$ is true: $\mathbf{k < 2^k}$.

III. We show $P(k+1)$ is true; namely that $k+1 < 2^{k+1}$:

$$\mathbf{k} + 1 \underset{\text{II}}{<} \mathbf{2^k} + 1 \underset{1 \leq 2^k}{\leq} 2^k + 2^k = 2(2^k) = 2^{k+1}$$

CHECK YOUR UNDERSTANDING 7.7

For every integer $n \geq 4$, $2n < n!$

Answer: See page A-23.

EXERCISES

Exercises 1-14. Establish the validity of the given statement.

1. For every integer $n \geq 1$, $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{3n^2 - n}{2}$.

2. For every integer $n \geq 1$, $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n - 1)(2n + 1)}{3}$.

3. For every integer $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$.

4. For every integer $n \geq 1$, $4 + 4^2 + 4^3 + \cdots + 4^n = \dfrac{4(4^n - 1)}{3}$.

5. For every integer $n \geq 1$, $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$.

6. For every integer $n \geq 1$, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.

7. For every integer $n \geq 1$, $\dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{(n + 1)(n + 2)} = \dfrac{n}{2n + 4}$.

8. For every integer $n \geq 1$, $\left(1 + \dfrac{1}{1}\right)\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right)\ldots\left(1 + \dfrac{1}{n}\right) = n + 1$.

9. For every integer $n \geq 1$, $\dfrac{(2n)!}{2^n n!}$ is an odd integer.

10. For every integer $n \geq 1$ and any real number $x \neq 1$, $x^0 + x^1 + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$.

11. For every integer $n \geq 0$, $2^n > n$.

12. For every integer $n \geq 5$, $2n - 4 > n$.

13. For every integer $n \geq 5$, $2^n > n^2$.

14. For every integer $n \geq 4$, $3^n > 2^n + 10$.

15. What is wrong with the following "Proof" that any two positive integers are equal:

Let $P(n)$ be the proposition: If a and b are any two positive integers such that $\max(a, b) = n$, *then* $a = b$.

I. $P(1)$ is true: If $\max(a, b) = 1$, then both a and b must equal 1.

II. Assume $P(k)$ is true: If $\max(a, b) = k$, then $a = b$.

III. We show $P(k+1)$ is true:

If $\max(a, b) = k + 1$ then $\max(a-1, b-1) = k$.

By II, $a - 1 = b - 1 \Rightarrow a = \dot{b}$.

CHAPTER SUMMARY

DIRECT PROOF	The **direct method** of proving $p \Rightarrow q$ is to assume that p is True and then apply mathematical reasoning to deduce that q is True.

PROOF BY CONTRADICTION	One can establish that a proposition p is True by showing that the assumption that p is False leads to a contradiction.

MATHEMATICAL INDUCTION	Let $P(n)$ denote a proposition that is either true or false, depending on the value of the integer n. I. Let $P(1)$ be True. II. Assume $P(k)$ is True. III. If it follows that $P(k+1)$ is also True. Then: $P(n)$ is valid **for all integers** $n \geq 1$ More generally: I. Let $P(n_0)$ be True for an integer n_0. II. Assume $P(k)$ is True, for $k > n_0$ III. If it follows that $P(k+1)$ is also True. Then: $P(n)$ is valid **for all integers** $n \geq n_0$

CHECK YOUR UNDERSTANDING SOLUTIONS

CHAPTER 1 A LOGICAL BEGINNING

CYU 1.1 (a) Since q: $3+5 = 8$ is True, $p \vee q$ is True (even though p: $7 = 5$ is False.

(b) Since $7 = 5$ is False, $p \wedge q$ is False (even though $3+5 = 8$ is True).

CYU 1.2 (a) Since p: $5 > 3$ is True, $\sim p$ is False.

(b) Since p is the proposition $\sim q$, $\sim p$ is the proposition q, which *is False.*

CYU 1.3 (a) $\sim(p \wedge q) = \sim(True \wedge True) = \sim True = False$

(b) $\sim(p \vee q) = \sim(True \vee True) = \sim True = False$

(c) $\sim(s \vee q) = \sim(False \vee True) = \sim True = False$

(d) $\sim(s \wedge q) = \sim(False \wedge True) = \sim False = True$

(e) $\sim p \wedge q = \sim True \wedge True = False \wedge True = False$

(f) $\sim p \vee q = \sim True \vee True = False \vee True = True$

(g) $\sim s \vee q = \sim False \vee True = True \vee True = True$

(h) $\sim s \wedge q = \sim False \wedge True = True \wedge True = True$

(i) $(p \vee s) \wedge (q \wedge s) = (True \vee False) \wedge (True \wedge False) = True \wedge False = False$

(j) $(p \vee s) \vee (q \wedge s) = (True \vee False) \vee (True \wedge False) = True \vee False = True$

(k) $\sim(p \vee s) \wedge (q \vee s) = \sim(True \vee False) \wedge (True \vee False)$
$= \sim True \wedge True = False \wedge True = False$

CYU 1.4

p	q	$p \to q$	$q \to p$	$(p \to q) \vee (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

CYU 1.5 Since $p \vee \sim p$ is a tautology: $[(p \to q) \vee q] \Rightarrow (p \vee \sim p)$.

CYU 1.6

(a)

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b)

p	q	$p \to q$	$\sim(p \to q)$	$\sim p \vee q$	$\sim(\sim p \vee q)$
T	T	T	F	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	F	T	F

CYU 1.7 (a) $(-2ax^2)^4 = (-2)^4a^4(x^2)^4 = 16a^4x^8$

(b) $\left(\frac{-x^2}{2}\right)^3 = \frac{(-x^2)^3}{2^3} = \frac{-x^6}{8} = -\frac{x^6}{8}$ (note that, in general: $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$)

(c) $(-x)^2\left(\frac{x}{2y}\right)^3 = x^2 \cdot \frac{x^3}{2^3y^3} = \frac{x^5}{8y^3}$

CYU 1.8 (a) $25x^2 - 1 = (5x)^2 - 1^2 = (5x+1)(5x-1)$

(b) $x^4 - 16 = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$

CYU 1.9 (a) $2x^2 + 7x - 4 = (2x-1)(x+4)$

(b) $18x^4 - 6x^3 - 60x^2 = 6x^2(3x^2 - x - 10) = 6x^2(3x+5)(x-2)$

CYU 1.10 (a)

$$25x^2 - 1 = 0$$
$$(5x+1)(5x-1) = 0$$
$$5x + 1 = 0 \text{ or } 5x - 1 = 0$$
$$x = -\frac{1}{5} \text{ or } x = \frac{1}{5}$$

(b)

$$2x^2 + 7x - 4 = 0$$
$$(2x-1)(x+4) = 0$$
$$x = \frac{1}{2} \text{ or } x = -4$$

CYU 1.11 $\frac{12\text{ oz}}{\text{min}} = \frac{12\cancel{\text{ oz}}}{\cancel{\text{min}}} \cdot \frac{60\cancel{\text{ min}}}{1\text{ hr}} \cdot \frac{1\text{ lb}}{16\cancel{\text{ oz}}} = \frac{12(60)}{16}\frac{\text{lb}}{\text{hr}} = 45\frac{\text{lb}}{\text{hr}}$

CYU 1.12 $\frac{0.3\text{ gal}}{\text{yd}^2} = \frac{0.3\cancel{\text{ gal}}}{\cancel{\text{yd}^2}} \cdot \frac{4\text{ qt}}{1\cancel{\text{ gal}}} \cdot \frac{1^2\cancel{\text{ yd}^2}}{3^2\text{ ft}^2} = \frac{(0.3)(4)}{3^2}\frac{\text{qt}}{\text{ft}^2} \approx 0.13\frac{\text{qt}}{\text{ft}^2}$

CYU 1.13

$$4\frac{\text{g}}{\text{cm}^3} = 4\frac{\text{g}}{\text{cm}^3} \cdot \frac{100^3\text{ cm}^3}{1^3\text{m}^3} \cdot \frac{1^3\text{ m}^3}{(3.28)^3\text{ ft}^3} \cdot \frac{1^3\text{ ft}^3}{12^3\text{ in}^3} \cdot \frac{.0353\text{ oz}}{1\text{ g}} = \frac{4(10^6)(.0353)}{(3.28)^3 12^3}\frac{\text{oz}}{\text{in}^3} \approx 2.32\frac{\text{oz}}{\text{in}^3}$$

CYU 1.14 Cost in dollars:

$$\$[\underbrace{3}_{\text{cost}}\overbrace{\boxed{(2\pi 2^2)}}^{\text{area of top and bottom}} + \underbrace{2}_{\text{cost}}\overbrace{\boxed{(8)2(\pi\cdot 2)}}^{\text{area of side}}] \approx \$276.46$$

Cost in Euros:

$$\$276.46 = \$276.46 \cdot \frac{1.07\text{ EUR}}{1\ \$} = (276.46)(1.07)\text{ EUR} \approx 295.81\text{ EUR}$$

CHAPTER 2 EQUATIONS AND INEQUALITIES

CYU 2.1 (a)

$$\begin{aligned} 5 - 8x + 2 &= -12x - 3 + 6x \\ -8x + 12x - 6x &= -3 - 5 - 2 \\ -2x &= -10 \\ x &= \frac{10}{2} = 5 \end{aligned}$$

(b)

$$\begin{aligned} \frac{3x}{5} - \frac{2-x}{3} + 1 &= x - 1 \\ 15\left(\frac{3x}{5} - \frac{2-x}{3} + 1\right) &= 15(x-1) \\ 3(3x) - 5(2-x) + 15 &= 15x - 15 \\ 9x - 10 + 5x + 15 &= 15x - 15 \\ 9x + 5x - 15x &= -15 + 10 - 15 \\ -x &= -20 \\ x &= 20 \end{aligned}$$

CYU 2.2

$$\left.\begin{aligned} (1)\quad 3x + 4y &= -1 \\ (2)\quad x + 2y &= 0 \end{aligned}\right\}$$

$$\begin{aligned} 3x + 4y &= -1 \\ \text{2 times equation (2):}\quad 2x + 4y &= 0 \\ \text{subtract:}\quad x &= -1 \end{aligned}$$

$$\begin{aligned} \text{substitute in (1):}\quad -1 + 2y &= 0 \\ 2y &= 1 \\ y &= \frac{1}{2} \end{aligned}$$

Answer: $x = -1, y = \frac{1}{2}$

CYU 2.3 Let w and l denote the width and length of the rectangle, respectively. We then have:

$$(1):\ l = 4w \quad\text{and}\quad (2):\ (l+4)(w-1) = 100$$

Substituting $4w$ for l in (2):

$$\begin{aligned} (4w+4)(w-1) &= 32 \\ 4w^2 - 4 &= 32 \\ 4w^2 &= 36 \\ w^2 &= 9 \Rightarrow w = 3 \qquad \text{From (1): } l = 12 \end{aligned}$$

Answer: The width of the rectangle is 3 inches and the length is 12 inches.

CYU 2.4

$$\frac{3x}{5}-\frac{2-x}{3}+1<\frac{x-1}{15}$$

$$15\left(\frac{3x}{5}-\frac{2-x}{3}+1\right)<15\left(\frac{x-1}{15}\right)$$

$$9x-5(2-x)+15<x-1$$

$$9x-10+5x+15<x-1$$

$$9x+5x-x<-1+10-15$$

$$13x<-6$$

$$x<-\frac{6}{13}$$

CYU 2.5 Let x denote Mary's grade on the final.

Mary's average on the three tests: $\frac{82+87+85}{3} = \frac{234}{3}$

Course grade: $x(40\%)+\frac{234}{3}(60\%) = x\left(\frac{40}{100}\right)+\frac{234}{3}\left(\frac{60}{100}\right) = \frac{2}{5}x+\frac{234}{5}$

Solving for x: $\frac{2}{5}x+\frac{234}{5} = 90$

$$2x+234 = 5(90)$$

$$x = \frac{450-234}{2} = 108$$

Conclusion: It is not possible for Mary to receive an A in the course.

CYU 2.6 $\begin{bmatrix}1&2&1\\0&1&2/3\\0&-5&-3\end{bmatrix}\to\begin{bmatrix}1&0&-1/3\\0&1&2/3\\0&3&1\end{bmatrix}\to\begin{bmatrix}1&0&-1/3\\0&1&2/3\\0&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

CYU 2.7 Referring to Definition 2.1 we note that $\begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&4\end{bmatrix}$, and $\begin{bmatrix}0&1&-3&0&5\\0&0&0&1&3\\0&0&0&0&0\end{bmatrix}$ satisfy all thee properties. $\begin{bmatrix}1&0&-3&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$ fails property (ii).

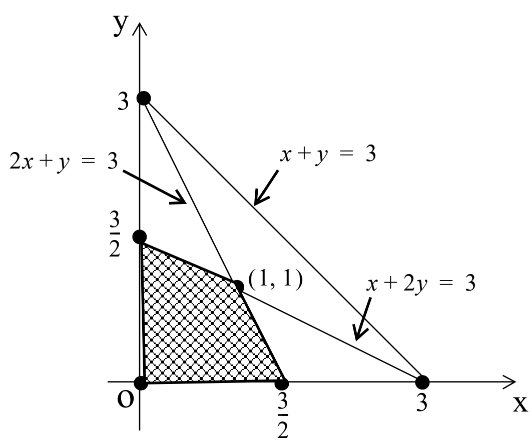
CYU 2.8

$$\left.\begin{aligned} x+y+z &= 6\\ 3x+2y-z &= 4\\ 3x+y+2z &= 11 \end{aligned}\right\} \leftrightarrow \begin{array}{c} x\ y\ z \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6\\ 3 & 2 & -1 & 4\\ 3 & 1 & 2 & 11 \end{array}\right] \end{array} \rightarrow \begin{array}{c} x\ y\ z \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3 \end{array}\right] \end{array} \leftrightarrow \left.\begin{aligned} x &= 1\\ y &= 2\\ z &= 3 \end{aligned}\right\}$$

CYU 2.9 (a) $\begin{array}{c} x\ y\ z \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1\\ 0 & 1 & 5 & 4\\ 0 & 0 & 0 & 2 \end{array}\right] \end{array}$ is inconsistent since the equation $0x + 0y + 0x = 2$ has no solution

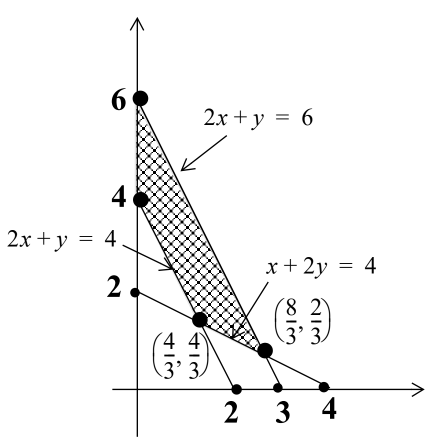
(b) $\begin{array}{c} x\ y\ z \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1\\ 0 & 1 & 5 & 4\\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ has infinitely many solutions since z is a free variable.

CYU 2.10



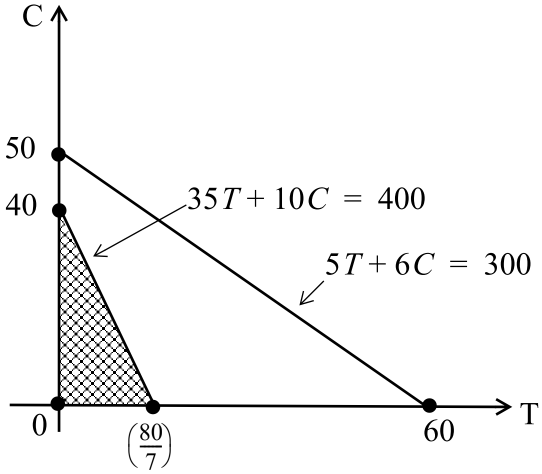
Vertex of Constraint Region	$z = x + 2y$
$\left(0, \frac{3}{2}\right)$	max: 3
$(1, 1)$	max: 3
$\left(\frac{3}{2}, 0\right)$	$\frac{3}{2}$
$(0, 0)$	min: 0

CYU 2.11



Vertex of Constraint Region	$z = 4x - y$
$(0, 4)$	-4
$(0, 6)$	min: -6
$\left(\frac{8}{3}, \frac{2}{3}\right)$	max: 10
$\left(\frac{4}{3}, \frac{4}{3}\right)$	4

CYU 2.12



Vertex of Constraint Region	$P = 75T + 50C$
$(0, 40)$	max: 2000
$\left(\frac{80}{7}, 0\right) \xrightarrow{\text{rounding down}} (11, 0)$	825
$(0, 0)$	0

CHAPTER 3 FUNCTIONS

CYU 3.1 $f(2x+1) = 3(2x+1)^2 - 5 = 3(4x^2 + 4x + 1) - 5 = 12x^2 + 12x - 2$

CYU 3.2 (a) $(f \circ g)(-2) = f[g(-2)] = f[4(-2)+3] = f(-5) = (-5)^2 + 2(-5) - 2$
$= 25 - 10 - 2 = 13$

(b) $(f \circ g)(x) = f[g(x)] = f(4x+3) = (4x+3)^2 + 2(4x+3) - 2$
$= 16x^2 + 24x + 9 + 8x + 6 - 2$
$= 16x^2 + 32x + 13$

CYU 3.3 (a) $f(a) = f(b) \Rightarrow \frac{1}{2}a - 5 = \frac{1}{2}b - 5 \Rightarrow \frac{1}{2}a = \frac{1}{2}b \Rightarrow a = b$

(b) Start with: $(f \circ f^{-1})(x) = x$ i.e: $f[f^{-1}(x)] = x$

substitute t for $f^{-1}(x)$: $f[t] = x$

Since $f(x) = \frac{1}{2}x - 5$: $\frac{1}{2}t - 5 = x$

Solve for t: $t = 2x + 10$

Substitute $f^{-1}(x)$ back for t: $f^{-1}(x) = 2x + 10$

(c) $(f \circ f^{-1})(x) = f[f^{-1}(x)] = f(2x+10) = \frac{1}{2}(2x+10) - 5 = x$

CYU 3.4 $3^{x+2} = 27^x \Rightarrow 3^{x+2} = (3^3)^x$

$$\begin{aligned} 3^{x+2} &= 3^{3x} \\ x+2 &= 3x \\ -2x &= -2 \\ x &= 1 \end{aligned}$$

CYU 3.5 $\log_2(x^2+2x) = \log_2 3$

$$\begin{aligned} x^2+2x &= 3 \\ x^2+2x-3 &= 0 \\ (x-3)(x+1) &= 0 \Rightarrow x = 3, x = -1 \end{aligned}$$

both are solutions since $x^2+2x>0$ for $x = 3$ and for $x = -1$

CYU 3.6 (a-i) Since $2^4 = 16, \log_2 16 = 4$ (a-ii) Since $\left(\frac{1}{3}\right)^{-2} = 9, \log_{\frac{1}{3}} 9 = -2$

(a-iii) Since $5^0 = 1, \log_5 1 = 0$ (b-i) 1.14 (b-ii) 0.43

CYU 3.7

$$\begin{aligned} 2^{3x-1} &= 5^{x+2} \\ (3x-1)\ln 2 &= (x+2)\ln 5 \\ 3x\ln 2 - \ln 2 &= x\ln 5 + 2\ln 5 \\ 3x\ln 2 - x\ln 5 &= 2\ln 5 + \ln 2 \\ x(3\ln 2 - \ln 5) &= 2\ln 5 + \ln 2 \\ x &= \frac{2\ln 5 + \ln 2}{3\ln 2 - \ln 5} = \frac{\ln 50}{\ln\frac{8}{5}} \end{aligned}$$

CYU 3.8 (a)
$$\begin{aligned}
\log_2(x+9)-\log_2 x &= 2\\
\log_2\frac{x+9}{x} &= 2\\
2^{\log_2\frac{x+9}{x}} &= 2^2\\
\frac{x+9}{x} &= 4\\
4x &= x+9\\
3x &= 9\\
x &= 3 \quad \text{a solution since both } x+9 \text{ and } x \text{ are positive at } x=3
\end{aligned}$$

(b)
$$\begin{aligned}
\log_2(2x+5)-\log_2(2x+1) &= \log_2(3)\\
\log_2\frac{2x+5}{2x+1} &= \log_2(3)\\
\frac{2x+5}{2x+1} &= 3\\
6x+3 &= 2x+5\\
4x &= 2\\
x &= \frac{1}{2} \quad \text{a solution since both } 2x+5 \text{ and } 2x+1 \text{ are positive at } x=\frac{1}{2}
\end{aligned}$$

CYU 3.9 (a) $\log_5 32 = \frac{\ln 32}{\ln 5} \approx 2.15$ (b) $\log_{\frac{1}{2}} 17 = \frac{\ln 17}{\ln\frac{1}{2}} \approx -4.09$

CYU 3.10 Let $P(t)$ denote the population at time t. Then $P(t) = P_0e^{kt}$. Since the initial population is 500, $P_0 = 500$, so that $P(t) = 500e^{kt}$. In 9 years, the population is $P(9) = 500 + \frac{15}{100}(500) = 575$. Thus:

$$\begin{aligned}
575 &= 500(e^k)^9\\
(e^k)^9 &= \frac{575}{500} = \frac{23}{20}\\
e^k &= \left(\frac{23}{20}\right)^{1/9}
\end{aligned}$$

$$\text{So: } P(t) = 500\left(\frac{23}{20}\right)^{t/9}$$

We now find t such that:

$$\begin{aligned}
P(t) &= 3(500) = 1500\\
1500 &= 500\left(\frac{23}{20}\right)^{t/9}\\
\left(\frac{23}{20}\right)^{t/9} &= 3\\
\frac{t}{9}\ln\frac{23}{20} &= \ln 3\\
t &= \frac{9\ln 3}{\ln\frac{23}{20}} \approx 70.75 \text{ years}
\end{aligned}$$

CYU 3.11 (a) Take 2023 as $t = 0$. Then $A_0 = 35$, and the formula becomes $A(t) = 35e^{-t/4}$.

The year 2026 corresponds to $t = 3$, and $A(3) = 35e^{-3/4} \approx 16.5$ grams,

(b) With $A(t) = 35e^{-t/4}$ we now find t for which $A(t) = \frac{1}{2}(35)$:

$$\frac{1}{2}(35) = 35e^{-t/4} \Rightarrow e^{-t/4} = \frac{1}{2} \Rightarrow -\frac{t}{4} = \ln\left(\frac{1}{2}\right) \Rightarrow \left(t = -4\ln\left(\frac{1}{2}\right) \approx 2.77 \text{ years}\right)$$

CHAPTER 4 DIFFERENTIAL CALCULUS

CYU 4.1 (a) As x approaches -1, $4x^2 + x$ approaches 3 ($4x^2$ tends to 4, and x to -1). Thus: $\lim\limits_{x \to -1} (4x^2 + x) = 3$

(b) As x approaches 2, $x + 3$ approaches 5 and $x + 2$ approaches 4. Thus: $\lim\limits_{x \to 2} \frac{x+3}{x+2} = \frac{5}{4}$

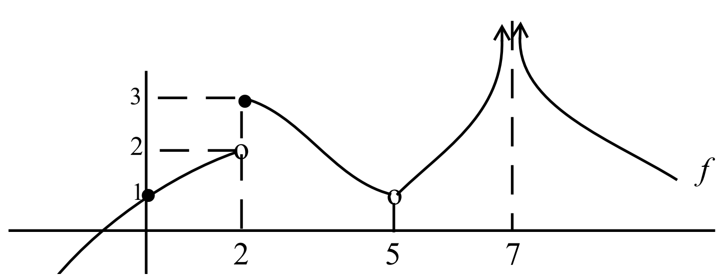
(c) As x approaches 3, $x(3x^2 + 1)$ approaches $3(3 \cdot 3^2 + 1) = 84$. Thus: $\lim\limits_{x \to 3} [x(3x^2 + 1)] = 84$

CYU 4.2 (a) $$\lim_{x \to 1} \frac{x^2 + 3x - 4}{x^2 - 1} = \lim_{x \to 1} \frac{(x+4)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \lim_{x \to 1} \frac{(x+4)}{(x+1)} = \frac{5}{2}$$

(b) $$\lim_{x \to 1} \frac{x^2 + 3x - 4}{x^2 - 1} = \lim_{x \to 1} \frac{(x+4)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \lim_{x \to 1} \frac{(x+4)}{(x+1)} = \frac{5}{2}$$

(c) $$\lim_{x \to 1} \frac{x^2 - 1}{\frac{x}{2} - \frac{1}{2}} = \lim_{x \to 1} \frac{2(x^2 - 1)}{x - 1} = \lim_{x \to 1} \frac{2(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} 2(x+1) = 4$$

CYU 4.3



(a) As you approach 0 from either side, the function values approach 1. Thus: $\lim_{x \to 0} f(x) = 1$.

(b) As you approach 2 from the left, the function values approach 2, but as you approach from the right, the function values approach 3. Thus: $\lim_{x \to 2} f(x)$ does not exist.

(c) As you approach 5 from either side, the function values approach 1 (never mind that the function is not defined at 5; for the limit does not care what happens there—it is only concerned about what happens as you **approach** 5. Thus: $\lim_{x \to 5} f(x) = 1$.

(d) As you approach 7 from either side, the function values get larger and larger, and cannot tend to any number. Thus: $\lim_{x \to 7} f(x)$ does not exist —we can also write $\lim_{x \to 5} f(x) = \infty$.

<u>**CYU 4.4**</u> (a) Since the function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is not defined at $x = 3$ it cannot be continuous at that point.

(b) Since $\lim_{x \to 3} \dfrac{x^2 - 9}{x + 3} = \lim_{x \to 3} \dfrac{(x + 3)(x - 3)}{x + 3} = \lim_{x \to 3} (x - 3) = 0$ and since

$f(3) = \dfrac{3 - 3}{3 + 3} = \dfrac{0}{6} = 0$, the function os continuous at $x = 3$.

<u>**CYU 4.5**</u>
$$
\begin{aligned}
f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} &= \lim_{h \to 0} \frac{[-3(2 + h)^2 + 6(2 + h) - 1] - (-12 + 12 - 1)}{h} \\
&= \lim_{h \to 0} \frac{-3(4 + 4h + h^2) + 12 + 6h}{h} \\
&= \lim_{h \to 0} \frac{-3h^2 - 6h}{h} = \lim_{h \to 0} (-3h - 6) = -6
\end{aligned}
$$

<u>**CYU 4.6**</u>
$$
\begin{aligned}
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x + h)^2 + (x + h) + 1 - (x^2 + x + 1)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h + 1 - x^2 - x - 1}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{h(2x + h + 1)}{h} \\
&= \lim_{h \to 0} (2x + h + 1) = 2x + 1
\end{aligned}
$$

In particular: $f'(-1) = -1$, $f'(0) = 1$, and $f'(1) = -1$

Find the tangent line to the graph of the function $f(x) = x^2 + x + 1$ at $x = 2$.

CYU 4.7 Since $(x^2+x+1)' = 2x+1$, $f'(2) = 5$. Consequently, the tangent line to the graph of $f(x) = x^2+x+1$ at $x = 2$ is of the form $y = 5x+b$.

Since the point $(2, f(2)) = (2, 7)$ lies on th line: $7 = 5(2)+b \Rightarrow b = -3$, and we have our tangent line: $y = 5x-3$.

CYU 4.8 (a) $f'(x) = (-4x^3+2x^2-3x+5)' = -12x^2+4x-3$

(b) $f'(x) = [x^2(3x^3+2x-5)]' = (3x^5+2x^3-5x^2)' = 15x^4+6x^2-10x$

(c) $f(x) = \dfrac{5x^3+2x-7}{2x^2}$

(c) $f'(x) = \left(\dfrac{5x^3+2x-7}{2x^2}\right)' = \left(\dfrac{5}{2}x+x^{-1}-\dfrac{7}{2}x^{-2}\right)' = \dfrac{5}{2}-\dfrac{1}{x^2}+\dfrac{7}{x^3}$

CYU 4.9 Since $f'(x) = \left(\dfrac{x^3-x}{x^2}\right)' = (x-x^{-1})' = 1+\dfrac{1}{x^2}$, $f'(2) = \dfrac{5}{4}$. Consequently, the tangent line to the graph of $f(x) = \dfrac{x^3-x}{x^2}$ at $x = 2$ is of the form $y = \dfrac{5}{4}x+b$.

Since the point $(2, f(2)) = \left(2, \dfrac{3}{2}\right)$ lies on th line: $\dfrac{3}{2} = \dfrac{5}{4}(2)+b \Rightarrow b = -1$, and we have our tangent line: $y = \dfrac{5}{4}x-1$.

CYU 4.10 Without: $[(2x-1)(x^2-3)]' = (2x^3-x^2-6x+3)' = 6x^2-2x-6$

With:
$$\begin{aligned}[(2x-1)(x^2-3)]' &= (2x-1)(x^2-3)'+(x^2-3)(2x-1)' \\ &= (2x-1)(2x)+(x^2-3)(2) = 6x^2-2x-6\end{aligned}$$

CYU 4.11 $f'(x) = \left(\dfrac{2x^2+3x}{2x^2+3}\right)' = \dfrac{(2x^2+3)(2x^2+3x)' - (2x^2+3x)(2x^2+3)'}{(2x^2+3)^2}$

$$= \frac{(2x^2+3)(4x+3) - (2x^2+3x)(4x)}{(2x^2+3)^2} \underset{\text{steps omitted}}{=} \frac{-6x^2+12x+9}{(2x^2+3)^2}$$

CYU 4.12 $f''(x) = \left(5x^3 - x + \dfrac{1}{x}\right)'' = (15x^2 - 1 - x^{-2})' = 30x + \dfrac{2}{x^3}$

$$f'''(x) = (30x + 2x^{-3})' = 30 - \frac{6}{x^4}$$

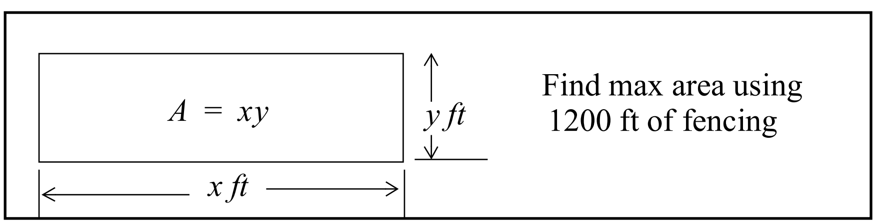
CYU 4.13 $P'(x) = (5.7x - 0.025x^2)' = 5.7 - 0.050x$

$$P'(x) = 0 \Rightarrow 5.7 - 0.050x = 0 \Rightarrow x = \frac{5.7}{0.050} = 114$$

Since $P''(x) = (5.7 - 0.050x)' = -0.050 < 0$, a maximum occurs at $x = 114$

Conclusion: Maximum Profit $= \$[5.7(114) - 0.025(114)^2](1{,}000) = \$324,900$

CYU 4.14



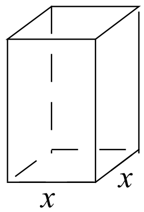
From $2x + 2y = 1200$, we have $x + y = 600 \Rightarrow y = 600 - x$. So:

$A = x(600 - x) = -x^2 + 600x \Rightarrow A' = -2x + 600 \Rightarrow A' = 0$ when $x = 300$

Since $A''(x) = (-2x + 600)' = -2 < 0$, maximum area occurs when $x = 300$.

Conclusion: Maximum area $= xy = (300)(600 - 300) = 90{,}000 \text{ ft}^2$

CYU 4.15



Volume $V = x^2y = 8 \text{ ft}^3$

Want to minimize the surface area $S = \underset{\substack{\uparrow \\ \text{bottom and top}}}{2x^2} + \underset{\substack{\uparrow \\ \text{the sides}}}{4xy}$

$x^2y = 8 \Rightarrow y = \dfrac{8}{x^2}$. So $S = 2x^2 + 4x\left(\dfrac{8}{x^2}\right) = 2x^2 + 32x^{-1}$. Setting $S' = 0$, we have:

$$(2x^2 + 32x^{-1})' = 0 \Rightarrow 4x - 32x^{-2} = 0 \Rightarrow x - 8x^{-2} = 0 \Rightarrow x = \frac{8}{x^2} \Rightarrow x^3 = 8 \Rightarrow x = 2$$

Since $S''(x) = (2x^2 + 32x^{-1})' = 4x - 32x^{-2}$, $S''(2) = 4(2) - \dfrac{32}{2^2} = 0$, the second derivative test is inconclusive. Turning to

$$S' = (2x^2 - 32x^{-1})' = 4x - \frac{32}{x^2} = \frac{4x^3 - 32}{x^2} = \frac{4(x^3 - 8)}{x^2}$$

we find that it is negative to the left of 2 and positive to the right of 2. Employing the first derivative test, we conclude that the minimum surface area of $S(2) = 2(2)^2 + \dfrac{32}{2} = 24\text{ft}$ occurs, when $x = 2$.

CYU 4.16

$\$24 \rightarrow 150$ cars	$\$(24 + x) \rightarrow 150 - 5x$ cars

maximize revenue

Revenue, $R(x)$, realized at a cost of $\$(24 + x)$ per car:

$$R(x) = (24 + x)(150 - 5x) = -5x^2 + 30x + 3600.$$

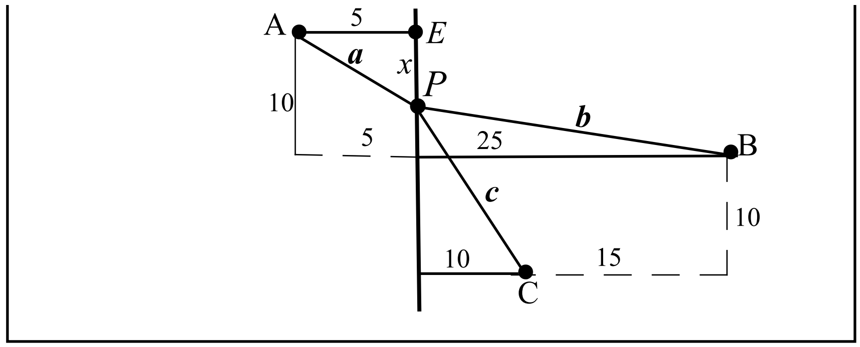
Finding the critical point:

$$R'(x) = 0 \Rightarrow (-5x^2 + 30x + 3600)' = 0 \Rightarrow -10x + 30 = 0 \Rightarrow x = 3$$

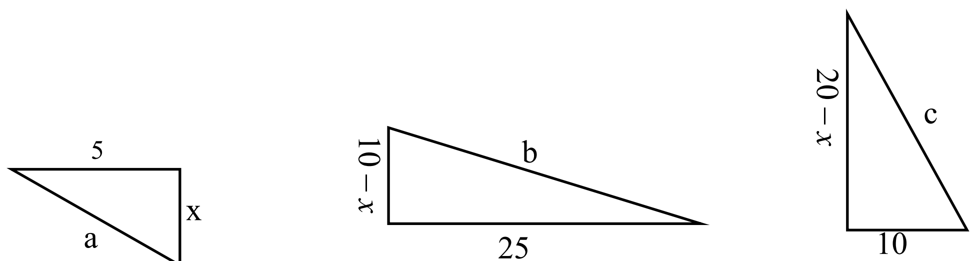
Since $R''(x) = (-10x + 30)' = -10$, $R''(3)$ is negative.

Employing the second derivative test, we conclude that the maximum revenue will be realized when the cars are rented for $\$(24 + 3) = \$27$ per day

CYU 4.17



We need to minimize the combined distance $s = a + b + c$ and do so by expressing it in terms of the above variable x. Focusing on the above three right triangles:

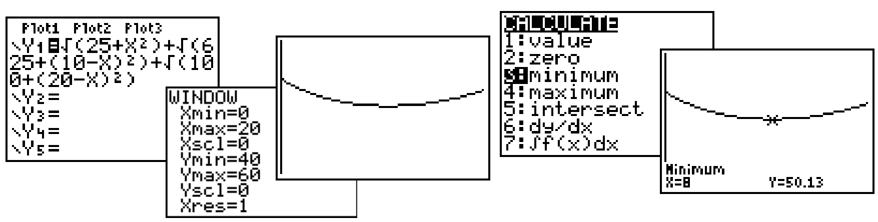


$$a^2 = 5^2 + x^2 \qquad b^2 = 25^2 + (10 - x)^2 \qquad c^2 = 10^2 + (20 - x)^2$$

$$\Rightarrow a = \sqrt{5^2 + x^2} \qquad \Rightarrow b = \sqrt{25^2 + (10 - x)^2} \qquad \Rightarrow c = \sqrt{10^2 + (20 - x)^2}$$

Hence: $s = \sqrt{5^2 + x^2} + \sqrt{25^2 + (10 - x)^2} + \sqrt{10^2 + (20 - x)^2}$

Then:



Conclusion: A minimum combined distance of approximately 50.13 miles will be achieved when P is positioned 8.00 miles South of point E.

CHAPTER 5 PERSONAL FINANCE

CYU 5.1 29 months $= (29 \cancel{\text{months}})\left(\frac{1\,\text{year}}{12\ \cancel{\text{months}}}\right) = \frac{29}{12}$ years

$$I = \$\left(5,000 \cdot \frac{5.2}{100} \cdot \frac{29}{12}\right) = \$628.33$$

Future vale: A $= \$(5000 + 628.33) = \$5,628.33$

CYU 5.2 (a) Using the simple interest formula:

$$A = \$(10,000 + I) = \$\left(10,000 + 10,000 \cdot \frac{5}{100} \cdot 10\right) = \$15,000.00$$

(b) From Theorem 5.1, with $n = 1$:

$$A = \$10,000(1 + .05)^{10} = \$16,288.95$$

(c) From Theorem 5.1, with $n = 4$:

$$A = \$10,000\left(1 + \frac{.05}{4}\right)^{40} = \$16,463.19$$

CYU 5.3 (a) This is a present value problem. So:

$$P = A\left(1 + \frac{r}{n}\right)^{-nt} = \$100,000\left(1 + \frac{.05}{2}\right)^{-40} = \$37,243.06$$

(b) This is a future value problem. So:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = \$37,243\left(1 + \frac{.05}{2}\right)^{40} = \$99,999.83$$

CYU 5.4 (a) Turning to Theorem 5.2:

$$A = \boldsymbol{P\left(1 + \frac{r}{n}\right)^{nt}} = \$25,000\left(1 + \frac{.07}{12}\right)^{12 \cdot 20} = \$100,968.47$$

(b) Turning to Definition 5.1:

$$A = \boldsymbol{Pe^{rt}} = \$25,000e^{.07(20)} = \$101,380.00$$

CYU 5.5 For (a), We set $A = 2P$ in Definition 5.1, and solve for t. We do the same for (b), but with $A = 3P$:

(a)
$$\begin{aligned} 2P &= Pe^{.09t} \\ e^{.09t} &= 2 \\ .09t &= \ln 2 \\ t &= \frac{\ln 2}{.09} \approx 7.70 \text{ years} \end{aligned}$$

(b)
$$\begin{aligned} 3P &= Pe^{.09t} \\ e^{.09t} &= 3 \\ .09t &= \ln 3 \\ t &= \frac{\ln 3}{.09} \approx 12.21 \text{ years} \end{aligned}$$

CYU 5.6 Turning to Definition 5.2:

$$P = Ae^{-rt} = \$50.000e^{-(.06)(25)} = \$11,156.51$$

CYU 5.7 From Theorem 5.3, with $a = 2$, $r = 5$, and $n = 10$ we have:

$$S_{10} = \frac{2(1 - 5^{10})}{1 - 5} = 4,882,812$$

CYU 5.8 From Theorem 5.4, with $A = 120, \frac{r}{n} = \frac{.04}{4}$, and $t = 8$ we have:

$$A = d\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right] = \$120\left[\frac{\left(1+\frac{.04}{4}\right)^{32}-1}{\frac{.04}{4}}\right] \approx \$4{,}499.29$$

CYU 5.9 From Theorem 5.5, with $A = \$50{,}000, \frac{r}{n} = \frac{.039}{4}$, and $t = 10$ we have:

$$d = \frac{A\left(\frac{r}{n}\right)}{\left(1+\frac{r}{n}\right)^{nt}-1} = \frac{\$50{,}000\frac{.039}{4}}{\left(1+\frac{.039}{4}\right)^{40}-1} \approx \$1{,}028.06$$

CYU 5.10 Ignoring the down payment issue momentarily, we substitute \$1,500 for P, 5.7% for r, and 30 for t, in Theorem 4.6:

$$P = d\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right] = \$1{,}500\left[\frac{1-\left(1+\frac{.057}{12}\right)^{-30\cdot 12}}{1+\frac{.057}{12}}\right] = \$258{,}442.26$$

But this is not the amount to be financed by the bank, as you have to make a 15% down payment on the cost of the property. The bank would only finance 85% of the actual property cost, with:

$$.85P = \$258{,}442.26 \Rightarrow P = \frac{\$258{,}442.26}{.85} = \$304{,}049.72$$

But anther issue comes into pay; namely is 15% of \$304,049.72 less than or equal to \$50,000? Fortunately yes: $(.15)(304{,}049.72) = 45{,}607.45$.

CYU 5.11 .Amount of loan $= \$375,000(.8) = \$300{,}000$

$$\text{Monthly payments: } \boldsymbol{d} = \frac{\boldsymbol{P\left(\frac{r}{n}\right)}}{\boldsymbol{1-\left(1+\frac{r}{n}\right)^{-nt}}} = \frac{300,000\left(\frac{.06}{12}\right)}{1-\left(1+\frac{.06}{12}\right)^{-12(20)}} = \$2{,}149.29$$

Total payment: $\$2{,}149.29(240) = \$515{,}829.60$

Interest paid: $\$(644{,}788.80 - 375{,}000) = \$269{,}788.80$

$$\text{Monthly payments without the down payment: } \boldsymbol{d} = \frac{375,000\left(\frac{.06}{12}\right)}{1-\left(1+\frac{.06}{12}\right)^{-12(20)}} = \$2{,}686.62$$

Total payment: $\$2{,}686.62(20)(12) = \$644{,}788.80$

Interest paid: $\$(644{,}788.80 - 375{,}000) = \$269{,}788.80$

The down payment will reduce the interest cost by

$$\$269{,}788.80 - 140{,}829.60 = \$128{,}959.20$$

CHAPTER 6 PROBABILITY

CYU 6.1 Letting, for example, the symbol (2,4) represent the event of drawing a 2 and then a 4 from the hat; and the symbol (4,5) represent the drawing of a 4 followed by a 5 we come to the following equiprobable sample space, S:

$$S = \left\{ \begin{array}{cccc} (1,2) & (1,3) & (1,4) & (1,5) \\ (2,1) & (2,3) & (2,4) & (2,5) \\ (3,1) & (3,2) & (3,4) & (3,5) \\ (4,1) & (4,2) & (4,3) & (4,5) \\ (5,1) & (5,2) & (5,3) & (5,4) \end{array} \right\}$$

drawing a 2 and then a 4

Note that there is no (4,4), for example, since the first number drawn is not replaced.

drawing a 4 and then a 5

With the above $4 \cdot 5 = 20$ element equiprobable sample space at hand, it is a trivial matter to find the specified probabilities:

no. of successes: in first row in third in fifth

(a) $$Pr(\text{first is odd}) = \frac{4+4+4}{20} = \frac{12}{20}$$

number of possibilities

in first row, in second row, 3rd row, 4th row

(b) $$Pr(\text{second} > \text{first}) = \frac{4+3+2+1}{20} = \frac{10}{20}$$

(no success in 5th row)

CYU 6.2 Out of 900 lady-fingers tested (the sample space), $900 - 32 = 868$ functioned properly (the successes). Based on this, we calculate the (empirical) probability of a success:

$$Pr(\text{Bang}) = \frac{868}{900} \approx 0.96$$

CYU 6.3 There are 52 cards, 12 of which are face cards. Thus:

$$Pr(\textbf{not}\text{ a face card}) = \frac{52-12}{52} = \frac{40}{52}$$

CYU 6.4 You can count the successes directly in your head, without benefit of Theorem 5.2: There are 26 red cards (half of the deck), and 6 black face cards (half of the face cards), for a total of 32 successes. Thus:

$$Pr(\text{Red or Face}) = \frac{32}{52}$$

You can also use Theorem 6.2:

$$Pr(\underbrace{R \cup F}_{\text{Red or Face}}) = P(R) + P(F) - P(R \cap F) = \overset{\text{Red}}{\frac{26}{52}} + \overset{\text{Face}}{\frac{12}{52}} - \overset{\text{both Red and Face}}{\frac{6}{52}} = \frac{32}{52}$$

CYU 6.5 To get the number of successes, just add the number of black face cards to the number of red aces (nothing is being counted twice):

$$Pr(\overbrace{\text{Black Face Card}} \text{ or } \overbrace{\text{Red Ace}}) = \frac{6+2}{52} = \frac{8}{52}$$

Using Theorem 6.3, we have:

$$Pr(\underbrace{\text{BFC} \cup \text{RA}}_{\text{Black Face Card or Red Ace}}) = P(\text{BFC}) + P(\text{RA}) = \frac{6}{52} + \frac{2}{52} = \frac{8}{52}$$

CYU 6.6 The sample space for the rolling of a pair of dice appears below, two times. On the left, we shaded in the restricted sample space subject to the (a)-condition: at least one die is a 5. On the right, we shaded in the restricted sample space subject to the (b)-condition: the sum is 7. In both cases, we also boxed in the specified successes. The rest, is counting.

2 successes   11 possibilities

$$\text{(a)} \quad \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & \boxed{(2,5)} & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & \boxed{(5,2)} & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,5) \end{matrix} \right\}$$

$$Pr(\text{sum is7}|\text{at least one 5}) = \frac{2}{11}$$

2 successes   6 possibilities

$$\text{(b)} \quad \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & \boxed{(2,5)} & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & \boxed{(5,2)} & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,5) \end{matrix} \right\}$$

$$Pr(\text{at least one 5}|\text{sum is 7}) = \frac{2}{6}$$

CYU 6.7 (a) $Pr(\text{R}_{1st} \text{ and } \text{R}_{2nd}) = Pr(R_{1st}) \cdot Pr(R_{2nd}|R_{1st}) = \frac{5}{16} \cdot \frac{4}{15} = \frac{1}{12}$

(b) $Pr(\text{R and B}) = P(\text{R}_{1st} \text{ and } \text{B}_{2nd}) \text{ or } P(\text{B}_{1st} \text{ and } \text{R}_{2nd})$

$$\frac{5}{16} \cdot \underbrace{\frac{4}{15} + \frac{4}{16}}_{\text{one marble is gone}} \cdot \frac{5}{15} = 2\left(\frac{5 \cdot 4}{16 \cdot 15}\right) = \frac{1}{6}$$

CYU 6.8 Using Theorem 5.6, and looking at the situation as 5 independent experiments (rolling one die five times), each with a probability of success $\frac{1}{6}$, we have:

$$Pr(\text{rolling a die five times and getting a 4 each time}) = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{6^5} \approx 0.0001$$

CYU 6.9 Here is the probability that you make at least one basket out of 2 shots:

2 independent events

$$Pr(\text{at least one}) = 1 - Pr(\text{two misses}) = 1 - [Pr(\text{miss on first}) \cdot Pr(\text{miss on second})] = 1 - (0.6)^2$$

And here is the probability that you will get at least one basket in each of five games:

5 independent events

$$Pr(\text{get large cuddly bear}) = [1-(0.6)^2][1-(0.6)^2][1-(0.6)^2][1-(0.6)^2][1-(0.6)^2]$$
$$= [1-(0.6)^2]^5 \approx 0.11$$

CYU 6.10 Here is one of the things you are trying to count: DD2431. In writing that sample down, we were aware of the fact that the second letter had to be the same as the first (choice of 1 for the second letter), and that every time we wrote down a digit, it was unavailable for future choices; in other words:

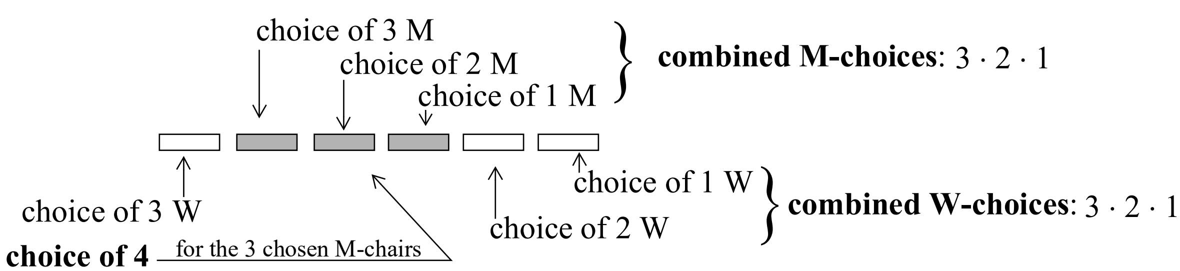
letter choices: 26 1 — digit choices: 10 9 8 7

Turning to the Fundamental Counting Principle, we have:

Number of plates $= 26 \cdot 1 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 131{,}040$

CYU 6.11 One possible success: WMMMWW. The main focus was to make sure that the three M's were next to each other. We started them off in seat 2, but could have started that string of M's at seat 1, or 2, or 3, or 4; for **an initial choice of 4**. This done, let's focus on our possible success, and arrange the three men in the three M seats (shaded below), and the three women in the remaining 3 seats:

:



Turning to the Fundamental Counting Principle, we find the number of different ways of sitting the six people, given that the three men must be seated next to each other:

$$\underbrace{4}_{\text{The “first man” can be seated in any of 4 chairs}}\overbrace{(3\cdot 2\cdot 1)}^{\text{number of ways the three men can be seated in the 3 adjacent chairs.}}\underbrace{(3\cdot 2\cdot 1)}_{\text{number of ways the three women can be seated in the remaining 3 chairs.}} = 144$$

CYU 6.12 We've already counted the number of possibilities in Example 5.17:

$$\text{no. of possibilities: } 26^2\cdot 10^4$$

Let's write down a success: CC2512. There is no problem with the first part of the "journey:" a choice of 26 for the first letter, followed by a choice of 1 for the second. The rest is not so difficult either: any of the 10 digits will do for the first digit, or second, or third; but the last digit has to be the same as the first (a choice of 1):

$$\text{no. of successes: } 26\cdot \overset{\text{same as}}{1}\cdot 10\cdot 10\cdot 10\cdot \overset{\text{same as}}{1}$$

Knowing the number of possibilities (sample space), and the number of successes, we arrive at our answer:

$$Pr(\text{Letters same; first and last digit same}) = \frac{26\cdot 1\cdot 10\cdot 10\cdot 10\cdot 1}{26^2\cdot 10^4} = \frac{1}{260}\approx 0.0038$$

CYU 6.13 Let's lay three blocks down; say: ACD, and realize that we had a choice of 4 for the first block, a choice of 3 for the second, and a choice of 2 for the third—totaling: $4\cdot 3\cdot 2$ possible outcomes of the experiment. The number of successes are given to us explicitly in the problem, and there are 3 of them: CAD, BAD, and DAB. Thus:

$$Pr(\text{CAD or BAD or DAB}) = \frac{3}{4\cdot 3\cdot 2} = \frac{3}{24} = 0.125$$

CYU 6.14 Resorting to Theorem 6.8, we quickly arrive at the answer:

There are $7! = 5{,}040$ different ways of arranging 7 books on a shelf.

CYU 6.15 The number of possibilities is the number of ways the 15 children can be ordered:

$$\text{no. of possibilities} = 15!$$

Now we have to think a bit about the number of successes:

choice of 1: the tallest

choice of **13!**: the number of ways the remaining 13 children can be ordered.

choice of 1: the smallest

$$\underline{1}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{1}$$

Conclusion:

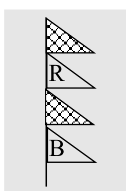
$$Pr(\text{tallest first and shortest last}) = \frac{1\cdot 13!\cdot 1}{15!} = \boxed{\frac{\mathbf{1}}{\mathbf{14\cdot 15}}} = \frac{1}{210} \approx 0.0048$$

$$\boxed{\frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\ldots\cdot\cancel{12}\cdot\cancel{13}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\ldots\cdot\cancel{12}\cdot\cancel{13}\cdot 14\cdot 15} = \frac{\mathbf{1}}{\mathbf{14\cdot 15}}}$$

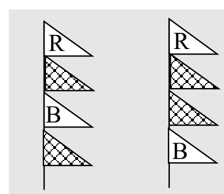
CYU 6.16 There are as many ways of awarding a gold, silver, and bronze medal to a group of 15 individuals as there are of picking three objects from 15, when "order counts:"

$$P(15,3) = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\ldots\cdot\cancel{12}\cdot 13\cdot 14\cdot 15}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\ldots\cdot\cancel{12}} = 13\cdot 14\cdot 15 = 2{,}730$$

CYU 6.17 Red can be hoisted first or second, but the two cases have to be treated separately, for if red is hoisted second, then blue must be hoisted last (there is got to be another flag between red and blue); whereas if red is hoisted first, then blue can be hoisted third or fourth. Here are the possible successes:



OR



choice of 1 (R hoisted second)
followed by a choice of $P(5,2)$
(pick two of the remaining 5 flags)

choice of 2 (R hoisted first and B third or B last)
followed by a choice of $P(5,2)$
(pick two of the remaining 5 flags)

Having accounted for all of the success we have:

$$\text{Pr(R above B but not adjacent)} = \frac{1\cdot P(5,2)+2P(5,2)}{P(7,4)} = \frac{3P(5,2)}{P(7,4)} = \frac{3\cdot\frac{5!}{3!}}{\frac{7!}{3!}} = \frac{3\cdot 4\cdot 5}{4\cdot 5\cdot 6\cdot 7} = \frac{1}{14}$$

CYU 6.18 There are as many poker hands as there are ways of grabbing 5 objects (the 5 cards dealt) from 52 (the deck), when "order does not count:"

$$C(52,5) = \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!} = \frac{\cancel{47!}\cdot\cancelto{2}{48}\cdot 49\cdot\cancelto{10}{50}\cdot 51\cdot 52}{\cancel{47!}\,\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5}} = 2\cdot 49\cdot 10\cdot 51\cdot 52 = 2{,}598{,}960$$

CYU 6.19

R_{25} W_9 P_4 B_7 $\longrightarrow$ grab 8

success: 2 R, 2W, 2P, and 2B

The number of possibilities is the number of ways 8 objects can be grabbed from 45 when "order does not count:" $C(45, 8)$. The number of successes is the number of ways 2 red jelly-beans can be grabbed from the 25, times the number of ways 2 white jelly-beans can be grabbed from the 9, times the number of ways 2 purple jelly-beans can be grabbed from the 4, times the number of ways 2 black jelly-beans can be grabbed from 7 (choices followed by choices—multiply): $C(25, 2) \cdot C(9, 2) \cdot C(4, 2) \cdot C(7, 2)$. Dividing the number of successes by the number of possibilities we have the answer (which you can check by pencil or calculator):

$$P(\text{two of each color}) = \frac{C(25, 2) \cdot C(9, 2) \cdot C(4, 2) \cdot C(7, 2)}{C(45, 8)} = \frac{1{,}360{,}800}{215{,}553{,}195} \approx 0.0063$$

CYU 6.20

$$E(\text{no. of Aces}) = 0 \cdot \frac{C(4, 0)C(48, 5)}{C(52, 5)} + 1 \cdot \frac{C(4, 1)C(48, 4)}{C(52, 5)} + 2 \cdot \frac{C(4, 2)C(48, 3)}{C(52, 5)}$$

$$+ 3 \cdot \frac{C(4, 3)C(48, 2)}{C(52, 5)} + 4 \cdot \frac{C(4, 4)C(48, 1)}{C(52, 5)} \approx 0.39$$

successes: grab any 3 of the 4 aces and 2 of the non-aces $\rightarrow \dfrac{C(4, 3)C(48, 2)}{C(52, 5)}$

possible outcomes: grab any 5 of 52 cards $\nearrow$

CYU 6.21

Time	1	2	3	4	5	6	7	8	9	Sum
Frequency	15	32	25	19	16	9	3	0	1	120
Probability	$\frac{15}{120}$	$\frac{32}{120}$	$\frac{25}{120}$	$\frac{19}{120}$	$\frac{16}{120}$	$\frac{9}{120}$	$\frac{3}{120}$	$\frac{0}{120}$	$\frac{1}{120}$	$\frac{120}{120} = 1$

$$E(\text{Time}) = 1 \cdot \frac{15}{120} + 2 \cdot \frac{32}{120} + 3 \cdot \frac{25}{120} + 4 \cdot \frac{19}{120} + 5 \cdot \frac{16}{120} + 6 \cdot \frac{9}{120} + 7 \cdot \frac{3}{120} + 8 \cdot 0 + 9 \cdot \frac{1}{120} \approx 3.28$$

In Example 6.29, the expected waiting time was 3.34 minutes, and here it is 3.28 minutes, so the situation did improve a bit.

CYU 6.22 $E(\text{Winnings}) = \$349 \cdot \frac{1}{1000} + \$199 \cdot \frac{1}{1000} + \$49 \cdot \frac{2}{1000} - \$1 \cdot \frac{996}{1000} = -\$0.35$

CHAPTER 7 METHODS OF PROOF

CYU 7.1 (a) For $n = 2k+1$ and $m = 2h+1$, we have:

$$n+m = (2k+1)+(2h+1) = 2k+2h+2 = 2(k+h+1)$$

even!

(b) The sum of an even integer with an odd integer is odd:

For $n = 2k$ and $m = 2h+1$, we have:

$$n+m = 2k+(2h+1) = 2k+2h+1 = 2(k+h)+1$$

odd!

CYU 7.2 $2m+n$ is even $\Leftrightarrow$ n is even:

$\Leftarrow$: Let $n = 2k$. Then $2m+n = 2m+2k = 2(m+k)$, which is even.

$\Rightarrow$: $2m+n = 2k \Rightarrow n = 2k-2m = 2(k-m)$, which is even.

CYU 7.3 $3n+2$ is odd $\Leftrightarrow$ n is odd:

$\Leftarrow$: Let $n = 2k+1$. Then $3n+2 = 3(2k+1)+2 = 6k+5 = 2(3k+2)+1$, which is odd.

$\Rightarrow$: (By contradiction) Given that $3n+2$ is odd. Can n be even? No:

If $n = 2k$ then: $3n+2 = 3(2k)+2 = 2(3k+1)$ which is even—contradicting the given condition that it is odd.

CYU 7.4 (a) False. For $m = 3$ and $n = 3$, $2m+n = 6+2 = 8$.

(b) True. If n is even, $n = 2k$, then:

$$\begin{aligned} n^2+3n+5 &= (2k)^2+3(2k)+5 \\ &= 4k^2+6k+4+1 = 2(2k^2+3k+2)+1 \text{, which is odd.} \end{aligned}$$

If n is odd, $n = 2k+1$, then:

$$\begin{aligned} n^2+3n+5 &= (2k+1)^2+3(2k+1)+5 \\ &= 4k^2+4k+1+6k+8 = 2(2k^2+5k+4)+1 \text{, which, again, is odd.} \end{aligned}$$

CYU 7.5 (a-i) If $n = ak$ and $m = ah$, then $n+m = ak+ah = a(k+h) \Rightarrow a|(n+m)$.

(a-ii) For any $c \in Z$, if $a = nk$, then $ca = c(nk) \Rightarrow n|ca$.

(b-i) True. If $b = ak$ and $c = ah$, then: $b+c = ak+ah = a(k+h) \Rightarrow a|(b+c)$.

(b-ii) True. If $a|(b+c)$, then $b+c = ak$, which is even since a is even. Since b is even, c must also be even.

CYU 7.6 Let $P(n)$ be the proposition:

$$2 + 4 + 6 + \ldots + n = n^2 + n$$

I. $P(1)$ is true: $2 = 1^2 + 1$ Check!

II. **Assume** $P(k)$ is true: $2 + 4 + 6 + \ldots + 2k = k^2 + k$

III. We show that $P(k+1)$ is true; which is to say, that:

$$2 + 4 + 6 + \ldots + 2k + 2(k+1) = \mathbf{(k+1)^2 + (k+1)} \quad :$$

$$\begin{aligned} 2 + 4 + 6 + \ldots + 2(k) + 2(k+1) &= [2 + 4 + 6 + \ldots + 2(k)] + 2(k+1) \\ &= [k^2 + k] + 2(k+1) = \mathbf{k^2 + 3k + 2} \end{aligned}$$

Also:

$$\mathbf{(k+1)^2 + (k+1)} = k^2 + 2k + 1 + k + 1 = \mathbf{k^2 + 3k + 2}$$

CYU 7.7 For $n \geq 4$, let $P(n)$ be the proposition: $2n < n!$

I. $P(4)$ is true: $2(4) < 4!$ Check!

II. **Assume** $P(k)$ is true: $2(k) < k!$

III. We show that $2(k+1) < (k+1)!$:

$$(k+1)! = k!(k+1) > 2k(k+1) > 2(k+1)$$

APPENDIX B

SELECTED ANSWERS

CHAPTER 1: A LOGICAL BEGINNING

1.1 PROPOSITIONS (PAGE 9)

1. True 3. True 5. True 7. False 9. False 11. False 13. False 15. True 17. False 19. False 21. True 23. True 25. True 27. False 29. True 31. False 33. Yes 35. No 37. Yes 39. No 45. Yes 47. No 49. No 51. Yes 53. No 55. No 57. No 59 No

1.2 PRELIMINARIES (PAGE 14)

1. -100 3. $\frac{25}{9}$ 5. $\frac{9}{2}$ 7. $\frac{5}{6}$ 9. $\frac{1}{648}$ 11. $-\frac{1}{ab}$ 13. $(3x+2)(3x-2)$

15. $(2x+5)(2x-5)$ 17. $(1+2x)(1-2x))$ 19. $(2x+\sqrt{5})(2x-\sqrt{5})$ 21. $(x+4)(x+3)$

23. $(x-4)(x-3)$ 25. $(2x+1)(3x+2)$ 27. $(2x+1)(3x-5)$ 29. $-x(3x+4)(2x-3)$

31. $-7, \frac{5}{2}$ 33. $\pm\frac{4}{5}$ 35. $\frac{\pm\sqrt{30}}{5}$ 37. -5 39. $-\frac{1}{3}, 0, 5$ 41. $-\frac{1}{2}, 0, 3$ 43. $-5, -2, \frac{1}{5}, \frac{5}{2}$

1.3 UNITS CONVERSIONS (PAGE 20)

1. 0.00061 mi 3. 0.028 yd/min 5. 1.97 in 7. 0.74 m^2 9. 9.56 yd^2

11. 2.0070×10^5 km/sec 13. 172.57 mi 15. 3.16 l/sec 17. 2.74 ft^3 19. 3.13 min

CHAPTER 2: EQUATIONS AND INEQUALITIES

2.1 LINEAR EQUATIONS AND INEQUALITIES (PAGE 28)

1. $\frac{13}{5}$ 3. $\frac{8}{5}$ 5. $\frac{8}{7}$ 7. $x = \frac{3}{2}, y = \frac{3}{4}$ 9. $x = \frac{5}{3}, y = \frac{7}{3}$ 11. $x = \frac{15}{13}, y = \frac{8}{13}$

13. $x > 9$ 15. $x \le -\frac{10}{13}$ 17. $x < 10$ 19. 17 and 8 21. quarters and 8 dimes

23. $l = 24$ in. and $w = 12$ in. 25. 32 and 8 years 27. \$125.00 29. \$84.00 31. 4 ft 33. No

2.2 SYSTEMS OF LINEAR EQUATIONS (PAGE 37)

1. $\left[\begin{array}{ccc|c} 3 & -3 & 1 & 2 \\ 5 & 5 & -9 & -1 \\ -3 & -4 & 1 & 0 \end{array}\right]$ 3. $\left.\begin{array}{r} 5x+y+4z = 3 \\ -2x-3y+z = 4 \\ \frac{1}{2}x-y = 0 \end{array}\right\}$ 5. $x = 1, y = 0, z = 2$

7. $x_1 = 1, x_2 = 3, x_3 = 2, x_4 = 2, x_5 = -1$

9. $x = \frac{1}{2}, y = -\frac{9}{2}, z = 2$ 11. $x = 10, y = -8, z = 12, w = 2$ 15. 4, 6, 11

17. 10 Q, 15 D, AND 7 N 19. 5, 7, 9, 21

2.3 TWO-DIMENSIONAL LINEAR PROGRAMMING (PAGE 46)

1. Max:12; Min: 0 3. Max: 16; Min: -24 5. Max: 75; Min: -30 7. Max: 80; Min: 30
9. MAX: 36; MIN: -80 11. Max: 240; Min: 152 13. Max: 49; Min: -15 15. Max: 120; Min: -140
17. 500 jackets and no pants 19. 26 pounds of X and 21 pounds of Y 21. 9 of Y and 0 of X

CHAPTER 3: FUNCTIONS

3.1 BASIC DEFINITIONS (PAGE 54)

1. $(g \circ f)(2) = 0, (f \circ g)(2) = 0$ 3. $(g \circ f)(2) = 11, (f \circ g)(2) = 12$ 5. $(g \circ f)(2) = \frac{2}{3}, (f \circ g) = \frac{4}{5}$

7. $(g \circ f)(2) = \frac{9}{2}, (f \circ g)(2) = \frac{7}{3}$ 9. $(g \circ f)(x) = x - 2, (f \circ g)(x) = x - 2$

11. $(g \circ f)(x) = 2x^2 + 3, (f \circ g)(x) = 4x^2 - 12x + 12$ 13. $(g \circ f)(x) = \frac{2x}{x+1}, (f \circ g)(x) = \frac{2x}{2x+1}$

19. $f^{-1}(x) = \frac{x}{6} - \frac{1}{3}$ 21. $f^{-1}(x) = \frac{x}{2} - 3$ 23. $f^{-1}(x) = 2x - 2$ 25. $f^{-1}(x) = \frac{3x}{1-x}$

3.2 EXPONENTIAL AND LOGARITHMIC FUNCTIONS (PAGE 62)

1. $-\frac{3}{5}$ 3. $-3, 1$ 5. 1 7. -4 9. $-\frac{1}{3}$ 11. 0 13. 9 15. 2.84 17. 1.95 19. $\frac{1}{2}$

21. 2 23. $-\frac{3}{2}$ 25. $\frac{\ln 12 + 5\ln 3}{3\ln 3}\left(\text{or: } \frac{\ln(12 \cdot 3^5)}{\ln(3^3)}\right)$ 27. $\frac{-\ln 3 - \ln 4}{2\ln 4 - \ln 3}\left(\text{or: } \frac{\ln 12}{\ln\frac{3}{16}}\right)$

29. $\frac{\ln 4 + \ln 3}{\ln 4 - 2\ln 3}\left(\text{or: } \frac{\ln 12}{\ln\frac{4}{9}}\right)$ 31. No solution 33. 8

3.3 EXPONENTIAL GROWTH AND DECAY (page 70)

1. (a) 69.31 min (b) 2568.05 (c) 1809.67 3. (a) 37655 (b) 33653 (c) 154.22
5. (a) 148 years (b) 216.14 grams 7. 121 years 9. 600 years

CHAPTER 4: DIFFERENTIAL CALCULUS

4.1 LIMITS (PAGE 81)

1. 4 3. $\frac{2}{3}$ 5. 0 7. 50 9. $\frac{5}{3}$ 11. –4 13. $\frac{4}{5}$ 15. $\frac{2}{3}$ 17. 0 19. $\frac{1}{5}$ 21. $\frac{3}{32}$ 23. $\frac{1}{5}$
25. 2 27. 0 29. limit: 2; not continuous at 2. 31. limit does not exist
33. limit: 4; continuous at 4 35. limit 4; not continuous at 4

4.2 TANGENT LINES AND THE DERIVATIVE (PAGE 90)

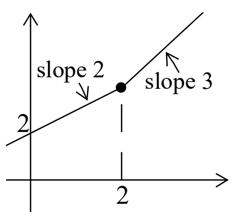
1. $f'(2) \approx 1, f'(4) \approx 0, f'(7) \approx 0$ **3.** 5 **5.** 16 **7.** –1 **9.** 0 **11.** 13 **13.** –5

15. $6x$ **17.** $-4x+1$ **19.** 0 **21.** $y = -3x-30$ **23.** $y = 2x$ **25.** $y = 6x-4$ **27.** $y = 11$

29. The graph of the function $f(x) = x$ is the line $y = x$ which has slope 1. Since the tangent line at every point on the graph of the function $f(x) = x$ is the line $y = x$, $f'(x) = 1$. More formally, for $f(x) = x$: $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h)-x}{h} = \lim\limits_{h \to 0} \frac{h}{h} = \lim\limits_{h \to 0} 1 = 1$.

31. The graph of the function $f(x) = mx+b$ is a line of slope m. It follows that the tangent line at every point on the graph of the function has slope m. Therefore $f'(x) = m$. More formally:
$\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \frac{m(x+h)+b-(mx+b)}{h} = \lim\limits_{h \to 0} \frac{mh}{h} = \lim\limits_{h \to 0} m = m$.

33. No limit at 3 and at 4. Neither continuous nor differentiable at 1,2,3, and 4.

35. 

$\lim\limits_{h \to 2^-} \frac{f(x+h)-f(x)}{h} = 2$ while $\lim\limits_{h \to 2^+} \frac{f(x+h)-f(x)}{h} = 3$.

4.3 DIFFERENTIATION FORMULAS (PAGE 100)

1. 2 **3.** $15x^4+12x^2$ **5.** $-\frac{6}{x^4}$ **7.** $21x^2+10x-4-\frac{4}{x^5}$ **9.** $1-\frac{2}{x^2}-\frac{6}{x^3}$

11. $2x-\frac{6}{x^3}+\frac{15}{x^4}$ **13.** $18x^5+32x^3+9x^2+8x+6$ **15.** $3+\frac{5}{x^2}$ **17.** $\frac{3x^2+24x+13}{(x+4)^2}$

19. $-\frac{30x}{(3x^2+1)^2}$ **21.** $\frac{5x^2-10x}{(2x+1)^2(3x-1)^2}$ **23.** $\frac{6x^3+9x^2+4x}{(3x+1)^2}$ **25.** $y = 5x-4$

27. $y = -5x-8$ **29.** $y = \frac{1}{4}x+\frac{3}{4}$ **31.** $6x-12$ **33.** $6x+4-\frac{2}{x^3}$ **35.** 8

37. $\frac{3}{2}$ **39.** 3 **41.** 13 **43.** $\frac{17}{6}$ **45.** $\frac{7}{4}$ **47.** 49

4.4 OPTIMIZATION PROBLEMS (PAGE 109)

1. 500 units **3.** $2\frac{\text{quarts}}{\text{acre}}$ **5.** $w = 30, L = 30$

7. 100 by 150 ft, with the 100 ft dimension parallel to the dividing fence. **9.** $45

11. $12,000 **13.** $300. **15.** $\$\left(\left[5\left(\frac{13}{\pi}\right)^{2/3} + 10\right] \approx 23.00\right)$

17. $r = \left(\frac{65}{2\pi}\right)^{1/3} \approx 2.2\text{ ft}, h = \left(\frac{260}{\pi}\right)^{1/3} \approx 4.4\text{ ft}$ **19.** 0.42 21. $12349.13

CHAPTER 5: PERSONAL FINANCE

5.1 Interest (page 121)

1. $I = \$875.00, FV = \$5{,}875$ 3. $A = \$5{,}938.43, I = \$938.43$

5. A= $11,875.31, I= $2875.31 7. $A = \$27{,}331.78, I = \$12{,}331.78$ 9. 40 years

10. 14.21 years 11. 21.66 years 13. A=$12,516.56

15. $12,433.22

5.2 Annuities (page 130)

1. 1,747,625 2. 29,667 3. 199.80 5. $46,071.13 7. $58,663.65 9. $66,262.41

11. $1,028.06 13. $2,045.56 15. $754.16 17. $13,562.57 19. $2,226.75

CHAPTER 6: PROBABILITY

6.1 Definitions and Examples (page 139)

1. {Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sept, Oct, Nov, Dec} **3.** {CW, VF, VW, SF, SW}

5. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

7. {AA, AB, AC, BA, BB, BC, CA, CB, CC } **9.** {ABC, ACB, BAC, BCA, CAB, CBA}

11. $\frac{26}{52}$ **13.** $\frac{6}{52}$ **15.** $\frac{6}{36}$ **17.** $\frac{6}{36}$ **19.** $\frac{8}{20}$ **21.** $\frac{8}{20}$ **23.** $\frac{9}{25}$

25. $\frac{8}{25}$ **27.** $\frac{1}{6}$ **29.** $\frac{4}{6}$ **31.** $\frac{1}{6}$ **33.** $\frac{5}{6}$ **35.** $\frac{4}{10}$ **37.** $\frac{2}{10}$

39. $\frac{1}{10}$ **41.** $\frac{6}{10}$ **43.** $\frac{3}{10}$ **45.** $\frac{7}{10}$ **47.** $\frac{1}{6}$ **49.** $\frac{1}{6}$

51. $\frac{1}{5}$ **53.** $\frac{2}{6}$ **55.** $\frac{2}{6}$ **57.** $\frac{6}{20} = 0.3$ **59.** $\frac{6}{21} \approx 0.29$

6.2 Unions and Complements of Events (page 147)

1. $\frac{12}{52}$ **3.** $\frac{36}{52}$ **5.** $\frac{22}{52}$ **7.** $\frac{32}{36}$ **9.** $\frac{24}{36}$ **11.** $\frac{12}{36}$ **13.** $\frac{2}{6}$

15. $\frac{4}{6}$ **17.** $\frac{4}{6}$ **19.** $\frac{7}{11}$ **21.** $\frac{3}{11}$ **23.** $\frac{1}{11}$ **25.** $\frac{2}{11}$ **27.** $\frac{9}{11}$

6.3 Conditional Probability and Independent Events (page 155)

1. (a) $\frac{13}{52}$ (b) $\frac{13}{26}$ (c) $\frac{13}{39}$ **3.** (a) $\frac{4}{11}$ (b) $\frac{4}{6}$ **5.** (a) $\frac{18}{36}$ (b) $\frac{18}{27}$ (c) $\frac{18}{18}$

7. (a) $\frac{1}{500}$ (b) $\frac{1}{450}$ (c) $\frac{1}{200}$ **9.** (a) $\frac{4}{165} \approx 0.02$ (b) $\frac{7}{33} \approx 0.21$ (c) $\frac{28}{165} \approx 0.17$ (d) $\frac{28}{55} \approx 0.51$

11. (a) $\frac{1}{6}$ (b) $\frac{1}{5}$ (c) $\frac{1}{5}$ (d) $\frac{1}{5}$ **13.** (a) $\left(\frac{4}{52}\right)^5 \approx 0.000003$ (b) $\left(\frac{8}{52}\right)^5 \approx 0.000086$

15. (a) $\left(\frac{1}{10}\right)^5 = 0.00001$ (b) $\left(\frac{1}{10}\right)^4 \approx 0.0001$ **17.** (a) $\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \approx 0.12$ (b) $1 - \left(\frac{5}{6}\right)^3 \approx 0.42$

(c) $1 - \left(\frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6}\right) \approx 0.69$ 19. (a) 1 (b) $\frac{14}{15} \cdot \frac{6}{7} \cdot \frac{2}{3} \approx 0.53$

6.4 The Fundamental Counting Principle (page 164)

1. $26 \cdot 25 \cdot 10$ **3.** $26 \cdot 10^4$ **5.** $50 \cdot 10^4$ **7.** $26 \cdot 25 \cdot 4^4$ **9.** $26 \cdot 10^3$

11. $26^2 \cdot 4$ **13.** 20,000,000 **15.** 270,000,000 **17.** 273,375 **19.** 1080

21. (a) $\frac{1}{10^2}$ (b) $\frac{9^6}{10^9} \approx 0.00053$ (c) $\frac{9^6 \cdot 8 \cdot 7 \cdot 6}{10^9} \approx 0.18$ (d) $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10^9} \approx 0.00036$

23. (a) $\frac{1}{4^3} \approx 0.016$ (b) $\frac{24}{4^4} \approx 0.093$ (c) $\frac{13}{4^3} \approx 0.20$ (d) $\frac{9}{4^3} \approx 0.14$ (e) $\frac{10}{4^2} \approx 0.63$

25. (a) $\frac{1}{100}$ (b) $\frac{1}{2}$ (c) $\frac{81}{100}$ (d) $\frac{19}{100}$ (e) $\frac{18}{100}$

6.5 Permutations and Combinations (page 174)

1. 24 **3.** 604,800 **5.** $P(11, 5) = 55{,}440$ **7.** $P(7, 5) = 2520$ **9.** $P(5, 4) = 120$

11. $\frac{240}{6!} = \frac{1}{3}$ **13.** $\frac{3! \cdot 3!}{6!} = \frac{1}{20}$ **15.** $\frac{C(9, 3)C(6, 2)}{C(15, 5)} = \frac{60}{143} \approx 0.42$

17. $\dfrac{P(9,3)P(6,2)}{P(15,5)} = \dfrac{6}{143} \approx 0.042$ **19.** $C(52, 13) = 635{,}013{,}559{,}600 \approx 6.35 \times 10^{11}$

21. $C(5, 3) = 10$ **23.** $\dfrac{C(12,4)C(9,0)}{C(21,4)} = \dfrac{11}{133} \approx 0.083$

25. $\dfrac{C(9,4)+C(7,4)+C(5,4)}{C(21,4)} = \dfrac{166}{5985} \approx 0.028$

27. $\dfrac{C(9,2)C(7,1)C(5,1)+C(9,1)C(7,2)C(5,1)+C(9,1)C(7,1)C(5,2)}{C(21,4)} = \dfrac{9}{19} \approx 0.47$

29. $\dfrac{C(9,3)C(6,2)}{C(15,5)} = \dfrac{60}{143} \approx 0.42$ **31.** $1 - \dfrac{C(9,4)C(6,1)+C(9,5)C(6,0)}{C(15,5)} = \dfrac{101}{143} \approx 0.71$

33. $\dfrac{C(48,2)}{C(52,2)} = \dfrac{188}{221} \approx 0.85$ **35.** $\dfrac{C(13,5)}{C(52,5)} = \dfrac{33}{66{,}640} \approx 0.0005$

37. $\dfrac{C(4,3)C(4,2)}{C(52,5)} = \dfrac{1}{108{,}290} \approx 0.000009$ **39.** $\dfrac{4}{C(52,5)} = \dfrac{1}{649{,}740} \approx 0.0000015$

41. $\dfrac{4}{C(52,13)} \approx 6.3 \times 10^{-12}$ **43.** $\dfrac{C(16,13)}{C(52,13)} \approx 8.8 \times 10^{-10}$

45 (a). 1 **(b)** $\dfrac{C(19,5)C(5,0)+C(19,4)C(5,1)}{C(24,5)} = \dfrac{1292}{1771} \approx 0.73$

(c) $\dfrac{C(19,0)C(5,5)+C(19,1)C(5,4)}{C(24,5)} = \dfrac{4}{1771} \approx 0.0023$

47. **(a)** $\dfrac{1}{C(50,5)} = \dfrac{1}{2{,}118{,}760} \approx 0.00000047$ **(b)** $\dfrac{5 \cdot 45}{C(50,5)} = \dfrac{45}{423{,}752} \approx 0.00011$

(c) $1 - \left[\dfrac{1}{C(50,5)} + \dfrac{5 \cdot 45}{C(50,5)}\right] \approx 0.99989$

<u>6.6 Expected Value</u> (page 182)

1. $\dfrac{3}{2}$ **3.** 1.75 **5.** $0.42 **7.** $0.00

9. Yes; expected winnings is approximately $0.12 **11.** –$0.70 **13.** –$10.67

CHAPTER 7: METHODS OF PROOF

Answers omitted, as answers would be solutions to the problems

INDEX