# Galois Representations Associated to Modular Forms for $\Gamma_0(35)$.

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## Background:

**Modular Curves:** Let $\mathbb{H}$ denote the complex upper half-plane and let $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Then $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the group of $2 \times 2$ matrices with integer coefficients and determinant 1, acts on $\overline{\mathbb{H}}$ by fractional linear transformations.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (z) = \frac{az+b}{cz+d}$$

The quotient of $\overline{\mathbb{H}}$ by the action of the subgroup $\Gamma_0(N)$, consisting of those matrices for which $N \mid c$, is called the modular curve $X_0(N)$.

**Modular Forms:** A weight $k$ modular form for $\Gamma_0(N)$ is a meromorphic function $f$ on $\mathbb{H}$ such that

$$f(\gamma z) = (cz+d)^k f(z) \qquad \forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$

When $k = 0$, this just means that $f(z)$ defines a function on $X_0(N)$. When $k = 2$, it means that $f(z)dz$ gives a well-defined differential on $X_0(N)$ which we denote by $\omega_f$.

**Theorem of Shimura:** Let $f(q) = \sum_{n \geq 1} a_n q^n$ be a weight 2 newform for $\Gamma_0(N)$ whose coefficients generate a degree $d$ extension $K$ of $\mathbb{Q}$. Then $J_0(N) := \mathrm{Jac}(X_0(N))$ has a factor $A_f$ of dimension $d$ defined over $\mathbb{Q}$. For any prime $\ell$, let $\rho_{f,\ell}$ be the representation induced by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $A_f[\ell]$.

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$$

If $p$ is a prime with $p \nmid N\ell$, and $\sigma_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is any Frobenius automorphism, then

$$\mathrm{Tr}\, \rho_{f,\ell}(\sigma_p) = a_p \qquad \det\, \rho_{f,\ell}(\sigma_p) = p.$$

In particular, if $K = \mathbb{Q}$, then $A_f$ is an elliptic curve, and the representation is into $\mathrm{GL}_2(\mathbb{F}_\ell)$.

## Project Goals:

The goal of this project is to verify Shimura's Theorem through explicit computation in the specific case of $\Gamma_0(35)$. In this case, there are three weight 2 newforms, with one newform defined over $\mathbb{Q}$ and two Galois conjugate newforms defined over $K = \mathbb{Q}(\sqrt{-17})$. Thus, the representation corresponding to the first newform may be constructed via an elliptic curve quotient of $X_0(35)$, while the representation corresponding to the other two requires calculations inside of a dimension 2 factor of the Jacobian $J_0(35)$.

## Summary of Results:

### Step 1: Find an Explicit Model for $X_0(35)$

One way to build functions on $X_0(N)$ is to use products/quotients of $\eta$-functions. In this case, we found the following two functions.

$$f = \frac{\eta_5\,\eta_7}{\eta_1\,\eta_{35}} \qquad g = \frac{\eta_1^2\,\eta_5^2}{\eta_7^2\,\eta_{35}^2}$$

An explicit model is simply an equation relating powers of $f$ and $g$. By comparing $q$-expansions (and solving a linear system), we arrived at:

$$g^2 f^3 - g f^6 + 5 g f^5 - 5 g f^3 + 5 g f + g + 49 f^3 = 0.$$

A substitution puts the equation into standard hyperelliptic form.

$$z = \frac{-f^6 + 5f^5 + 2f^3 g - 5f^3 + 5f + 1}{f^2 - 3f - 1} \Longrightarrow$$

$$z^2 = \left(f^2 + f - 1\right)\left(f^6 - 5f^5 - 9f^3 - 5f - 1\right)$$

### Step 2: Match Holomorphic Differentials with Weight 2 Newforms

In terms of the parameters $f$ and $z$, a basis for the holomorphic differentials is given by:

$$\left\{ \frac{df}{z}, f\,\frac{df}{z}, f^2\,\frac{df}{z} \right\}$$

On the other hand, from the LMFDB website, one newform, $f_1$, is defined over $\mathbb{Q}$, and two Galois conjugate newforms, $f_2$ and $\bar{f}_2$, are defined over $K = \mathbb{Q}(\alpha)$ where $\alpha^2 + \alpha - 4 = 0$.

$$
\begin{aligned}
f_1 &= q + q^3 - 2q^4 - q^5 + q^7 - 2q^9 - 3q^{11} - 2q^{12} + 5q^{13} + \cdots \\
f_2 &= q + \alpha q^2 + (-\alpha - 1)q^3 + (-\alpha + 2)q^4 + q^5 - 4q^6 - q^7 + (\alpha - 4)q^8 + \cdots
\end{aligned}
$$

Once again, we compare $q$-expansions and solve a linear system to do the matching.

$$\omega_1 = -\left(1 + f^2\right)\frac{df}{z} \qquad \omega_2 = \left(1 - \alpha f - f^2\right)\frac{df}{z}$$

**Step 3: Equation for Elliptic Curve Corresponding to $\omega_1$**

From the divisor of $\omega_1$ and Riemann-Hurwitz genus formula, it follows that the elliptic curve $E$ corresponding to $f_1$ is the quotient of $X_0(35)$ by an involution fixing the four points defined by $f^2 = -1$. There is only one such involution, $\iota : X_0(35) \to X_0(35)$, given by

$$\iota(f) = \frac{-1}{f} \qquad \iota(z) = \frac{z}{f^4}\,.$$

To describe the quotient, we must find functions that are fixed by this involution:

$$X = f - \frac{1}{f} \qquad Y = \frac{z}{f^2}.$$

Thus, we arrive at the following equation for the elliptic curve quotient corresponding to $f_1$.

$$Y^2 = X^4 - 4X^3 - 2X^2 - 16X - 19$$

A further substitution brings the equation into standard Weierstrass form.

$$X = -\frac{x + 204}{x - 48} \qquad Y = \frac{84y}{(x - 48)^2} \qquad \Longrightarrow \qquad E:\ y^2 = x^3 + 11232x - 78192$$

**Step 4: Verifying Shimura's Theorem**

First we verify the theorem for $\ell = 3$ by computing the 3-torsion polynomial of $E$.

$$p_3(x) = 3x^4+67392x^2-938304x-126157824 = 3(x-48)(x+36)(x^2+12x+24336)$$

From this we find that $E[3]$ is defined over $\mathbb{Q}\left(\sqrt{-3}\right)$. In fact, we may choose the following basis "vectors" for $E[3] \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

$$P = (48, 756) \qquad Q = (-36, 420\sqrt{-3})$$

The representation $\rho_{f_1,3}$ is now computed by applying each Galois automorphism to these basis vectors, which gives the columns of the matrix. In this case, there are only two automorphisms, the identity automorphism and the one determined by $\tau(\sqrt{-3}) = -\sqrt{-3}$.

$$\rho_{f_1,3}(id) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho_{f_1,3}(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now we may choose any prime $p \nmid N\ell = 3 \cdot 5 \cdot 7$. First, take $p = 11$. In this case the Frobenius automorphism, $\sigma_{11}$, is equal to $\tau$. Thus the trace and determinant are 0 and $-1$ respectively. But $a_{11} = -3$ and $p = 11$. So, Shimura is verified!!

$$\operatorname{Tr} \rho_{f_1,3}(\sigma_{11}) = 0 \equiv -3 = a_{11} \qquad \det \rho_{f_1,3}(\sigma_{11}) = -1 \equiv 11 = p$$

The second example would be $p = 13$, for which $\sigma_p = id$. Again, the theorem is verified since

$$\operatorname{Tr} \rho_{f_1,3}(\sigma_{13}) = 2 \equiv 5 = a_{13} \qquad \det \rho_{f_1,3}(\sigma_{11}) = 1 \equiv 13 = p.$$

By Quadratic Reciprocity, $\sigma_p = \tau$ exactly when $p \equiv 2 \bmod 3$ (just like $p = 11$), and $\sigma_p = id$ exactly when $p \equiv 1 \bmod 3$ (just like $p = 13$). Hence, it is easy to verify Shimura's Theorem for all other primes, as seen in the following table.

| $p$ | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 |
|---|---|---|---|---|---|---|---|---|
| $a_p$ | -3 | 5 | 3 | 2 | -6 | 3 | -4 | 2 |

The theorem may be further verified by choosing other initial primes $\ell$. For example, the torsion polynomial for $\ell = 5$ has degree 12 and is irreducible.

## Continuing Work:

We are in the process of similarly verifying Shimura's Theorem for the two Galois conjugate newforms defined over $K = \mathbb{Q}(\sqrt{-17})$. This requires that we do computations inside the Jacobian $J_0(35)$, for which we are using the Mumford representation.

## References


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